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Estimation of Generalized Inverted Exponential Distribution based on Adaptive Type-II Progressive Censoring Data

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Abstract: In this paper, we consider generalized inverted exponential (GIE) model. Three issues represent the purpose of this paper. First, based on adaptive progressively Type-II censored data, we derive the maximum likelihood estimators (MLE) of the model parameters as well as the reliability and hazard rate functions. Next, the Bayes estimates are evaluated by applying Markov chain Monte Carlo method under the balanced squared error (BSEL) loss function and balanced linear exponential (BLINEX) loss function. Based on the asymptotic distributions of the MLEs and MCMC samples, we compute asymptotic confidence interval and symmetric credible interval along with the coverage probability. We analyze a real data set to illustrate the results derived. Simulation studies are conducted to compare the performances of the Bayes estimators with the maximum likelihood estimators. Finally, a numerical example is presented to illustrate the methods developed.

Keywords: Generalized inverted exponential distribution; Adaptive type-II progressive censoring data ; Maximum likelihood estimation, Bayesian estimations; Markov chain Monte Carlo.

1 Introduction

In some life testing, it is common that not all the items under test will be observed until failure. That is, some of the items will be withdrawn or removed from the life test. When this happens, the samples resulting from such life test considered a censored samples. Censoring is a technique that truncates the experiment in a well-planned manner before the failure of all the items is put on the test. Censoring can be done with respect to a pre-specified time, pre-specified number of failures or a combination of both. The main motivation of using different censoring in reliability and life testing analysis, specially in industrial life testing, is to save on test time or the number of items that are tested until failure. Among the different censoring schemes, Type I and Type II censoring schemes are the most used ones in reliability and life testing experiments, see for example Dey and Kundu [1]. Under Type-I censoring scheme, the life testing experiment will be stopped at a pre-fixed time T, while under Type-II censoring scheme, the life testing experiment will be terminated at the time when the rth failure is observed. Progressive Type-II censoring scheme has been discussed by Nelson [2] as a generalization of Type-II censoring scheme. The review article by Balakrishnan [3] addressed progressive censoring scheme and its different applications. In progressively censored life tests, some units are removed during the conduction of the experiment indicating that we do not observe the failure time of any unit. Under this censoring scheme, n units are placed on a life-testing experiment and when the first failure occurs, R_1 of the $n-1$ surviving units are withdrawn from the experiment. When the next failure occurs, R_2 of the $n-2-R_1$ surviving units are withdrawn from the experiment, and so on. Finally, at the time of the m-th failure, all the remaining $R_m = n-m-\sum_{i=1}^{m-1} R_i$ surviving units are withdrawn from the experiment. Prior to the experiment, a number $m(<n)$ is fixed and the censoring scheme R_i, $i=1,2,..,m$ with $R_i \geq 0$ is specified. The set of observed lifetimes $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$ is a progressively Type-II right censored sample. We obtain the usual Type-II censored sample if one set $R_i = 0$, $i=1,2,..,m-1$ and $R_m = n-m$, and the complete sample case (no censoring), if we set $m=n$ and $R_i = 0$, for $i=1,2,..,m-1$. For extensive reviews of the literature on

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progressive censoring see Balakrishnan and Aggarwala [4], Balakrishnan [3], Balakrishnan and Cramer [5] and Soliman et al. [6].
The progressive censoring scheme still has disadvantages, including that the running time T of the experiment is still unknown and it may take long time for m units to fail. Thus, Ng et al. [7] proposed a mixture of Type-I censoring and Type-II progressive censoring scheme, called the adaptive Type-II progressively censoring scheme. In their procedure the effective number of failures m is fixed in advance and the experimental time is allowed to run over time T which is an ideal total time test. According to this model, the progressive censoring scheme $R_1, R_2, ..., R_m$ is available, but the values of some of the R_i may change accordingly during the experiment. If the mth progressively censored observed failures occur before time T ($X_{m:m:n} < T$), the experiment stops at this time $X_{m:m:n}$, and we will have a usual type-II progressive censoring scheme with the pre-fixed progressive censoring scheme $R_1, R_2, ..., R_m$. Otherwise, if $X_{J:m:n} < T < X_{J+1:m:n}$ where $J+1 < m$ and $X_{J:m:n}$ is the Jth failure time occurs before time T, then we will not withdraw any items from the experiment by setting $R_{J+1}, R_{J+2}, ..., R_{m-1} = 0$ and at the time of the mth failure all remaining surviving items $R_m = n - m - \sum_{i=1}^{J} R_i$ are removed from the experiment. Thus, the effectively applied scheme is $(R_1, R_2, ..., R_J, 0, 0, ..., 0, n - m - \sum_{i=1}^{J} R_i)$.
Here, the value of T, the experimenter is free to change the value of T to adjust the optimum of shorter experimental time and a higher chance of observing many failures. According to Lin et al. [8], it guarantees not only acquiring m observed failure times for efficiency of statistical inference but also controlling the total time on the test to be not too far away from the ideal time T. The extreme case when $T \to \infty$, which means that time is not the main consideration for the experimenter, we will have a usual progressive Type-II censoring scheme with pre-fixed $R_i^{'}s$. Furthermore, when $T \to 0$, we will have a conventional Type-II censoring scheme with $R_1, R_2, ..., R_{m-1} = 0$ and $R_m = n - m$.
Several authors, including Hemmati and Khorram [9], Ashour and Nassar [10], Ismail [11], Al Sobhi and Soliman [12], Hemmati and Khorram [13], Nassar and Abo-Kasem [14], Abd-Elmougod and Mahmoud [15] and a more recently EL-Sagheer et al. [16]. Although adaptive Type-II progressively censoring scheme was modified by Cramer and Iliopoulos [17] assuming that T is a random variable. In this study we assume that the lifetimes follow the GIE distribution. The two-parameter GIE distribution was introduced in the literature by Abouammoh and Alshingiti [18]. The study adopts it because it has many applications in several areas of life such as accelerated life testing, horse racing, supermarket queues, sea currents, wind speeds and others, see Nadarajah and Kotz [19]. Also, it has a great ability to synthesize different forms of failure rates. The probability density function (PDF) and cumulative distribution function (CDF) of GIE are given, respectively, by

$$f(x; \alpha, \lambda) = \frac{\alpha\lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\alpha-1}, x \geq 0, \alpha, \lambda > 0. \tag{1}$$

$$F(x; \alpha, \lambda) = 1 - (1 - e^{-\lambda/x})^{\alpha}, x \geq 0, \alpha, \lambda > 0, \tag{2}$$

where $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. The generalized inverted exponential distribution with parameters α and λ will be denoted by GIE(α, λ). The corresponding reliability function and the failure rate function are given by

$$S(x) = (1 - e^{-\lambda/x})^{\alpha}, x \geq 0, \alpha, \lambda > 0, \tag{3}$$

$$H(x; \alpha, \lambda) = \frac{\alpha\lambda}{x^2 (e^{\lambda/x} - 1)} \tag{4}$$

Several studies examined the characteristics and inferences of the GIE distribution using different types of data. First, based on complete sample, Abouammoh and Alshingiti [18] derived some distributional properties and reliability characteristics as well as maximum likelihood estimators (MLEs) of GIE distribution. Estimations using both the MLEs and least squares method for the unknown parameters of the GIE distribution under progressively type-II censored sample are derived by Krishna and Kumar [20]. Based on the same censoring scheme, the necessary and sufficient conditions for existence, uniqueness and existentialists of the MLEs of the parameters have been discussed by Dey and Dey [21]. In Dey and Pradhan [22] as well as Garg et al. [23], the authors discussed some Bayesian inference for the GIE parameters under hybrid random censoring. Furthermore, Krishna et al. [24] addressed estimation of stress–strength parameter $\theta = P(Y < X)$ using progressively first-failure censoring, when X and Y both follow two-parameter generalized inverted exponential distribution. Recently, based on progressively first-failure type-II right-censored data, Ahmed [25] investigated an expectation–maximization (EM) algorithm to obtain maximum likelihood estimates of unknown parameters and to construct asymptotic confidence intervals. He also obtained the Bayes estimation and prediction of generalized inverted exponential distribution. He constructed an exact interval and an exact confidence region for the parameters.

To our knowledge, inference of the unknown parameters of GIE distribution in the presence of adaptive progressive censoring has not been explored yet. Hence the main objective of this article is to make inferences on the proposed model parameters adopting both maximum likelihood as well as Bayesian methods. When the two parameter are unknown, as expected, the explicit expressions of the Bayes estimates cannot be obtained. We propose Markov chain Monte Carlo (MCMC) techniques to compute point and interval estimations of the unknown parameters. The simulation study with the numerical investigations of Bayes estimations are attempted using Metropolis-Hastings algorithm technique via Gibbs sampling.
The remainder of this paper is organized as follows: The maximum likelihood estimates (MLEs) of unknown parameters and reliability characteristic are obtained in Section Two. The asymptotic confidence intervals based on the observed Fisher's information matrix are revisited in this section, as well. We cover Bayes estimates relative to balanced squared error loss function and balanced LINEX loss function using the MCMC techniques in Section Three. Simulation study with numerical comparisons of the estimates and data analysis are presented in Section Four. Section Five involves conclusion.

2 The ML estimation

It is assumed that the lifetimes of the units tested have a GIE(α,λ) distribution and suppose that a random sample of n units is put on a life test experiment. The joint density function of an adaptive progressive Type-II censoring sample $\underline{X}=(X_{1,m,n},\ldots,X_{m,m,n})$ with censoring scheme $\underline{R}=(R_1,R_2,\ldots,R_J,0,0,\ldots,0,n-m-\sum_{i=1}^{J}R_i)$ is then given as

$$f(x_1,\ldots,x_m)=d_J\prod_{i=1}^{m}f(x_{i:m:n};\alpha,\lambda)\prod_{i=1}^{J}[1-F(x_{i:m:n};\alpha,\lambda)]^{R_i}\times[1-F(x_{m:m:n};\alpha,\lambda)]^{R_J^*}, \tag{5}$$

where $f(x_{i:m:n};\alpha,\lambda)$ and $F(x_{i:m:n};\alpha,\lambda)$ are respectively given by (1) and (2),
where

$$d_J=\prod_{i=1}^{m}(n-i+1-\sum_{k=1}^{\min\{i-1,J\}}R_k)\text{ and }R_J^*=n-m-\sum_{i=1}^{J}R_i. \tag{6}$$

Substituting from Equations (1) and (2) in (5), we obtain the likelihood function of α and λ as

$$L(\alpha,\lambda|\underline{x})=d_J\alpha^m\lambda^m\prod_{i=1}^{m}\frac{e^{-\lambda/x_i}}{x_i^2}(1-e^{-\lambda/x_i})^{(\alpha-1)}\prod_{i=1}^{J}(1-e^{-\lambda/x_i})^{\alpha R_i}]\times(1-e^{-\lambda/x_m})^{\alpha R_J^*}, \tag{7}$$

Thus, the log-likelihood function may be written in the following form

$$\ell(\alpha,\lambda|\underline{x})=\log(d_J)+m\log(\alpha)+m\log(\lambda)-2\sum_{i=1}^{m}\log x_i-\lambda\sum_{i=1}^{m}x_i^{-1}$$
$$+(\alpha-1)\sum_{i=1}^{m}\log(1-e^{-\frac{\lambda}{x_i}})+\alpha\sum_{i=1}^{J}R_i\log(1-e^{-\frac{\lambda}{x_i}})+\alpha R_J^*\log(1-e^{-\frac{\lambda}{x_m}}). \tag{8}$$

Calculating the first partial derivatives of Equation (8) with respect to α and λ, and equating to zero, we get the likelihood equations as

$$\frac{m}{\alpha}+\sum_{i=1}^{m}\log(1-e^{-\frac{\lambda}{x_i}})+\sum_{i=1}^{J}R_i\log(1-e^{-\frac{\lambda}{x_i}})+R_J^*\log(1-e^{-\frac{\lambda}{x_m}})=0, \tag{9}$$

and

$$\frac{m}{\lambda}-\sum_{i=1}^{m}x_i^{-1}+(\alpha-1)\sum_{i=1}^{m}x_i^{-1}(e^{\frac{\lambda}{x_i}}-1)^{-1}+\alpha\sum_{i=1}^{J}R_ix_i^{-1}(e^{\frac{\lambda}{x_i}}-1)^{-1}+\alpha R_J^*x_m^{-1}(e^{\frac{\lambda}{x_m}}-1)^{-1}=0. \tag{10}$$

From Equation (9), we have

$$\hat{\alpha}_{ML}=\frac{-m}{\sum_{i=1}^{m}\log(1-e^{-\frac{\hat{\lambda}_{ML}}{x_i}})+\sum_{i=1}^{J}R_i\log(1-e^{-\frac{\hat{\lambda}_{ML}}{x_i}})+R_J^*\log(1-e^{-\frac{\hat{\lambda}_{ML}}{x_m}})}, \tag{11}$$

and we can obtain $\hat{\lambda}$ as the solution of

$$\frac{m}{\hat{\lambda}_{ML}}-\sum_{i=1}^{m}x_i^{-1}+(\hat{\alpha}_{ML}-1)\sum_{i=1}^{m}x_i^{-1}(e^{\frac{\hat{\lambda}_{ML}}{x_i}}-1)^{-1}+\hat{\alpha}_{ML}\sum_{i=1}^{J}R_i x_i^{-1}(e^{\frac{\hat{\lambda}_{ML}}{x_i}}-1)^{-1}$$
$$+\hat{\alpha}_{ML}R_J^{*}x_m^{-1}(e^{\frac{\hat{\lambda}_{ML}}{x_m}}-1)^{-1}=0. \tag{12}$$

It should be noted here that if $X_{m:m:n}<T$, the log-likelihood equations presented by Dey and Dey [21] can be obtained from Equations (9-11). Also, selecting an appropriate value for T is very important, and will be discussed in the numerical part of this paper.
Substituting from (11) in (12), the resulting equation is a non-linear equation in $\hat{\lambda}_{ML}$, which cannot be solved analytically. One must use a numerical procedure such as a Newton–Raphson type algorithm to obtain $\hat{\lambda}_{ML}$. Once we obtaine $\hat{\lambda}_{ML}$, the MLE $\hat{\alpha}_{ML}$ of the parameter α can be obtained from (11). Furthermore, the initial value for the parameter λ can be obtained by using the graphical method, see Balakrishnan and Kateri [26].
Using invariance property of maximum likelihood estimation, the MLEs of the reliability and hazard rate functions from Equations (3) and (4) are obtained as

$$\hat{S}_{ML}(t)=(1-e^{-\hat{\lambda}_{ML}/t})^{\hat{\alpha}_{ML}} \text{ and } \hat{H}_{ML}(t)=\frac{\hat{\alpha}_{ML}\hat{\lambda}_{ML}}{t^2(e^{\lambda/t}-1)}, t>0. \tag{13}$$

2.1 *Asymptotic confidence intervals for α and λ*

The most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the MLEs. It is known that under some regularity conditions, according to Lawless [27], the asymptotic distribution of the MLE of α and λ is approximately distributed as bivariate normal: $(\hat{\theta}_{ML}-\theta)\longrightarrow N_2(0,I^{-1}(\theta))$, where $I^{-1}(\theta)$ is the inverse of the observed information matrix of the unknown parameters $\theta=(\alpha,\lambda)$, given as

$$I^{-1}(\theta)=\begin{pmatrix}-\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\alpha\partial\lambda}\\ -\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\lambda^2}\end{pmatrix}^{-1}_{(\alpha,\lambda)=(\hat{\alpha}_{ML},\hat{\lambda}_{ML})}=\begin{pmatrix}var(\hat{\alpha}_{ML}) & cov(\hat{\alpha}_{ML},\hat{\lambda}_{ML})\\ cov(\hat{\alpha}_{ML},\hat{\lambda}_{ML}) & var\left(\hat{\lambda}_{ML}\right)\end{pmatrix}. \tag{14}$$

The second partial derivatives of log-likelihood function are

$$\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\alpha^2}=-\frac{m}{\alpha^2}, \tag{15}$$

$$\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\lambda\partial\alpha}=\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\alpha\partial\lambda}=\sum_{i=1}^{m}x_i^{-1}(e^{\frac{\lambda}{x_i}}-1)^{-1}+\sum_{i=1}^{J}R_i x_i^{-1}(e^{\frac{\lambda}{x_i}}-1)^{-1}+R_J^{*}x_m^{-1}(e^{\frac{\lambda}{x_m}}-1)^{-1}, \tag{16}$$

and

$$\frac{\partial^2\ell(\alpha,\lambda|\underline{x})}{\partial\lambda^2}=-\frac{m}{\lambda^2}-(\alpha-1)\sum_{i=1}^{m}x_i^{-2}e^{\frac{\lambda}{x_i}}(e^{\frac{\lambda}{x_i}}-1)^{-2}-\alpha\sum_{i=1}^{J}R_i x_i^{-2}e^{\frac{\lambda}{x_i}}(e^{\frac{\lambda}{x_i}}-1)^{-2}-\alpha R_J^{*}x_m^{-2}e^{\frac{\lambda}{x_m}}(e^{\frac{\lambda}{x_m}}-1)^{-2}. \tag{17}$$

A $100(1-\gamma)\%$ two-sided approximate confidence intervals for the parameters α and λ can be given by

$$\hat{\alpha}_{ML}\mp Z_{\gamma/2}\sqrt{var(\hat{\alpha}_{ML})} \text{ and } \hat{\lambda}_{ML}\mp Z_{\gamma/2}\sqrt{var(\hat{\lambda}_{ML})}. \tag{18}$$

where $var(\hat{\alpha}_{ML})$ and $var(\hat{\lambda}_{ML})$ are the estimated variances of $\hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$, which are given by the first and the second, diagonal element of $I^{-1}(\theta)$, and $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

2.2 Approximate confidence intervals for $S(t)$ and $H(t)$

To construct the asymptotic confidence interval of the reliability and hazard functions, we need to find their variances. Thus, we use the delta method, introduced by Greene [28], to calculate the approximate confidence intervals for $S(t)$ and $H(t)$. Let

$$G_1 = \left(\frac{\partial S(t)}{\partial \alpha}, \frac{\partial S(t)}{\partial \lambda}\right) = ((1-e^{-\frac{\lambda}{t}})^{\alpha} \log(1-e^{-\frac{\lambda}{t}}), \alpha e^{-\frac{\lambda}{t}} t^{-1}(1-e^{-\frac{\lambda}{t}})^{\alpha-1}), \tag{19}$$

and

$$G_2 = \left(\frac{\partial H(t)}{\partial \alpha}, \frac{\partial H(t)}{\partial \lambda}\right) = (\lambda(e^{\frac{\lambda}{t}}-1)^{-1}t^{-2}, \alpha(e^{\frac{\lambda}{t}}-1)^{-2}t^{-2}[(e^{\frac{\lambda}{t}}-1)-\lambda t^{-1}e^{\frac{\lambda}{t}}]. \tag{20}$$

Then, the approximate estimates of $\widehat{var}(\hat{S}(t))$ and $\widehat{var}(\hat{H}(t))$ are given, respectively, by

$$\widehat{var}\left[\hat{S}(t)\right] \simeq \left[G_1^t I^{-1}(\alpha,\lambda) G_1\right]_{(\alpha,\lambda)=(\hat{\alpha}_{ML},\hat{\lambda}_{ML})}, \widehat{var}\left[\hat{H}(t)\right] \simeq \left[G_2^t I^{-1}(\alpha,\lambda) G_2\right]_{(\alpha,\lambda)=(\hat{\alpha}_{ML},\hat{\lambda}_{ML})}, \tag{21}$$

where G^t is the transpose G. These results indicate the approximate confidence intervals for $S(t)$ and $H(t)$ as

$$\hat{S}(t) \mp Z_{\gamma/2}\sqrt{\widehat{var}[\hat{S}(t)]} \text{ and } \hat{H}(t) \mp Z_{\gamma/2}\sqrt{\widehat{var}[\hat{H}(t)]}. \tag{22}$$

3 Bayes Estimation

Based on an adaptive progressive type-II censored sample, we provide the Bayes estimates of the unknown parameters as well as $S(t)$ and $H(t)$ of GIE(α,λ) distribution relative to both the balanced squared error loss and balanced LINEX loss functions. However, any other loss function can be easily incorporated as a special case. Also, we use the MCMC sampling procedure to calculate the credible intervals.

3.1 Prior and Posterior

When both the parameters α and λ are unknown it is difficult to obtain joint bivariate prior distribution. Hence, we assume that the parameters α and λ are independent and follow the gamma prior distributions with the following PDFs

$$\pi_1(\alpha|a,b) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha},\ \alpha>0 \text{ and } \pi_2(\lambda|c,d) = \frac{b^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda},\ \lambda>0.$$

The hyper-parameters a, b, c, and d are known and non-negative. In this case, the joint prior distribution of α and λ turns out to be

$$g(\alpha,\lambda) \propto \alpha^{a-1}\lambda^{c-1}e^{-d\lambda}e^{-b\alpha},\ \alpha>0, \lambda>0, a>0, b>0, c>0, d>0. \tag{23}$$

From (5) and (23), the joint posterior distribution of α and λ given the observed data is obtained as

$$q(\alpha,\lambda|\underline{x}) = A\alpha^{m+a-1}\lambda^{m+c-1}e^{-(b\alpha+d\lambda)}\left(1-e^{-\lambda/x_m}\right)^{\alpha R_J^*}\prod_{i=1}^{m}\frac{\left(1-e^{-\lambda/x_i}\right)^{\alpha-1}}{x_i^2 e^{\lambda/x_i}}\prod_{i=1}^{J}\left(1-e^{-\lambda/x_i}\right)^{\alpha R_i}, \tag{24}$$

where

$$A^{-1} = \int_0^{\infty}\int_0^{\infty}\alpha^{m+a-1}\lambda^{m+c-1}e^{-(b\alpha+d\lambda)}\left(1-e^{-\lambda/x_m}\right)^{\alpha R_J^*}\prod_{i=1}^{m}\frac{\left(1-e^{-\lambda/x_i}\right)^{\alpha-1}}{x_i^2 e^{\lambda/x_i}}\prod_{i=1}^{J}\left(1-e^{-\lambda/x_i}\right)^{\alpha R_i} d\alpha d\lambda. \tag{25}$$

3.2 Bayes Estimation under Balanced loss Function

In this subsection, we derive the Bayes estimators for the unkown parameters of a GIED under balanced loss functions. For estimating a scalar parameter θ, Jozani et al. [29] took an extended view of balanced loss in the following form

$$L^{q}_{\rho,\omega,\theta_0}\left(\theta,\hat{\theta}\right)=\omega\rho\left(\theta_0,\hat{\theta}\right)+(1-\omega)\rho\left(\theta,\hat{\theta}\right), \tag{26}$$

where ρ is any loss function, the weight ω takes values in $[0,1)$, and θ_0 is a chosen a prior target estimate of θ, obtained for instance using the criterion of maximum likelihood estimator, least-squares, or unbiasedness among others. In the Bayesian inference, the most commonly used loss function is the squared error loss. This loss function is symmetrical and gives equal weight to overestimation as well as underestimation. By choosing $\rho\left(\theta,\hat{\theta}\right)=\left(\theta-\hat{\theta}\right)^2$, equation (26) reduces balanced squared error loss function, in the form:

$$L^{q}_{\rho,\omega,\theta_0}\left(\theta,\hat{\theta}\right)=\omega\left(\theta_0-\hat{\theta}\right)^2+(1-\omega)\left(\theta-\hat{\theta}\right)^2, \tag{27}$$

and the corresponding Bayes estimate of the unknown parameter θ is given by:

$$\hat{\theta}_{BS}=\omega\theta_0+(1-\omega)E\left(\theta|\mathrm{X}\right). \tag{28}$$

Using symmetric loss functions may be inappropriate in several circumstances, particularly when positive and negative errors have different consequences. One of the most commonly used asymmetric loss function is the LINEX (linear exponential) loss function, see Varian [30]. The balanced LINEX loss function with shape parameter $c\neq 0$ is obtained with the choice of $\rho\left(\theta,\hat{\theta}\right)=e^{c\left(\theta_0-\hat{\theta}\right)}-c\left(\theta-\hat{\theta}\right)-1$ in Equation (26) and given by

$$L^{q}_{\rho,\omega,\theta_0}\left(\theta,\hat{\theta}\right)=\omega\left[e^{c\left(\theta_0-\hat{\theta}\right)}-c\left(\theta-\hat{\theta}\right)-1\right]+(1-\omega)\left[e^{c\left(\theta_0-\hat{\theta}\right)}-c\left(\theta-\hat{\theta}\right)-1\right], \tag{29}$$

and the corresponding Bayes estimate under above-mentioned balanced LINEX loss function can be obtained as

$$\hat{\theta}_{BL}=-\frac{1}{c}\log\left[\omega e^{-c\theta_0}+(1-\omega)E\left(e^{-c\theta}|\mathrm{X}\right)\right]. \tag{30}$$

The sign of c means the direction of asymmetry and its magnitude indicates the degree of asymmetry. For $c<0$, the positive error is better than the negative error. For $c>0$, the negative error is better than the positive error. Furthermore, the loss parameter of balanced LINEX loss function is chosen such that it does not respectively equal zero (0), if it equals zero, the loss function turns to be approximately the balanced squared error loss. The original motivation for balanced loss functions came from a desire to account for both estimation error and goodness of fit (Zellner [31]). Also, it is clear that the Bayes estimates under balanced loss functions are more general including both MLE and Bayes estimates as special cases.
Under the BSEL function (see Soliman et al. [32].), the respective Bayes estimates of a function $g(\alpha;\lambda)=\alpha,\ \lambda,\ S(t)$ or $H(t)$, is given by

$$\hat{g}_{BS}=\omega\hat{g}_{ML}+(1-\omega)\int_0^\infty\int_0^\infty g(\alpha;\lambda)q(\alpha,\lambda|\underline{\mathrm{x}})d\alpha d\lambda, \tag{31}$$

where $\hat{g}_{ML}$ is the MLE of $g(\alpha;\lambda)$. Also, the Bayes estimate of the function $g(\alpha;\lambda)=\alpha,\ \lambda,\ S(t)$ or $H(t)$ under the BLINEX loss function is written as

$$\hat{g}_{BL}=\frac{-1}{c_1}\log[\omega e^{-c_1\hat{g}_{ML}}+(1-\omega)\int_0^\infty\int_0^\infty e^{-c_1 g}q(\alpha,\lambda|\underline{\mathrm{x}})d\alpha d\lambda], \tag{32}$$

From Equations (31) and (32), it is seen that the Bayes estimators of $\alpha,\ \lambda,\ S(t)$ and $H(t)$ are ratio of two integrals for which analytical expressions are unavailable. We apply here the approaches of the MCMC method to calculate the Bayesian estimates of $\alpha,\ \lambda,\ S(t)$ and $H(t)$, and to construct the corresponding credible intervals. In the next section we discuss Metropolis-Hastings (M-H) within the Gibbs algorithm to generate random numbers from (24).

3.3 The M-H within Gibbs sampling algorithm

In this section we use the M-H algorithm which is potentially very powerful tool for approximating posterior expectations. After Metropolis et al. [33] introduced Metropolis algorithm, Hastings [34] provided relevant theory for this sampling method and discussed potential applications of this algorithm in life testing experiments. Furthermore, Gibbs sampling method is a special case of the MCMC method. It can be used to generate a sequence of samples from the full conditional probability distributions of two or more random variables. Gibbs sampling requires decomposing the joint posterior distribution into full conditional distributions for each parameter and then sampling from them. We can estimate desired posterior expectations using this method by proceeding in a following manner. From Eq. (24) the joint posterior density of α and λ can be written as

$$q_{\alpha,\lambda}(\alpha,\lambda|\underline{x}) \propto \alpha^{m+a-1}\lambda^{m+c-1}e^{-(b\alpha+d\lambda)}\left(1-e^{-\lambda/x_m}\right)^{\alpha R_J^*}\prod_{i=1}^{m}\frac{\left(1-e^{-\lambda/x_i}\right)^{\alpha-1}}{x_i^2 e^{\lambda/x_i}}\prod_{i=1}^{J}\left(1-e^{-\lambda/x_i}\right)^{\alpha R_i}, \tag{33}$$

From Equation (33), the posterior conditional density function of α given λ can be obtained as

$$\begin{aligned} q_{\alpha}(\alpha|\lambda,\underline{x}) &\propto \alpha^{m+a-1}e^{-b\alpha}\left(1-e^{-\lambda/x_m}\right)^{\alpha R_J^*}\prod_{i=1}^{m}\left(1-e^{-\lambda/x_i}\right)^{\alpha}\prod_{i=1}^{J}\left(1-e^{-\lambda/x_i}\right)^{\alpha R_i}, \\ &\propto \alpha^{m+a-1}\exp[-\alpha T_m], \end{aligned} \tag{34}$$

where

$$T_m = b-\left[R_J^*\log\left(1-e^{-\lambda/x_m}\right)+\sum_{i=1}^{m}\log\left(1-e^{-\lambda/x_i}\right)+\sum_{i=1}^{J}R_i\log\left(1-e^{-\lambda/x_i}\right)\right], \tag{35}$$

where R_J^* given by (6). From (35), for any $\lambda>0$, and $x_i>0$, $i=1,2,\ldots,m$, one can see that

$$\log\left(1-e^{-\lambda/x_m}\right)<0,\sum_{i=1}^{m}\log\left(1-e^{-\lambda/x_i}\right)<0,\sum_{i=1}^{J}R_i\log\left(1-e^{-\lambda/x_i}\right)<0.$$

So $T_m>0$ for $b>0$.
Similarly, the posterior conditional density function of λ given α can be obtained as

$$q_{\lambda}(\lambda|\alpha,\underline{x}) \propto \lambda^{m+c-1}e^{-(b\alpha+d\lambda)}\left(1-e^{-\lambda/x_m}\right)^{\alpha R_J^*}\prod_{i=1}^{m}\frac{\left(1-e^{-\lambda/x_i}\right)^{\alpha-1}}{x_i^2 e^{\lambda/x_i}}\prod_{i=1}^{J}\left(1-e^{-\lambda/x_i}\right)^{\alpha R_i}. \tag{36}$$

We observe that the conditional distribution of α given λ and data is a gamma distribution with shape parameter $(m+a)$ and scale parameter T_m. However it is quite difficult to derive the conditional posterior distribution of λ given α and data to some known form. Thus, to generate samples from this distribution we assume that the proposal distributions are normal. Then, we generate samples using the following steps.
Step 1: Choose an initial guess of (α,λ), say $(\alpha^{(0)},\lambda^{(0)})$ and set $i=1$.
Step 2: Generate $\lambda^{(i)}$ according the following steps:
(i)- Generate λ^*fromnormal $N(\lambda^{(i-1)};var(\lambda))$ proposal distribution where $var(\lambda)$ denotes the variance of λ.
(ii)- Compute $r=min\left\{1,\dfrac{q_{\alpha}(\lambda^*|\alpha^{(i-1)},\underline{x})}{q_{\alpha}(\lambda^{(s-1)}|\alpha^{(i-1)},\underline{x})}\right\}$,
(iii)- Then generate a sample u from the $U(0;1)$ distribution
(iv)- If $u\leq d$, set $\lambda^{(i)}=\lambda^*$; otherwise $\lambda^{(i)}=\lambda^{(i-1)}$;
Step 3: Generate $\alpha^{(i)}$ from Gamma$(m+a,T_m)$, where T_m is given by (34).
Step 4: Compute $S^{(i)}(t)=(1-\exp(-\lambda^{(i)}/t))^{\alpha^{(i)}}$ and $H^{(i)}(t)=\frac{\alpha^{(i)}\lambda^{(i)}}{t^2(exp(\lambda^{(i)}/t)-1)}$.
Step 5: Set $i=i+1$.
Step 5: Repeat steps $(2-5)$ N times to obtain desired number of samples.
We discard the initial N_0 number of burn-in samples and obtain estimates using the remaining $N-N_0$ samples. The Bayes estimates of $g=g(\alpha,\lambda)$ under the BSE loss function can now be computed as follows

$$\hat{g}_{BS}=\omega\hat{g}(\alpha,\lambda)_{ML}+\frac{(1-\omega)}{N-N_0}\sum_{i=M+1}^{N}g(\alpha^{(i)},\lambda^{(i)}). \tag{37}$$

In addition, the approximate Bayes estimate of the g under BLINEX loss function is then given by

$$\hat{g}_{BL} = \frac{-1}{c_1}\log[\omega e^{-c_1\hat{g}(\alpha,\lambda)_{ML}}\frac{(1-\omega)}{N-N_0}\sum_{i=M+1}^{N} e^{-c_1 g(\alpha^{(i)},\lambda^{(i)})} \tag{38}$$

Substituting from (37) and (38) by $g(\alpha,\lambda)=\alpha,\lambda,S(t)$ or $H(t)$, the Bayes estimates of $\alpha,\lambda,S(t)$ and $H(t)$ under both BSEL and BLINEX loss functions can be obtained.

To compute the credible intervals of $\alpha,\lambda,S(t)$ and $H(t)$, order $\alpha_{(N_0+1)},\ldots,\alpha_{(N)}$, $\lambda_{(N_0+1)},\ldots,\lambda_{(N)}$, $S_{(N_0+1)}(t),\ldots,S_{(N)}(t)$ and $H_{(N_0+1)}(t),\ldots,H_{(N)}(t)$ as $\alpha_{(1)}<\ldots<\alpha_{(N-N_0)}$, $\lambda_{(1)}<\ldots<\lambda_{(N-N_0)}$, $S_{(1)}(t)<\ldots<S_{(N-N_0)}(t)$ and $H_{(1)}(t)<\ldots<H_{(N-N_0)}(t)$, respectively. Then the $100(1-\gamma)\%$ symmetric credible intervals of α and λ become

$$\left[\alpha_{((N-N_0)(\gamma/2))},\alpha_{((N-N_0)(1-\gamma/2))}\right],\left[\lambda_{((N-N_0)(\gamma/2))},\lambda_{((N-N_0)(1-\gamma/2))}\right], \tag{39}$$

Also, the $100(1-\gamma)\%$ symmetric credible intervals of $S(t)$ and $H(t)$ become

$$\left[S_{((N-N_0)(\gamma/2))}(t),S_{((N-N_0)(1-\gamma/2))}(t)\right],\left[H_{((N-N_0)(\gamma/2))}(t),H_{((N-N_0)(1-\gamma/2))}(t)\right]. \tag{40}$$

Table 1: ML, Bayesian estimates and posterior summarize of α, λ, $S(t)$ and $H(t)$.

Method	ML			Bayes ($\omega=0$)						
Parameter	Point	*LB*	*UB*	Mean	Median	Mode	S.D	Ske.	*LB*	*UB*
α	213.48	38.863	465.823	130.781	116.169	86.945	68.9257	1.9491	45.584	317.424
λ	13.7422	10.590	16.895	12.1337	12.1559	12.2003	1.2102	0.2492	10.6619	16.8431
$s(t)$	0.8013	0.7062	0.8963	0.2405	0.2387	0.2351	0.0456	0.2587	0.6619	0.8431
$H(t)$	0.7615	0.4774	1.0458	0.8285	0.8215	0.8075	0.1383	0.3006	0.5804	1.1158

Table 2: Bayesian estimates of α, λ, $S(t)$ and $H(t)$ under balanced loss function.

Parameter	$\omega=0.3$			$\omega=0.6$			$\omega=0.9$		
	BSEL	BLINEX		BSEL	BLINEX		BSEL	BLINEX	
		$c=-0.1$	$c=0.1$		$c=-0.1$	$c=0.1$		$c=-0.1$	$c=0.1$
α	155.665	574.589	72.668	180.443	568.993	78.2641	205.221	555.13	92.1271
λ	12.608	12.6885	12.5254	13.0943	13.1537	13.0288	13.5802	13.5983	13.559
$s(t)$	0.7719	0.7719	0.7718	0.7845	0.7845	0.7844	0.7971	0.7971	0.7971
$H(t)$	0.8082	0.8089	0.8075	0.7882	0.7887	0.7877	0.76824	0.7683	0.7681

4 Numerical experiments and data analysis

In this section we analyze a real data set for illustration purposes and a Monte Carlo simulation study is performed to compare the proposed estimation methods.

4.1 Data Analysis

In this subsection, we consider the following real data set as described in Badar and Priest [35].

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997
2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629	2.880	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585			

The previous data set represents the strength measured in GPa (giga-Pascals), for single carbon fibers and impregnated

1000-carbon fiber tows. Ahmed [25] described that generalized inverted exponential distribution fits the conductor data set well, and the MLEs of α and λ based on the complete data set are $\hat{\alpha}_{ML} = 205.884$ and $\hat{\lambda}_{ML} = 13.8827$. For analyzing the data set under adaptive progressively censored scheme, we have generated an artificial adaptive progressively censored sample from the real data set, with $m = 40$, $T = 2.5$ and censoring scheme $R_1 = 29, R_2 = \cdots = R_{40} = 0$. Thus, the adaptive progressive censored sample is given as

1.312,	1.479,	1.803,	1.861,	1.944,	1.966,	1.997,	2.006,	2.021,	2.027,	2.098,	2.14,
2.179,	2.224,	2.24,	2.253,	2.274,	2.301,	2.359,	2.382,	2.382,	2.434,	2.435,	2.478,
2.511,	2.554,	2.57,	2.629,	2.633,	2.648,	2.684,	2.697,	2.726,	2.773,	2.818,	3.067,
3.09,	3.128,	3.585,	3.585								

We obtain associated MLEs and the corresponding 95% confidence intervals (LB, UB) for α, λ, $S(t = 2)$ and $H(t = 2)$. Next, we compute the Bayes estimates of α, λ, $S(t)$ and $H(t)$. Because we lack prior information, we prefer to use the non-informative priors for α and λ, gamma with $(a = b = c = d = 0)$. The corresponding Bayes estimates relative to both BSEL and BLINEX loss functions with ($\omega = 0.3, .6, 0.9$) are considered and M-H sampling technique. In particular, if $\omega = 0$, BSEL means the squared error loss (SEL). Also, under BLINEX loss, values of c such as $c_1 = -0.1$, 0.1 are considered. We implement the Gibbs with in M-H algorithm to sample from the posterior distribution of α and λ. We run the chain for 11000 times and discard the first 1000 values as burn in. The number of burn in is needed because the chains are initialized with values not actually drawn from the posterior distribution. The simulated values of α and λ obtained at the beginning of a MCMC run are not distributed from the posterior distributions. However, after some iterations have been performed (the burn-in period), the effect of the initial values wears off and the distribution of the new iterates approaches the true posterior distribution. All Bayes estimates and the corresponding 95% credible intervals (LB, UB) are computed based on 10000 MCMC and the results are presented in Tables 1 and 2. Trace plots and corresponding histogram of 10000 iterations relate to the adaptive type II censored sample are listed in Figures. 1 and 2, respectively. They manifest that the plots syndicate good mixing performance and the Gibbs with in M-H sampling is convergent. Also, the density curves of α, λ, $S(t)$ and $H(t)$ are almost symmetric. Moreover, the MCMC results of the posterior mean, median, mode, standard deviation(S.D) and skewness (Ske.) of the parameters, α, λ, $S(t)$ and $H(t)$ are displayed in Table 1. Here, we can take the posterior mean as the best estimate for symmetric distribution and the posterior mode for a skewed distribution.

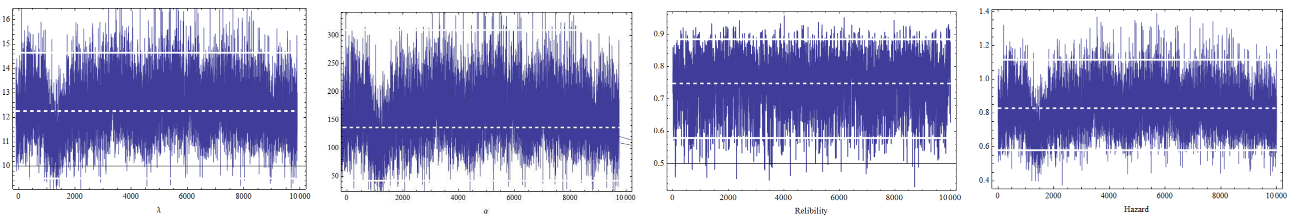

Fig. 1: Fig. 1: MCMC output of $\alpha, \lambda, S(t)$ and $H(t)$Dashed lines (...) represent the posterior means and soled lines (—) represent lower, and upper bounds 95% probability interval.

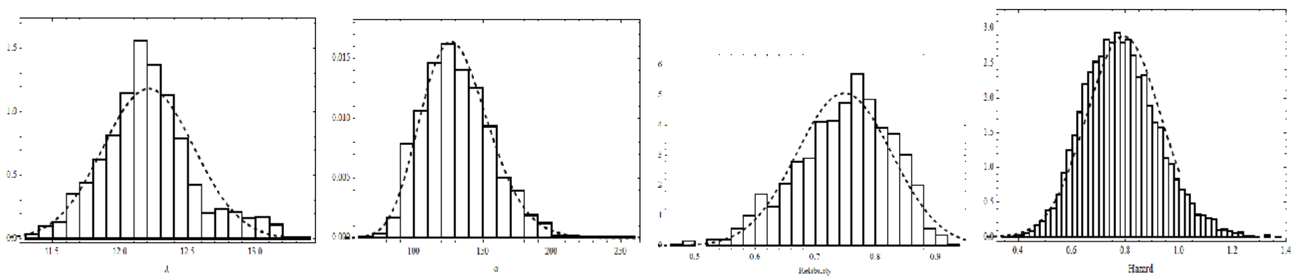

Fig. 2: Histogram and kernel density estimates of $alpha, \lambda, S(t)$ and $H(t)$.

4.1.1 Simulation Study and Comparisons

This subsection examines and compares the behaviors of the maximum likelihood and Bayes estimators for the two parameters of the GIE distribution based on adaptive progressive type II censored for different sample sizes. The simulation study stimulates us to generate the adaptive progressively type II censored sample from the GIE with arbitrarily true values $\alpha = 0.63$ and $\lambda = 2$. We consider three different effective sample sizes m ($m = 20, 30, 35$) from a sample of size n ($n = 40, 50, 60$) and progressive censoring schemes R_i values, taken from the model (1) and $T = 0.5$. The samples were generated using the algorithm described in Ng et al. [7]. We will denote the scheme: $(n = 40, m = 20,$ $R_i = (20,0,0,0,0,0,0,,0,0,0,0,0,0,0,0,0,0,0,0,0))$, by $(20;19^0)$. First we computed the ML estimates and the 95% approximate confidence intervals for α, λ, $S(t)$ and $H(t)$ at mission time $t = 4$, using the methods described in Section 2. The true values of $S(t)$ and $H(t)$ are computed to be $S(t) = 0.4444$ and $H(t) = 0.1214$. We compute the the average estimates (AE) and mean squared errors (MSE) based on 1000 replications.
For Bayesian inference, we use informative priors for both α and λ. The chosen hyper-parameters are $a_1 = 0.5, a_2 = 0.5, b_1 = 1, b_2 = 1$. Bayes estimators are computed using the BSE and BLINEX loss functions with $\omega = 0, 0.3, 0.6, 0.9$ and $c_1 = -0.5, 0.5$. We use the M–H algorithm, given in Section 3, to compute Bayes estimates based on the samples from the posterior distributions of α and λ. It is common to disregard a number of the early samples as a burn-in period to ensure the Markov chain has converged to the proper stationary distribution. In our implementation, we generate a total of $N = 11000$ samples with 1000 of them disregarded as a burn-in. Hence, we used the generated samples to construct both point and interval estimates of α, λ, $S(t)$ and $H(t)$. For different censoring schemes, the average estimates and mean squared error (MSE) for the MLE and the Bayes estimates under two different loss functions for α, λ, $S(t)$ and $H(t)$ are obtained and reported in Tables 3-6. Average estimates is listed first and the corresponding MSE is listed second in all tables. In addition, we computed the 95% Bayesian credible intervals for α, λ, $S(t = 4)$ and $H(t = 4)$ based on MCMC samples. Furthermore, the coverage probability which is the probability that the interval involving the true parameter is computed for the asymptotic confidence interval and Bayes credible interval of α, λ, $S(t = 4)$ and $H(t = 4)$. The results are presented in Table 7.

5 Conclusion

In this article we considered the Bayesian and non-Bayesian inference of the two-parameter generalized inverted exponential lifetime distribution, when the data are adaptive type-II progressive censoring. ML and Bayesian estimations of the model parameters as well as the reliability and hzarard rat functions are obtained. We found that when both parameters are unknown, the explicit expressions of the Bayes estimates cannot be obtained. We proposed Markov chain Monte Carlo (MCMC) techniques to compute point and interval estimations of the unknown parameters. The Gibbs sampling with Metropolis–Hastings sampling technique was confirmed to obtain the Bayesian estimators . The prior belief of the model was represented by the independent gamma priors on the shape and scale parameters. The Bayes estimates relative to BSEL and BLINEX loss functions. We also compared the confidence intervals obtained using asymptotic distribution of the MLEs with the credible intervals obtained by sampling from the posterior distribution. To illustrate the use of the proposed methods of estimation, numerical example was explained using a real life data, and Monte Carlo simulations were performed to compare the different proposed methods. The results exhibit the following:

(i) Tables 3, 4, 5 and 6 demonstrate that the Bayesian estimates relative to both BSEL and BLINEX loss are better than the ML estimates, for all different censoring schemes and most cases of n and m. Moreover, we notice that the MSE decreases when n and m increase.

(ii) The censoring scheme $R = (n - m, 0, ..., 0)$ is most efficient for all choices and it usually provides the smallest MSE for most estimators.

(iii) When ω goes to one all the Bayes estimates go to the corresponding MLEs.

(iv) In all cases, from Tables 3- 6, we observe that when the value of the shape parameter (c) of BLINEX loss function increases, the MSEs of the Bayes estimates using the BLINEX loss function decrease.

(v) Table 7 reveals that the coverage probabilities of the asymptotic confidence intervals and credible intervals based on MCMC samples for λ, α, $S(t)$ and $H(t)$ are close to the desired level of 0.95.

Table 3: Average estimates (AE) (first row) and MSE (second row) of α and λ, with $\omega = 0, 0.3$.

n	m	Scheme		MLE	Bayes ($\omega = 0.$)			Bayes ($\omega = 0.3$)		
					BSEL	BLINEX		BSEL	BLINEX	
						$c=-0.5$	$c=0.5$		$c=-0.5$	$c=0.5$
40	20	$(20, 19^0)$	α	0.7574	0.6485	0.6611	0.6371	0.6812	0.6914	0.6713
				0.1201	0.0461	0.0529	0.0409	0.0615	0.0696	0.0544
			λ	2.2394	1.8942	1.9633	1.8285	1.9978	2.0545	1.9386
				0.5383	0.2739	0.3032	0.2596	0.3101	0.3507	0.2762
		$(10^0, 20, 9^0)$	α	0.7623	0.6392	0.6525	0.6270	0.6761	0.6872	0.6654
				0.1345	0.0491	0.0561	0.0439	0.0666	0.0753	0.0591
			λ	2.2621	1.8949	1.9649	1.8283	2.005	2.0638	1.9438
				0.5896	0.2882	0.3181	0.2738	0.3297	0.3749	0.2918
		$(19^0, 20)$	α	0.7527	0.6368	0.6499	0.6249	0.6716	0.682	0.6614
				0.1066	0.0424	0.0484	0.0379	0.0555	0.0621	0.0495
			λ	2.2558	1.8957	1.9661	1.829	2.0037	2.0615	1.944
				0.4545	0.2321	0.2522	0.2258	0.2558	0.2885	0.2303
50	30	$(20, 29^0)$	α	0.6853	0.6283	0.6345	0.6223	0.6454	0.65	0.6408
				0.0361	0.0223	00237	0.0212	0.0251	0.0263	0.024
			λ	2.1547	1.9196	1.9709	1.8702	1.9901	2.0302	1.9492
				0.2966	0.1988	0.2122	0.1924	0.2973	0.2234	0.1994
		$(15^0, 20, 14^0)$	α	0.7169	0.6467	0.6541	0.6396	0.6678	0.6735	0.6621
				0.0588	0.0288	0.0309	0.0270	0.0352	0.0375	0.0331
			λ	2.1818	1.9356	1.9856	1.8871	2.0095	2.0493	1.9684
				0.3551	0.2105	0.2260	0.2019	0.2303	0.2512	0.2126
		$(29^0, 20)$	α	0.7196	0.6526	0.6602	0.6453	0.6727	0.6784	0.6671
				0.0513	0.0295	0.0317	0.0277	0.0342	0.0362	0.0323
			λ	2.2223	1.974	2.0267	1.9231	2.0485	2.0897	2.006
				0.3803	0.2311	0.2519	0.2181	0.2535	0.2773	0.2331
60	35	$(25, 34^0)$	α	0.705	0.6519	0.6577	0.6464	.6679	0.6721	0.6636
				0.04	0.023	0.025	0.0221	0.0271	0.0285	0.0258
			λ	2.1662	1.9569	2.0014	1.9135	2.0197	2.0542	1.9841
				0.2795	0.1851	0.1987	0.1767	0.1967	0.2119	0.1837
		$(17^0, 25, 17^0)$	α	0.6858	0.6311	0.6371	0.6253	0.6457	0.652	0.6431
				0.0374	0.0243	0.0257	0.0233	0.0271	0.0282	0.0259
			λ	2.1616	1.9536	1.9969	1.9116	2.016	2.0495	1.9817
				0.2388	0.1645	0.1747	0.1592	0.1707	0.1827	0.1612
		$(34^0, 25)$	α	0.6915	0.635	0.6411	0.6292	0.6519	0.6565	0.6475
				0.0394	0.0259	0.0274	0.0246	0.0286	0.0299	0.0273
			λ	2.1609	1.9488	1.9913	1.9073	2.0124	2.0456	1.9783
				0.2821	0.2044	0.2193	0.1951	0.2108	0.2263	0.1981

Table 4: Average estimates (AE) (first row) and MSE (second row) of α and λ, with $\omega = 0.6, 0.9$.

n	m	Scheme		$\omega=0.6$			$\omega=0.9$		
				BSEL	BLINEX		BSEL	BLINEX	
					$c=-0.5$	$c=0.5$		$c=-0.5$	$c=0.5$
40	20	$(20,19^0)$	α	0.7139 0.0828	0.7204 0.0896	0.707 0.0755	0.7466 0.1098	0.7484 0.1121	0.7445 0.1067
			λ	2.1013 0.3833	2.1379 0.4206	2.0587 0.3406	2.2049 0.4934	2.215 0.5067	1.1916 0.4733
		$(10^0,20,9^0)$	α	0.7131 0.0911	0.7203 0.0986	0.7055 0.0831	0.75 0.1225	0.752 0.1251	0.7477 0.1191
			λ	2.1152 0.4131	2.1536 0.4557	2.0703 0.3632	2.2254 0.5385	2.236 0.5583	2.211 0.5138
		$(19^0,20)$	α	0.7063 0.0738	0.713 0.0793	0.6994 0.0681	0.7411 0.975	0.7429 0.0994	0.739 0.0952
			λ	2.1117 0.3164	2.149 0.3483	2.069 0.2815	2.2198 0.4138	2.23 0.4256	2.2065 0.3965
50	30	$(20,29^0)$	α	0.6625 0.0291	0.6653 0.0299	0.6597 0.0282	0.6796 0.0341	0.6803 0.0344	0.6788 0.0339
			λ	2.0607 0.2336	2.0858 0.2477	2.0332 0.2189	2.1312 0.2779	2.138 0.2828	2.1232 0.2715
		$(15^0,20,14^0)$	α	0.6888 0.0438	0.6924 0.0457	0.6851 0.0419	0.7099 0.0547	0.7109 0.0553	0.7088 0.0539
			λ	2.0833 0.2703	2.1085 0.2891	2.0553 0.2497	2.1572 0.3306	2.164 0.3372	1.1488 0.3214
		$(29^0,20)$	α	0.6928 0.0404	0.6963 0.0419	0.6893 0.0390	0.7129 0.0483	0.7138 0.0487	0.7119 0.0478
			λ	2.123 0.2951	2.1488 0.3149	2.0942 0.2736	2.1974 0.3558	2.2044 0.3625	2.189 0.3468
60	35	$(25,34^0)$	α	0.6838 0.0319	0.6864 0.0329	0.6812 0.0309	0.6997 0.0378	0.7004 0.0381	0.699 0.0374
			λ	2.0825 0.2226	2.1040 0.2353	2.0589 0.2091	2.1453 0.2629	2.1510 0.2673	2.1385 0.2573
		$(17^0,25,17^0)$	α	0.6639 0.0307	0.6666 0.0317	0.6612 0.0299	0.6803 0.0355	0.6810 0.0358	0.6796 0.0353
			λ	2.0784 0.1908	2.0992 0.2010	2.0557 0.1802	2.1408 0.2245	2.1464 0.2281	2.1343 0.2201
		$(34^0,25)$	α	0.6689 0.0324	0.6716 0.0334	0.6661 0.0315	0.6858 0.0375	0.6865 0.0378	0.6851 0.0372
			λ	2.076 0.2317	2.0968 0.2442	2.0533 0.2189	2.1397 0.2672	2.1453 0.2713	2.1331 0.2620

Table 5: Average estimates (AE) (first row) and MSE (second row) of $S(t=4)$ and $H(t=4)$, with $\omega=0,0.3$.

n	m	Scheme		MLE	$\omega=0$			$\omega=0.3$		
					BSEL	BLINEX		BSEL	BLINEX	
						$c=-0.5$	$c=0.5$		$c=-0.5$	$c=0.5$
40	20	$(20,19^0)$	$S(t)$	0.5200	0.5327	0.5241	0.5314	0.5289	0.5299	0.5280
				0.0042	0.0037	0.0038	0.0037	0.0038	0.0038	0.0037
			$H(t)$	0.1542	0.113	0.1382	0.1374	0.1427	0.143	0.1424
				0.0031	0.0019	0.0020	0.0019	0.0022	0.0023	0.0022
		$(10^0,20,9^0)$	$S(t)$	0.5445	0.5585	0.5598	0.5572	0.5443	0.5553	0.5434
				0.0035	0.0028	0.0028	0.0027	0.0029	0.0029	0.0029
			$H(t)$	0.1462	0.1249	0.1252	0.1246	0.1313	0.1315	0.121
				0.0031	0.0018	0.0018	0.0018	0.0021	0.0021	0.0021
		$(19^0,20)$	$S(t)$	0.5617	0.5702	0.5714	0.569	0.5677	0.5685	0.5668
				0.0031	0.0024	0.0025	0.0024	0.0026	0.0026	0.0025
			$H(t)$	0.1372	0.1248	0.1251	0.1245	0.1285	0.1287	0.1258
				0.0029	0.0018	0.0018	0.0017	0.0021	0.0021	0.0021
50	30	$(20,29^0)$	$S(t)$	0.5599	0.558	0.5589	0.5571	0.5586	0.5592	0.5579
				0.0027	0.0025	0.0025	0.0025	0.0025	0.0025	0.0025
			$H(t)$	0.1229	0.1166	0.1168	0.1165	0.1185	0.1186	0.1184
				0.0013	0.0011	0.0011	0.0011	0.0011	0.0012	0.0012
		$(15^0,20,14^0)$	$S(t)$	0.5188	0.5257	0.5266	0.5248	0.5236	0.5243	0.523
				0.0023	0.0021	0.0021	0.0021	0.0021	0.0021	0.0022
			$H(t)$	0.1461	0.1327	0.1329	0.1324	0.1367	0.1369	0.1365
				0.0015	0.0011	0.0011	0.0012	0.0012	0.0012	0.0013
		$(29^0,20)$	$S(t)$	0.5399	0.5452	0.5461	0.5443	0.5436	0.5442	0.543
				0.0025	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022
			$H(t)$	0.1370	0.1255	0.1257	0.1253	0.1274	0.1275	0.1272
				0.0015	0.0011	0.0011	0.0011	0.0012	0.0012	0.0012
60	35	$(25,34^0)$	$S(t)$	0.5131	0.5140	0.5149	0.5132	.5138	0.5143	0.5132
				0.0026	0.0026	0.0026	0.0026	0.0026	0.0026	0.0025
			$H(t)$	0.1412	0.134	0.1341	0.1338	0.1361	0.1362	0.136
				0.0014	0.0011	0.0012	0.0011	0.0012	0.0012	0.0012
		$(17^0,25,17^0)$	$S(t)$	0.5760	0.5759	0.5766	0.5752	0.5759	0.5764	0.5754
				0.0021	0.0018	0.0018	0.0017	0.0018	0.0018	0.0018
			$H(t)$	0.1150	0.1089	0.1088	0.1086	0.1106	0.1107	0.1105
				0.0013	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
		$(34^0,25)$	$S(t)$	0.5403	0.5400	0.5447	0.5432	0.5428	0.543	0.5423
				0.0021	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018
			$H(t)$	0.1300	0.1200	0.1240	0.1236	0.1257	0.1258	0.1256
				0.0012	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011

Table 6: Average estimates (AE) (first row) and MSE (second row) of $S(t=4)$ and $H(t=4)$ with $\omega=0.6, 0.9$.

n	m	Scheme		$\omega=0.6$			$\omega=0.9$		
				BSEL	BLINEX		BSEL	BLINEX	
					$c=-0.5$	$c=0.5$		$c=-0.5$	$c=0.5$
40	20	$(20,19^0)$	$S(t)$	0.5251	0.5257	0.5246	0.5213	0.5214	0.5211
				0.0039	0.0039	0.0039	0.0041	0.0041	0.0041
			$H(t)$	0.1476	0.1478	0.1475	0.1526	0.1526	0.1525
				0.0026	0.0026	0.0025	0.0029	0.003	0.0029
		$(10^0,20,9^0)$	$S(t)$	0.5401	0.5407	0.5496	0.5459	0.5461	0.5458
				0.0031	0.0031	0.0031	0.0034	0.0034	0.0034
			$H(t)$	0.1377	0.1378	0.1375	0.1441	0.1441	0.144
				0.0025	0.0025	0.0024	0.0029	0.0029	0.0029
		$(19^0,20)$	$S(t)$	0.5651	0.5656	0.5646	0.5625	0.5627	0.5624
				0.0027	0.0027	0.0027	0.003	0.003	0.003
			$H(t)$	0.1322	0.1324	0.1321	0.1359	0.136	0.1359
				0.0024	0.0024	0.0024	0.0028	0.0028	0.0028
50	30	$(20,29^0)$	$S(t)$	0.5591	0.5595	0.5588	0.5597	0.5598	0.5596
				0.0026	0.0026	0.0025	0.0027	0.0027	0.0027
			$H(t)$	0.1204	0.1205	0.1203	0.1223	0.1223	0.1223
				0.0012	0.0012	0.0012	0.0013	0.0013	0.0013
		$(15^0,20,14^0)$	$S(t)$	0.5215	0.5219	0.5212	0.5195	0.5195	0.5194
				0.0022	0.0022	0.0022	0.0033	0.0023	0.0023
			$H(t)$	0.1408	0.1409	0.1406	0.1448	0.1448	0.1448
				0.0013	0.0013	0.0013	0.0015	0.0015	0.0015
		$(29^0,20)$	$S(t)$	0.542	0.5424	0.5416	0.5404	0.5405	0.5405
				0.0023	0.0023	0.0023	0.0025	0.0025	0.0025
			$H(t)$	0.1292	0.1293	0.1291	0.1311	0.1311	0.131
				0.0013	0.0013	0.0013	0.0014	0.0014	0.0014
60	35	$(25,34^0)$	$S(t)$	0.5135	0.5138	0.5132	0.5132	0.5133	0.5131
				0.0025	0.0025	0.0025	0.0026	0.0026	0.0026
			$H(t)$	0.1383	0.1384	0.1382	0.1405	0.1405	0.1404
				0.0013	0.0013	0.0013	0.0014	0.0014	0.0014
		$(17^0,25,17^0)$	$S(t)$	0.576	0.5763	0.5757	0.576	0.5761	0.5759
				0.0019	0.0019	0.0019	0.002	0.002	0.002
			$H(t)$	0.1125	0.1125	0.1124	0.1143	0.1143	0.1143
				0.0012	0.0012	0.0012	0.0013	0.0013	0.0013
		$(34^0,25)$	$S(t)$	0.5417	0.542	0.5414	0.5406	0.5407	0.5432
				0.0019	0.0019	0.0019	0.0019	0.0019	0.0019
			$H(t)$	0.1275	0.1276	0.1275	0.1294	0.1294	0.1236
				0.0011	0.0011	0.0011	0.0012	0.0012	0.0012

Table 7: 95% coverage probabilities for α, λ, $S(t=4)$, and $H(t=4)$.

Method			ML				Bayes			
n	m	Scheme	α	λ	$S(t)$	$H(t)$	α	λ	$S(t)$	$H(t)$
40	20	$(20,19^0)$	0.966	0.956	0.93	0.93	0.934	0.916	0.948	0.952
		$(10^0,20,9^0)$	0.92	0.908	0.932	0.932	0.928	0.908	0.976	0.954
		$(19^0,20)$	0.962	0.964	0.948	0.948	0.940	0.924	0.974	0.950
50	30	$(20,29^0)$	0.962	0.962	0.936	0.936	0.936	0.900	0.952	0.954
		$(15^0,20,14^0)$	0.980	0.954	0.940	0.940	0.942	0.908	0.970	0.968
		$(29^0,20)$	0.980	0.936	0.926	0.926	0.944	0.915	0.960	0.960
60	35	$(25,34^0)$	0.960	0.956	0.932	0.932	0.942	0.902	0.936	0.944
		$(17^0,25,17^0)$	0.972	0.974	0.938	0.938	0.936	0.914	0.956	0.946
		$(34^0,25)$	0.966	0.946	0.940	0.940	0.922	0.920	0.962	0.956

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