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Characterization on Modified Weibull Distribution

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Abstract: The present paper is devoted to derive some recurrence relations for single and product moments of generalized order statistics (Gos) for modified Weibull distribution (MWD). Based on these recurrence relations, some characterizations for this distribution are discussed.

Keywords: Modified Weibull Distribution; Generalized Order Statistics; Recurrence Relations; Single and Product Moments; Characterization.
Mathematics Subject Classification: 47H17; 47H05; 47H09 .

1 Introduction

Generalized order statistics (Gos) is very important in life-testing and reliability expriments, because it represents well-known models in statistics, such as ordinary order statistics and kthrecord values.

Let F denote a continuous distribution function with density function f . The random variables $X(i,n,\widetilde{m},k); i=1,2,...,n$ are called Gos based on F, if their joint density function is given by

$$f^{X(1,n,\widetilde{m},k),...,X(n,n,\widetilde{m},k)}(x_1,...,x_n)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}[1-F(x_i)]^{m_i}f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n), \tag{1}$$

on the cone $F^{-1}(0)<x_1\leq...\leq x_n<F^{-1}(1-)$ of $\mathbb{R}^n$, with parameters, $n\in\mathbb{N}, n\geq 2, k>0, \widetilde{m}=(m_1,m_2,...,m_{n-1})\in\mathbb{R}^{n-1}$, $M_r=\sum\limits_{j=r}^{n-1}m_j$ such that $\gamma_r=k+n-r+M_r>0$ for all $r\in\{1,2,...,n-1\}$. Moreover,

$$C_{r-1}=\prod_{j=1}^{n-1}\gamma_j, r=1,2,...,n-1, \gamma_r-\gamma_{r+1}-1=m_r. \tag{2}$$

The following two cases are considered:
For $m_1=m_2=...=m_{n-1}=m$.
The marginal pdf of the r-th gos $X(r,n,m,k)$ is given by

$$\frac{C_{r-1}}{\Gamma(r)}[1-F(x)]^{\gamma_r-1}f(x)g_m^{r-1}[F(x)], \tag{3}$$

where

$$g_m(x)=\begin{cases}-\ln(1-x), & m=-1\\ \frac{1-(1-x)^{m+1}}{m+1}, & m\neq -1,\end{cases} \tag{4}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\leq r\leq s\leq n$ is

$$\frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)}[1-F(x)]^{\gamma_r-1}f(x)g_m^{r-1}[F(x)][h_m(F(y))-h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s-1}f(y), \tag{5}$$

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where C_{r-1} is defined in (2) and $\gamma_j = k+(m+1)(n-j)$.
For $\gamma_i \neq \gamma_j, i, j = 1,2,...,n-1$.
The marginal pdf of the r-th gos $X(r,n,m,k)$ is given by

$$C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [1-F(x)]^{\gamma_i - 1}, \tag{6}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r \leq s \leq n$ is

$$C_{s-1} \left[\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{1-F(y)}{1-F(x)} \right)^{\gamma_i} \right] \left[\sum_{i=r+1}^{r} a_i(r) (1-F(x))^{\gamma_i} \right] \frac{f(x)}{1-F(x)} \frac{f(y)}{1-F(y)}, -\infty < x < y < \infty. \tag{7}$$

where

$$a_i^{(r)}(s) = \prod_{j=r+1, j\neq i}^{s} \frac{1}{\gamma_j - \gamma_i}, r+1 \leq i \leq s \leq n, a_i(r) = \prod_{j=1, j\neq i}^{r} \frac{1}{\gamma_j - \gamma_i} \ ,1 < i < r, \text{ and } a_i(s) = a_i^{(o)}(s). \tag{8}$$

Recuurence relations are very useful in obtaining moments, moment generating function and characterizting distributions. Recurrence relations for Gos have been discussed by several authors, see [1,2,3,4,5,6] and [7].

[8] represented the modified Weibull distribution (MWD) as a generalization of the linear failure rate distribution (LFRD). The pdf and the cdf of the MWD(λ, β, γ) are respectively given, by

$$f(x) = \left(\lambda + \beta\gamma x^{\gamma-1}\right) \exp\left(-\lambda x - \beta x^{\gamma}\right), \ \ \lambda, \beta \geq 0, \ \gamma > 0, x > 0, \tag{9}$$

and

$$F(x) = 1 - \exp\left(-\lambda x - \beta x^{\gamma}\right), \tag{10}$$

the characterizing differential equation is also given by

$$f(x) = \left(\lambda + \beta\gamma x^{\gamma-1}\right) [1 - F(x)], \tag{11}$$

where λ is a scale parameter, while β and γ are the shape parameters. Eq. (9) involves the following:

(i)If $\lambda = 0$, $\beta = c$ and $\gamma = a$, then MWD reduces to $Weibull(c,a)$.See [9].
(ii)If $\lambda = 0$, $\beta = \theta$ and $\gamma = 1$, then MWD reduces to $Exponential(\theta)$.See [10].
(iii)If $\lambda = 0$, $\beta = \theta$ and $\gamma = 2$, then MWD reduces to $Rayleigh(\theta)$. See [11].
(iv)If $\alpha = \lambda, \gamma = 2$ and $\beta = \frac{\theta}{2}$, then MWD reduces to LFRD(λ, θ). See [12].

To identify various views on recurrence relations of Gos for different distributions, see [13,14,15,16,17] and [18]. Sections Two and Three present new recurrence relations for MDW. Section Four involves characterizations of MWD.

2 Recurrence Relations for the Single Moments

In this section, we obtain recurrence relations for single moment of Gos from MWD with pdf and cdf given by Eq. (9) and Eq. (10), respectively.
For $m_1 = m_2 = ... = m_{n-1} = m$.

The single moment of Gos is written as follows:

$$\begin{aligned} \mu_{(r,n,m,k)}^{(i)} &= E\left(X_{(r,n,m,k)}^{(i)}\right) \\ &= \frac{C_{r-1}}{\Gamma(r)} \int_0^{\infty} x^i [1-F(x)]^{\gamma_r - 1} f(x) g_m^{r-1} [F(x)] \, dx, \end{aligned} \tag{12}$$

where C_{r-1} and $g_m(x)$ are defined in Eq. (2) and Eq. (4). The single moment of Gos given in Eq. (12) satisfies the following:

Relation 1. For $n \geq 1$ and $1 \leq r \leq n-1$, the following recurrence relation between the single moments of the Gos from MWD holds

$$\mu_{(r,n,m,k)}^{(j)} = \frac{j}{\gamma_r} E\left(\frac{X^{j-1}}{\lambda + \beta\gamma X^{\gamma-1}}\right) + \mu_{(r-1,n,m,k)}^{(j)}. \tag{13}$$

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Proof. Beginning with Eq. (12), we write

$$\mu_{(r,n,m,k)}^{(j)} = \frac{C_{r-1}}{\Gamma(r)} \int_0^{\infty} x^i \left[1-F(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1} \left[F(x)\right] dx, \tag{14}$$

where C_{r-1} is defined in Eq. (2) and using Eq.(11) we get,

$$\begin{aligned} \mu_{(r,n,m,k)}^{(j)} &= \frac{C_{r-1}}{\Gamma(r)} \int_0^{\infty} x^j \exp\{-\gamma_r(\lambda x + \beta x^{\gamma})\}] \left[\lambda + \beta\gamma x^{\gamma-1}\right] \\ &\times \left[\frac{1}{m+1}(1-[\exp\{-(\lambda x+\beta x^{\gamma})\}]^{m+1})\right]^{r-1} dx. \end{aligned} \tag{15}$$

Integrating by parts, we have

$$\begin{aligned} \mu_{(r,n,m,k)}^{(j)} &= \frac{jC_{r-1}}{\gamma_r \Gamma(r)} \int_0^{\infty} x^{j-1} \left[\frac{1}{m+1}(1-[\exp\{-(\lambda x+\beta x^{\gamma})\}]^{m+1})\right]^{r-1} \\ &\times \exp\{-(\lambda x+\beta x^{\gamma})\}^{\gamma_r - 1} \frac{f(x)}{\lambda+\beta\gamma x^{\gamma-1}} dx \\ &+ \frac{(r-1)C_{r-1}}{\gamma_r \Gamma(r)} \int_0^{\infty} x^j \left[\frac{1}{m+1}(1-[\exp\{-(\lambda x+\beta x^{\gamma})\}]^{m+1})\right]^{r-2} \\ &\times \exp\{-(\gamma_r+1)(\lambda x+\beta x^{\gamma})\} \left[\lambda+\beta\gamma x^{\gamma-1}\right] [\exp\{-(\lambda x+\beta x^{\gamma})\}]^m dx. \end{aligned} \tag{16}$$

After some simplifications, we obtain Eq. (13).■

For $\gamma_i \neq \gamma_j, i, j = 1, 2, ..., n-1$.

The single moment of Gos can be written as

$$\begin{aligned} \mu_{(r,n,\tilde{m},k)}^{(i)} &= E\left(X_{(r,n,\tilde{m},k)}^{(t)}\right) \\ &= C_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^{\infty} x^t \left[1-F(x)\right]^{\gamma_i - 1} f(x) dx, \end{aligned} \tag{17}$$

where C_{r-1} and $a_i(r)$ are defined in Eq. (2) and Eq. (8).

Relation 2. For $n \geq 1$ and $1 \leq r \leq n-1$, the following recurrence relation between the single moments of the Gos from MWD holds

$$\begin{aligned} \mu_{(r,n,\tilde{m},k)}^{(t)} = \gamma_r \Bigg[& \frac{\lambda}{t+1} \left\{\mu_{(r,n,\tilde{m},k)}^{(t+1)} - \mu_{(r-1,n,\tilde{m},k)}^{(t+1)}\right\} \\ & + \frac{\beta\gamma}{t+\gamma} \left\{\mu_{(r,n,\tilde{m},k)}^{(t+\gamma)} - \mu_{(r-1,n,\tilde{m},k)}^{(t+\gamma)}\right\} \Bigg]. \end{aligned} \tag{18}$$

Proof. Using Eq. (17) and Eq. (11), we have

$$\begin{aligned} \mu_{(r,n,\tilde{m},k)}^{(i)} &= C_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^{\infty} x^t \left[1-F(x)\right]^{\gamma_i - 1} \left[\lambda+\beta\gamma x^{\gamma-1}\right] \left[1-F(x)\right] dx, \\ &= I + II, \end{aligned} \tag{19}$$

where

$$I = \lambda C_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^{\infty} x^t \left[1-F(x)\right]^{\gamma_i} dx, \tag{20}$$

and

$$II = \beta\gamma C_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^{\infty} x^{t+\gamma-1} \left[1-F(x)\right]^{\gamma_i} dx, \tag{21}$$

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integrating by parts Eq. (20) and Eq. (21), we get

$$I=\frac{\lambda C_{r-1}}{t+1}\sum_{i=1}^{r}\gamma_i a_i(r)\int_0^\infty x^{t+1}\left[1-F(x)\right]^{\gamma_i-1}f(x)dx, \tag{22}$$

using

$$a_i(r-1)=(\gamma_r-\gamma_i)\,a_i(r) \quad \text{and} \quad \gamma_r C_{r-2}=C_{r-1}, \tag{23}$$

we rewrite

$$\begin{aligned} I=&\frac{\lambda\gamma_r}{t+1}C_{r-1}\sum_{i=1}^{r}a_i(r)\int_0^\infty x^{t+1}\left[1-F(x)\right]^{\gamma_i-1}f(x)dx\\ &-\frac{\lambda\gamma_r}{t+1}C_{r-2}\sum_{i=1}^{r-1}a_i(r-1)\int_0^\infty x^{t+1}\left[1-F(x)\right]^{\gamma_i-1}f(x)dx, \end{aligned} \tag{24}$$

so,

$$I=\frac{\lambda\gamma_r}{t+1}\left[\left\{\mu^{(t+1)}_{(r,n,\tilde{m},k)}-\mu^{(t+1)}_{(r-1,n,\tilde{m},k)}\right\}\right]. \tag{25}$$

Also,using the same manner, we obtain

$$II=\frac{\beta\gamma\gamma_r}{t+\gamma}\left[\left\{\mu^{(t+\gamma)}_{(r,n,\tilde{m},k)}-\mu^{(t+\gamma)}_{(r-1,n,\tilde{m},k)}\right\}\right]. \tag{26}$$

Substituting Eq. (25) and Eq. (26)) into Eq. (19), Eq. (18) is obtained. ■

3 Recurrence Relations for the Product Moments

This section is devoted obtaining the results for product moments of Gos from MWD considering two cases:

Case I: $m_1=m_2=...=m_{n-1}=m$

The product moment of Gos is written as

$$\begin{aligned} \mu^{(i,j)}_{(r,s:n,m,k)}&=E\left(X^{(i)}_{(r,n,m,k)}Y^{(j)}_{(s,n,m,k)}\right)\\ &=\frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)}\int_0^\infty\int_x^\infty x^iy^j\left[1-F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\\ &\times\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[1-F(y)\right]^{\gamma_s-1}f(y)dydx, \end{aligned} \tag{27}$$

where C_{r-1} and $g_m(x)$ are defined in Eq.(2) and Eq.(4).

Relation 3. For $n\geq 1$, the following recurrence relation between the product moments of Gos from the MWD holds.

$$\mu^{(i,j)}_{(r,r+1:n,m,k)}=E(X^{i+j})+\frac{j}{\gamma_{r+1}}E\left(X^i\frac{Y^{j-1}}{\lambda+\beta\gamma Y^{\gamma-1}}\right) \qquad \text{if } s=r+1, \tag{28}$$

and

$$\mu^{(i,j)}_{(r,s:n,m,k)}=\frac{j}{\gamma_s}E\left(\frac{X^iY^{j-1}}{\lambda+\beta\gamma Y^{\gamma-1}}\right)+\mu^{(i,j)}_{(r,s-1:n,m,k)} \qquad \text{if } s>r+1, \tag{29}$$

Proof. For $s=r+1$ and using Eq. (27),we have

$$\begin{aligned} \mu^{(i,j)}_{(r,r+1:n,m,k)}&=E\left(X^{(i)}_{(r,n,m,k)}Y^{(j)}_{(r+1,n,m,k)}\right)\\ &=\frac{C_r}{\Gamma(r)}\int_0^\infty\int_x^\infty x^i\left[\lambda+\beta\gamma x^{\gamma-1}\right]\left[\exp\{-(m+1)(\lambda x+\beta x^\gamma)\}\right]\\ &\times\left[\frac{1}{m+1}\left(1-\left[\exp\{-(\lambda x+\beta x^\gamma)\}\right]^{m+1}\right)\right]^{r-1}I(y)dx, \end{aligned} \tag{30}$$

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where

$$I(y)=\int_x^\infty y^j\left[\lambda+\beta\gamma y^{\gamma-1}\right]\exp\left[-\gamma_{r+1}\left(\lambda y+\beta y^\gamma\right)\right]dy. \tag{31}$$

Integrating by parts $I(y)$, we obtain

$$I(y)=\frac{x_j}{\gamma_{r+1}}\exp\left[-\gamma_{r+1}\left(\lambda x+\beta x^\gamma\right)\right]+\frac{j}{\gamma_{r+1}}\int_x^\infty y^j\exp\left[-\gamma_{r+1}\left(\lambda y+\beta y^\gamma\right)\right]dy. \tag{32}$$

Simplifying and substituting Eq.(32) into Eq. (30), we obtain Eq. (28).

When $s>r+1$ and using Eq. (29),we have

$$\begin{aligned}\mu^{(i,j)}_{(r,s:n,m,k)}&=E\left(X^{(i)}_{(r,n,m,k)}Y^{(j)}_{(s,n,m,k)}\right)\\&=\frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)}\int_0^\infty\int_x^\infty x^iy^j\left[\lambda+\beta\gamma x^{\gamma-1}\right]\left[\exp\left\{-\gamma_r\left(\lambda x+\beta x^\gamma\right)\right\}\right]\\&\times\left[\frac{1}{m+1}\left(1-\left[\exp\left\{-\left(\lambda x+\beta x^\gamma\right)\right\}\right]^{m+1}\right)\right]^{r-1}I(y)dx,\end{aligned} \tag{33}$$

where

$$\begin{aligned}I(y)&=\int_x^\infty y^j\left[\lambda+\beta\gamma y^{\gamma-1}\right]\left[\exp\left\{-\gamma_s\left(\lambda x+\beta y^\gamma\right)\right\}\right]\\&\times\left[\frac{1}{m+1}\left\{\exp\left\{-(m+1)(\lambda x+\beta x^\gamma)\right\}-\exp\left\{-(m+1)(\lambda y+\beta y^\gamma)\right\}\right\}\right]^{s-r-1}dy.\end{aligned} \tag{34}$$

Integrating by parts $I(y)$, we obtain

$$\begin{aligned}I(y)&=\frac{j}{\gamma_s}\int_x^\infty y^{j-1}\left[\frac{1}{m+1}\left\{\exp\left\{-(m+1)(\lambda x+\beta x^\gamma)\right\}-\exp\left\{-(m+1)(\lambda y+\beta y^\gamma)\right\}\right\}\right]^{s-r-1}\\&\times\exp\left[-\gamma_s(\lambda y+\beta y^\gamma)\right]dy+\frac{(s-r-1)(m+1)}{\gamma_s}\\&\times\int_x^\infty y^j\left[\frac{1}{m+1}\left\{\exp\left\{-(m+1)(\lambda x+\beta x^\gamma)\right\}-\exp\left\{-(m+1)(\lambda y+\beta y^\gamma)\right\}\right\}\right]^{s-r-2}\\&\times\left[\lambda+\beta\gamma y^{\gamma-1}\right]\exp\left[-(\gamma_{s-1})(\lambda y+\beta y^\gamma)\right]dy,\end{aligned} \tag{35}$$

Simplifying and substituting Eq.(35) into Eq. (33), we obtain Eq. (29). ■

Case II: $\gamma_i\neq\gamma_j, i,j=1,2,...,n-1$.

The product moment of Gos can be written as

$$\begin{aligned}\mu^{(t,z)}_{(r,s:n,\widetilde{m},k)}&=E\left(X^{(t)}_{(r,n,\widetilde{m},k)}Y^{(z)}_{(s,n,\widetilde{m},k)}\right)\\&=C_{s-1}\int_{-\infty}^\infty\int_x^\infty x^ty^z\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\\&\times\left[\sum_{i=r+1}^{r}a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)}dydx,\end{aligned} \tag{36}$$

where C_{s-1} and $a_i^{(r)}(s)$ are defined in Eq. (2) and Eq. (8).

Relation 4. For $n\geq1$ and $1\leq r<s\leq n$, the following recurrence relation between the product moments of the Gos from MWD holds

$$\begin{aligned}\mu^{(t,z)}_{(r,s:n,\widetilde{m},k)}&=\frac{\lambda\gamma_s}{z+1}\left\{\mu^{(t,z+1)}_{(r,s:n,\widetilde{m},k)}-\mu^{(t,z+1)}_{(r,s-1:n,\widetilde{m},k)}\right\}\\&+\frac{\beta\gamma\gamma_s}{z+\gamma}\left\{\mu^{(t,z+\gamma)}_{(r,s:n,\widetilde{m},k)}-\mu^{(t,z+\gamma)}_{(r,s-1,n,\widetilde{m},k)}\right\}.\end{aligned} \tag{37}$$

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Proof. It can be proved in the same method as Relation (2),

$$\begin{aligned}\mu^{(t,z)}_{(r,s:n,\tilde{m},k)} &= C_{s-1}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^t y^z \left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right] \\ &\times\left[\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]\,[1-F(y)]} \\ &\times\left[\lambda+\beta\gamma y^{\gamma-1}\right][1-F(y)]\,dydx \\ &= I+II,\end{aligned} \tag{38}$$

where

$$\begin{aligned}I &= \lambda C_{s-1}\int_{0}^{\infty}\frac{x^t f(x)}{[1-F(x)]}\left[\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right] \\ &\times\int_{x}^{\infty} y^z\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} dy,\end{aligned} \tag{39}$$

and

$$\begin{aligned}II &= \beta\gamma C_{s-1}\int_{0}^{\infty}\frac{x^t f(x)}{[1-F(x)]}\left[\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right] \\ &\times\int_{x}^{\infty} y^{z+\gamma-1}\left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right) dy,\end{aligned} \tag{40}$$

integrating by parts of Eq. (39) and Eq. (40) , we get

$$\begin{aligned}I &= \frac{\lambda C_{s-1}}{z+1}\int_{0}^{\infty}\frac{x^t f(x)}{[1-F(x)]}\left[\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right] \\ &\times\int_{x}^{\infty} y^{z+1}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\frac{(1-F(y))^{\gamma_i-1}}{(1-F(x))^{\gamma_i}} f(y)dy.\end{aligned} \tag{41}$$

Using

$$a_i(s-1) = (\gamma_s-\gamma_i)\,a_i(s) \quad \text{and} \quad \gamma_s C_{s-2} = C_{s-1}, \tag{42}$$

we obtain

$$\begin{aligned}I &= \frac{\lambda C_{s-1}}{z+1}\int_{0}^{\infty}\frac{x^t f(x)}{[1-F(x)]}\left[\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right] \\ &\times\int_{x}^{\infty} y^{z+1}\sum_{i=r+1}^{s}\gamma_s a_i^{(r)}(s)\frac{(1-F(y))^{\gamma_i-1}}{(1-F(x))^{\gamma_i}} f(y)dy, \\ &-\frac{\lambda C_{s-1}}{z+1}\int_{0}^{\infty}\frac{x^t f(x)}{[1-F(x)]}\left[\sum_{i=1}^{r} a_i(r)\,(1-F(x))^{\gamma_i}\right] \\ &\times\int_{x}^{\infty} y^{z+1}\sum_{i=r+1}^{s}(\gamma_s-\gamma_i) a_i^{(r)}(s)\frac{(1-F(y))^{\gamma_i-1}}{(1-F(x))^{\gamma_i}} f(y)dy.\end{aligned} \tag{43}$$

Then,

$$I = \frac{\lambda\gamma_s}{z+1}\left[\mu^{(t,z+1)}_{(r,s:n,\tilde{m},k)} - \mu^{(t,z+1)}_{(r,s-1:n,\tilde{m},k)}\right]. \tag{44}$$

Similarly, we obtain

$$II = \frac{\beta\gamma\gamma_s}{z+\gamma}\left[\mu^{(t,z+\gamma)}_{(r,s:n,\tilde{m},k)} - \mu^{(t,z+\gamma)}_{(r,s-1:n,\tilde{m},k)}\right]. \tag{45}$$

Simplifying and substituting Eq. (44) and Eq. (44) into Eq. (38), we obtain Eq. (37).■

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4 Characterizations of Modified Weibull Distribution

Characterizations of MWD are presented in subsec. (4.1) and (4.2), respectively.

4.1 Characterization of MWD based on single moments

Theorem 1: Let X be a nonnegative random variable with absolutely continuous cdf $F(x)$ and pdf $f(x)$, with $F(0)=0$ and $0<F(x)<1$, for all $x>0$. Then for $n\geq 1$ and $r=1,2,...,n-1$.
(i) For $m_1=m_2=...=m_{n-1}=m$, X has the MWD(λ,β,γ) iff

$$\mu_{(r,n,m,k)}^{(j)}=\frac{j}{\gamma_r}E\left(\frac{X^{j-1}}{\lambda+\beta\gamma X^{\gamma-1}}\right)+\mu_{(r-1,n,m,k)}^{(j)}, \tag{46}$$

holds

(ii) For $\gamma_i\neq\gamma_j, i,j=1,2,...,n-1, X$ has the MWD(λ,β,γ) iff

$$\begin{aligned}\mu_{(r,n,\widetilde{m},k)}^{(t)}=\gamma_r\Bigg[&\frac{\lambda}{t+1}\left\{\mu_{(r,n,\widetilde{m},k)}^{(t+1)}-\mu_{(r-1,n,\widetilde{m},k)}^{(t+1)}\right\}\\&+\frac{\beta\gamma}{t+\gamma}\left\{\mu_{(r,n,\widetilde{m},k)}^{(t+\gamma)}-\mu_{(r-1,n,\widetilde{m},k)}^{(t+\gamma)}\right\}\Bigg],\end{aligned} \tag{47}$$

where $\widetilde{m}=(m_1,m_{2,...,}m_{n-2})$ holds.

Proof. The necessary part can be proved immediately from Relation (1). For the sufficient part, consider Eq. (46) is satisfied. Then, Eq. (46) can be written as:

$$\begin{aligned}&\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^j\left[1-F(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx\\&=\frac{j}{\gamma_r}\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty\frac{x^{j-1}}{\lambda+\beta\gamma x^{\gamma-1}}\left[1-F(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx\\&+\frac{C_{r-2}}{\Gamma(r-1)}\int_0^\infty x^j\left[1-F(x)\right]^{\gamma_r+m}f(x)g_m^{r-2}\left[F(x)\right]dx.\end{aligned} \tag{48}$$

Integrating the second part of right side of Eq. (48) by parts gives

$$\begin{aligned}&\frac{C_{r-2}}{\Gamma(r-1)}\int_0^\infty x^j\left[1-F(x)\right]^{\gamma_r+m}f(x)g_m^{r-2}\left[F(x)\right]dx\\&=\frac{-jC_{r-2}}{(r-1)\Gamma(r-1)}\int_0^\infty x^{j-1}\left[1-F(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx\\&+\frac{\gamma_rC_{r-2}}{(r-1)\Gamma(r-1)}\int_0^\infty x^j\left[1-F(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx.\end{aligned} \tag{49}$$

Substituating Eq. (49) into Eq. (48) and simplifying, we get

$$\frac{j}{\gamma_r}\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^{j-1}\left[1-F(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]\left[\frac{f(x)}{\lambda+\beta\gamma x^{\gamma-1}}-\left[1-F(x)\right]\right]dx=0. \tag{50}$$

Making use of Muntz-Szasz Theorem [19], we obtain

$$f(x)=\left(\lambda+\beta\gamma x^{\gamma-1}\right)\left[1-F(x)\right], \tag{51}$$

which proves X has the MWD(λ,β,γ)

(ii) For $\gamma_i \neq \gamma_j, i, j = 1,2,...,n-1$, the fundamental part is proved in Relation (2). For the sufficient part, if Eq. (47) holds, then

$$\begin{aligned} & C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^{\infty} x^t\,[1-F(x)]^{\gamma_i-1} f(x)dx \\ & = \frac{\lambda\gamma_r}{t+1}\left[C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^{\infty} x^{t+1}\,[1-F(x)]^{\gamma_i-1} f(x)dx\right. \\ & \left.-C_{r-2}\sum_{i=1}^{r-1} a_i(r-1)\int_0^{\infty} x^{t+1}\,[1-F(x)]^{\gamma_i-1} f(x)dx\right] \\ & +\frac{\beta\gamma\gamma_r}{t+\gamma}\left[C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^{\infty} x^{t+\gamma}[1-F(x)]^{\gamma_i-1} f(x)dx\right. \\ & \left.-C_{r-2}\sum_{i=1}^{r-1} a_i(r-1)\int_0^{\infty} x^{t+\gamma}[1-F(x)]^{\gamma_i-1} f(x)dx\right]. \end{aligned} \tag{52}$$

Using

$$a_i(r-1) = (\gamma_r - \gamma_i)\,a_i(r) \quad \text{and} \quad \gamma_r C_{r-2} = C_{r-1}, \tag{53}$$

we obtain

$$\begin{aligned} & C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^{\infty} x^t\,[1-F(x)]^{\gamma_i-1} f(x)dx \\ & = \frac{\lambda C_{r-1}}{t+1}\sum_{i=1}^{r} a_i(r)\gamma_i\int_0^{\infty} x^{t+1}\,[1-F(x)]^{\gamma_i-1} f(x)dx \\ & +\frac{\beta\gamma C_{r-1}}{t+\gamma}\sum_{i=1}^{r} a_i(r)\gamma_i\int_0^{\infty} x^{t+\gamma}[1-F(x)]^{\gamma_i-1} f(x)dx. \end{aligned} \tag{54}$$

Integrating by parts of the right hand side of Eq. (54), we obtain

$$C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^{\infty} x^t\,[1-F(x)]^{\gamma_i-1}\left[f(x)-\lambda\,[1-F(x)]-\beta\gamma x^{\gamma-1}\,[1-F(x)]\right]dx = 0. \tag{55}$$

Applying Muntz-Szasz Theorem [19] yields , we have

$$f(x) = \left(\lambda+\beta\gamma x^{\gamma-1}\right)[1-F(x)], \tag{56}$$

which proves X has the MWD(λ,β,γ).

4.2 Characterization of MWD based on product moments

Theorem 2: Let X be a nonnegative random variable with absolutely continuous cdf $F(x)$ and pdf $f(x)$, with $F(0)=0$ and $0<F(x)<1$, for all $x>0$. Then for $n\geq 1$and $r=1,2,...,n-1$. (i) For $m_1=m_2=...=m_{n-1}=m$, X has the MWD(λ,β,γ) iff

$$\mu^{(i,j)}_{(r,r+1:n,m,k)} = E(X^{i+j}) + \frac{j}{\gamma_{r+1}}E\left(X^i\frac{Y^{j-1}}{\lambda+\beta\gamma Y^{\gamma-1}}\right) \qquad \text{if } s=r+1, \tag{57}$$

and

$$\mu^{(i,j)}_{(r,s:n,m,k)} = \frac{j}{\gamma_s}E\left(\frac{X^iY^{j-1}}{\lambda+\beta\gamma Y^{\gamma-1}}\right) + \mu^{(i,j)}_{(r,s-1:n,m,k)} \qquad \text{if } s>r+1, \tag{58}$$

holds.

(ii) For $\gamma_i \neq \gamma_j, i, j = 1, 2, ..., n-1, X$ has the MWD(λ, β, γ) iff

$$\begin{aligned} \mu_{(r,s:n,\widetilde{m},k)}^{(t,z)} &= \frac{\lambda \gamma_s}{z+1} \left\{ \mu_{(r,s:n,\widetilde{m},k)}^{(t,z+1)} - \mu_{(r,s-1:n,\widetilde{m},k)}^{(t,z+1)} \right\} \\ &\quad + \frac{\beta \gamma \gamma_s}{z+\gamma} \left\{ \mu_{(r,s:n,\widetilde{m},k)}^{(t,z+\gamma)} - \mu_{(r,s-1,n,\widetilde{m},k)}^{(t,z+\gamma)} \right\}. \end{aligned} \tag{59}$$

holds.

Proof. For $s = r+1$. The necessary part can be proved immediately from Relation (3). For the sufficient part, if Eq. (57) holds, then

$$\begin{aligned} &\frac{C_r}{\Gamma(r)} \int_0^\infty \int_x^\infty x^i y^j [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [1-F(y)]^{\gamma_{r+1}-1} f(y) dy dx \\ &= \frac{C_{r-1}}{\Gamma(r)} \int_0^\infty x^{i+j} [1-F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}[F(x)] dx \\ &+ \frac{jC_r}{\gamma_{r+1}\Gamma(r)} \int_0^\infty \int_x^\infty \frac{x^i y^{j-1}}{\lambda + \beta\gamma y^{\gamma-1}} [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [1-F(y)]^{\gamma_{r+1}-1} f(y) dy dx. \end{aligned} \tag{60}$$

Integrating by parts the first part of Eq. (60), we get

$$\begin{aligned} &\frac{C_r}{\gamma_{r+1}\Gamma(r)} \int_0^\infty \int_x^\infty x^{i+j} [1-F(x)]^{m+\gamma_{r+1}} f(x) g_m^{r-1}[F(x)] dx \\ &+ \frac{j}{\gamma_{r+1}} \frac{C_r}{\Gamma(r)} \int_0^\infty \int_x^\infty x^i y^{j-1} [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [1-F(y)]^{\gamma_{r+1}} dy dx \\ &= \frac{C_{r-1}}{\Gamma(r)} \int_0^\infty x^{i+j} [1-F(x)]^{\gamma_r-1} f(x) g_m^{r-1}[F(x)] dx \\ &+ \frac{jC_r}{\gamma_{r+1}\Gamma(r)} \int_0^\infty \int_x^\infty \frac{x^i y^{j-1}}{\lambda + \beta\gamma y^{\gamma-1}} [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [1-F(y)]^{\gamma_{r+1}-1} f(y) dy dx. \end{aligned} \tag{61}$$

Then,

$$\begin{aligned} &\frac{j}{\gamma_{r+1}} \frac{C_r}{\Gamma(r)} \int_0^\infty \int_x^\infty x^i y^{j-1} [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [1-F(y)]^{\gamma_{r+1}-1} \\ &\qquad \times \left[\frac{f(y)}{\lambda + \beta\gamma y^{\gamma-1}} - [1-F(y)] \right] dy dx = 0. \end{aligned} \tag{62}$$

Applying Muntz- Szasz Theorem [19], we obtain

$$f(x) = \left(\lambda + \beta\gamma y^{\gamma-1}\right) [1-F(y)], \tag{63}$$

which leads to X has the MWD(λ, β, γ)

For $s > r+1$,the necessary part can be proved immediately from Relation (3). For the sufficient part, if Eq. (58) is satisfied, then

$$\begin{aligned} &\frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)} \int_0^\infty \int_x^\infty x^i y^j [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ &\times [1-F(y)]^{\gamma_s-1} f(y) dy dx \\ &= \frac{jC_{s-1}}{\gamma_s\Gamma(r)\Gamma(s-r)} \int_0^\infty \int_x^\infty \frac{x^i y^{j-1}}{\lambda + \beta\gamma y^{\gamma-1}} [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ &\times [1-F(y)]^{\gamma_s-1} f(y) dy dx, \\ &+ \frac{C_{s-2}}{\Gamma(r)\Gamma(s-r-1)} \int_0^\infty \int_x^\infty x^i y^j [1-F(x)]^m f(x) g_m^{r-1}[F(x)] [h_m(F(y)) - h_m(F(x))]^{s-r-2} \\ &\times [1-F(y)]^{\gamma_s+m} f(y) dy dx. \end{aligned} \tag{64}$$

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Integrating by parts the second part of right hand side of Eq. (64), we obtain

$$\begin{aligned}&\int_x^\infty y^j\left[1-F(y)\right]^{\gamma_s+m}\left[h_m(F(y))-h_m(F(x))\right]^{s-r-2}f(y)dy\\ &=\frac{-j}{(s-r-1)}\int_x^\infty y^{j-1}\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[1-F(y)\right]^{\gamma_s}dy\\ &+\frac{\gamma_s}{(s-r-1)}\int_x^\infty y^j\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\left[1-F(y)\right]^{\gamma_s-1}f(y)dy,\end{aligned} \tag{65}$$

Simplifying and substituting Eq.(65) into Eq. (64), and simplified, we get

$$\begin{aligned}&\frac{jC_{s-1}}{\gamma_s\Gamma(r)\Gamma(s-r)}\int_0^\infty\int_x^\infty x^i y^{j-1}\left[1-F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\\ &\times\left[1-F(y)\right]^{\gamma_s-1}\left[\frac{f(y)}{\lambda+\beta\gamma y^{\gamma-1}}-\left[1-F(y)\right]\right]dydx=0.\end{aligned} \tag{66}$$

Applying Muntz-Suasz theorem [19] to Eq. (66), we obtain

$$f(y)=\left[\lambda+\beta\gamma y^{\gamma-1}\right]\left[1-F(y)\right]. \tag{67}$$

(ii) For $\gamma_i\neq\gamma_j, i,j=1,2,\ldots,n-1$,The necessary part can be proved immediately from Relation (4). For the sufficient part, if Eq. (59) holds, then

$$\begin{aligned}&C_{s-1}\int_{-\infty}^\infty\int_x^\infty x^t y^z\left[\sum_{i=r+1}^s a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\\ &\times\left[\sum_{i=1}^r a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx\\ &=\frac{\lambda\gamma_s}{z+1}C_{s-2}\int_{-\infty}^\infty\int_x^\infty x^t y^{z+1}\left[\sum_{i=r+1}^s a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\\ &\times\left[\sum_{i=1}^r a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx\\ &-\frac{\lambda\gamma_s}{z+1}C_{s-2}\int_{-\infty}^\infty\int_x^\infty x^t y^{z+1}\left[\sum_{i=r+1}^s a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\\ &\times\left[\sum_{i=1}^r a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx\\ &+\frac{\beta\gamma}{z+\gamma}\gamma_s C_{s-2}\int_{-\infty}^\infty\int_x^\infty x^t y^{z+\gamma}\left[\sum_{i=r+1}^s a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\\ &\times\left[\sum_{i=1}^r a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx\\ &-\frac{\beta\gamma}{z+\gamma}\gamma_s C_{s-2}\int_{-\infty}^\infty\int_x^\infty x^t y^{z+\gamma}\left[\sum_{i=r+1}^s a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\\ &\times\left[\sum_{i=1}^r a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx,\end{aligned}$$

using

$$a_i(s)(s-1)=(\gamma_s-\gamma_i)a_i(s)\quad\text{and}\quad\gamma_s C_{s-2}=C_{s-1}.$$

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So,

$$\begin{aligned}
&C_{s-1}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^t y^z \left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right] \\
&\times\left[\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx \\
&=\frac{\lambda\gamma_s}{z+1}C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^t y^{z+1}\left[\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right] \\
&\times\left[\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx \\
&+\frac{\beta\gamma}{z+\gamma}\gamma_s C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^t y^{z+\gamma}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\gamma_i\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right] \\
&\times\left[\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}dydx.
\end{aligned}\tag{68}$$

Integrating by parts of the right hand of Eq. (68), we get

$$\int_{x}^{\infty} y^{z+1}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\gamma_i\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\frac{f(y)}{1-F(y)}dy=(z+1)\int_{x}^{\infty} y^{z}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}dy, \tag{69}$$

$$\int_{x}^{\infty} y^{z+\gamma}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\gamma_i\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\frac{f(y)}{1-F(y)}dy=(z+\gamma)\int_{x}^{\infty} y^{z+\gamma-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}dy. \tag{70}$$

Then, substituting Eq. (69) and Eq. (70) into Eq. (68), we obtain

$$\begin{aligned}
&C_{s-1}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^t y^z\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\left[\sum_{i=1}^{r} a_i(r)\left(1-F(x)\right)^{\gamma_i}\right] \\
&\times\left[\frac{f(x)}{1-F(x)}\right]\left[\frac{f(y)}{1-F(y)}-\alpha-\beta\gamma y^{\gamma-1}\right]dydx=0.
\end{aligned}\tag{71}$$

Applying Muntz-Suasz theorem [19] to Eq. (71), we have

$$f(y)=(\lambda+\beta\gamma y^{\gamma-1})\exp(-[\lambda y+\beta y^{\gamma}]).\blacksquare \tag{72}$$

which proves X has the MWD(λ,β,γ).

Remarks:

1- Setting $\lambda=0,\beta=\theta$ and $\gamma=2$ in Relations (1), (2), (3) and (4), the results for Rayleigh distribution in [11] are deduced.

2- Putting $\beta=\frac{\theta}{2}$ and $\gamma=2$ in Relations (1), (2), (3) and (4), the results for linear exponential distribution and its characterization in [12] are deduced.

3- Setting $m=0$, $k=1$ in Relations (1), (2), (3) and (4), recurrence relations of ordinary order statistics from MWD are derived.

4- Putting $m=-1$, $k=1$ in Relations (1), (2), (3) and (4), our results agree with the results of [13].

5-Putting $m=-1$, $k=1$ and $\lambda=0$ in Relations (1), (2), (3) and (4), our results agree with the results of [16].

6- Setting $m=-1$, $k=1$, $\alpha=\lambda$ and $\gamma=2$ in Relations (1), (2), (3) and (4), the results of [16] for linear failure rate distribution are deduced.

7- Putting $m=-1$, $k=1$, $\lambda=0$ and $\gamma=2$ in Relations (1), (2), (3) and (4), the results of [16] are obtained.

8- Setting $m=-1$, $k=1$,$\alpha=\lambda$ and $\beta=0$ in Relations (1), (2), (3) and (4), the results in [17] are deduced.

9- Setting $m=-1$, $k=1$,$\alpha=\lambda$ and $\beta=0$ in Relations (1), (2), (3) and (4), the results of [20] are deduced.

10- Setting $m=-1,k=1$, $\alpha=\lambda$ and $\beta=\frac{\upsilon}{2}$, $\gamma=2$ in Theorem (1) and Theorem (2), characterizations of exponential distribution are deduced, see [21].

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5 Discussion

The present paper addresses the generalized order statistics from the MWD. Recurrence relations between the single and product moments are derived. Characterization of the MWD based on a recurrence relation for single and product moments are discussed. Special cases are also deduced.

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