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Parameter Estimation for a Mixture of Inverse Chen and Inverse Compound Rayleigh Distributions Based on Type-II Hybrid Censoring Scheme

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Abstract: The Bayesian estimation procedure for two-component mixture of the inverse Chen and inverse compound Rayleigh distributions(ICICRD) based on Type II hybrid censoring scheme is discussed. We derive maximum likelihood estimators and the approximate confidence intervals using asymptotic variance and covariance matrix. The Bayesian point estimation relative to symmetric squared error(SE) loss function and asymmetric linear exponential (LINEX) and general entropy (GE) loss functions, and highest posterior density credible interval of the parameters are obtained. We perform Monte Carlo simulation to compare the performance of the different estimates. Furthermore, we consider the problem of predicting the future order statistics. Numerical results using generated data sets are presented.

Keywords: Mixture model, Hybrid censored sample, Bayesian estimation, Maximum likelihood estimation, Bayesian Prediction.

1 Introduction

In reliability literature or life testing experiments, the data are often censored according to the different censoring schemes and the experimenter may not be in a position to observe the life times of all items put on test regarding cost, time or the data collection. Type-I and Type-II censoring schemes are the two most popular censoring schemes which are used in practice.

The mixture of Type I and Type II censoring scheme is known as hybrid censoring scheme, and it can be described as follows: Suppose n identical units are put to test and the test is terminated when the pre-chosen number R out of n items fails or when a pre-determined time T on the test is reached.

Epstein [1] proposed Type-I hybrid censoring scheme and considered lifetime experiments assuming that the lifetime of each unit follows an exponential distribution. Many authors have discussed statistical inference problems for various distribution under Type I hybrid, see Gupta and Kundu [2], Ebrahimi [3], Chen and Bhattacharya [4], Childs et al. [5], Kundu [6], Rastogi and Tripathi [7], Singh et al. [8], Hyun et al. [9] and Sultana et al. [10].

Childs et al. [5] proposed a new hybrid censoring scheme known as Type-II hybrid censoring scheme to cover the disadvantage of Type-I hybrid censoring scheme, as follows: Let n identical items put on test, then terminate the experiment at the random time $T^* = max\{x_{R:n}, T\}$,where R and T are prefixed number and $x_{R:n}$ indicates the time of Rth failure in a sample of size n.

Under the Type-II hybrid censoring scheme, we can observe the following three types of observations:

Case I: $\{x_{1:n} < \cdots < x_{R:n}\}$ if $x_{R:n} > T$.

Case II:$\{x_{1:n} < \cdots < x_{R:n} < x_{R+1:n} < \cdots < x_{d:n} < T < x_{d+1:n}\}$ if $R \leq d < n$ and $x_{d:n} < T < x_{d+1:n}$.

Case III: $\{x_{1:n} < \cdots < x_{n:n} < T\}$.

For some of the related references, see Banerjee and Kundu [11], Singh et al. [12,13], Salah[14], Mahmoud et al. [15] as well as Yadav and Yang [16]. Chen [17] proposed a new two parameter lifetime distribution with bathtub shaped or increasing failure rate function. The bathtub shape hazard function provides an appropriate conceptual model for some electronic and mechanical products.

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Inferences on the Chen distribution have been examined by Wu et al. [18], Wu [19], Sarhan et al. [20] as well as Rastogi and Tripathi [7]. Srivastava and Srivastava [21] have derived a new distribution called Inverse Chen (IC) distribution with its maximum likelihood estimators (MLEs) and their asymptotic confidence intervals, survival function and hazard rate.

The Inverse Chen (IC) distribution has the following cumulative distribution function (cdf) and the density function (pdf) given, respectively, by

$$F_1(x) = e^{\lambda_1\left(1-e^{x^{-\beta_1}}\right)}, \qquad x>0,\ \lambda_1>0, \beta_1>0 \tag{1}$$

$$f_1(x) = \lambda_1\beta_1 x^{-(\beta_1+1)} e^{\left[x^{-\beta_1}+\lambda_1\left(1-e^{x^{-\beta_1}}\right)\right]}, \qquad x>0,\ \lambda_1>0, \beta_1>0 \tag{2}$$

and, the reliability function is given by

$$R_1(x) = 1-e^{\lambda_1\left(1-e^{x^{-\beta_1}}\right)}, \qquad x>0,\ \lambda_1>0, \beta_1>0 \tag{3}$$

The compound Rayleigh distribution provides a population model which is useful in several areas of statistics, including life testing, reliability and survival analysis. This distribution is a special case of the three parameter Burr XII distribution. In the last two decades, statisticians paid attention to the development of this distribution [see Abushal [22], Al-Hossain [23], and Abd-Elmougod and Mahmoud [24]]. The introduced model will be named Inverse Compound Rayleigh (ICR) distribution. Its cdf and pdf are given, respectively, by

$$F_2(x) = \beta_2^{\lambda_2}\left(\beta_2+\frac{1}{x^2}\right)^{-\lambda_2}, \qquad x>0,\ \lambda_2>0, \beta_2>0 \tag{4}$$

$$f_2(x) = 2\lambda_2\beta_2^{\lambda_2}x^{-3}\left(\beta_2+\frac{1}{x^2}\right)^{-(\lambda_2+1)}, \qquad x>0,\ \lambda_2>0, \beta_2>0 \tag{5}$$

Then the reliability of ICR distribution is given by

$$R_2(x) = 1-\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{x^2}\right)^{-\lambda_2}, \qquad x>0,\ \lambda_2>0, \beta_2>0 \tag{6}$$

Mixture distributions have gained great interest for the analysis, so mixture distributions play a vital role in many practical applications. Direct applications of finite mixture models are medicine, botany, life testing, reliability, $\cdots$ etc. Indirect applications include outliers, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric density estimation. Finite mixture models are studied theortically and practically by many authors [see Everitt and Hand [25], Titterington et al. [26], Mclachlan and Basford [27], Lindsay [28] and Mclachlan and Peel [29]]. Also, mixture distributions have been extensively considered by researchers using both classical and Bayesian techniques [for example , Abu-Zinadah [30], Erisoglu et al. [31], Feroze and Aslam [32], Daniyal and Rajab [33], Mahmoud et al. [34], and Zhu et al. [35]].

If the population consists of a mixture of two independent subpopulation representing failure types, then the distribution function of the mixed population can be expressed by

$$F(x) = \sum_{j=1}^{2} p_j F_j(x), \qquad j=1,2 \tag{7}$$

where $F(x)$ is the cdf of the mixed population, $F_j(x)$ is the cdf of the j-th subpopulation defined by (1) for $j=1$ and (4) for $j=2$, and the mixing proportions p_j are such that $0\le p_j\le 1,\quad j=1,2,\ p_1+p_2=1$.

Also, the corresponding density function is given by

$$f(x) = \sum_{j=1}^{2} p_j f_j(x), \qquad j=1,2 \tag{8}$$

where $f_j(x)$ are given by (2) and (5)

Thus, the reliability function is given by

$$R(x) = \sum_{j=1}^{2} p_j R_j(x), \qquad j=1,2 \tag{9}$$

where $R_j(x)$ are given by (3) and (6).

One of the most important problems in life-testing experiment is prediction. Bayesian prediction plays an important role in different areas of applied statistics. Several researchers have focused on the problem of Bayesian prediction of future observations based on Type-I and Type-II hybrid censored data from different lifetime models; [see Ebrahimi[36], Balakrishnan and Shafay [37,38], Singh et al. [39] and Sadek[40]]. Some authors have considered the Bayesian prediction problem of the mixture of distributions, see for example [AL-Hussaini et al. [41], Jaheen [42]and Mahmoud et al. [43]].

The rest of this paper is organized as follows: Estimation by the method of maximum likelihood and asymptotic confidence interval are derived in Section 2. In Section 3,we have developed Bayesian estimation using Informative and non-Informative prior under different loss function. Credible intervals for the parameters and Bayesian prediction intervals for future order statistic are derived in Section 4 and 5, respectively. Numerical comparisons concerning the resulting estimations via Monte Carlo simulation and simulated data are analyzed in Section 6. Conclusion is presented in Section 7.

2 Maximum Likelihood Estimation

Suppose that n identical units from population with pdf (8) are placed on a life-test. In type-II hybrid censoring scheme, R, T are known in advance and the termination time of experiment is $t = max(x_{R:n}, T)$. where $x_{R:n}$ is the Rth order statistic of the sample of size n. Suppose r units have failed during the interval (0,t): r_1 units from the first subpopulation and r_2 units from the second subpopulation, such that $r = r_1 + r_2$. Assume also that x_{ij} denotes the failure time of the j^{th} unit belonging to the i^{th} subpopulation, where $i = 1,2,\ j = 1,2,\ldots,r_i$. For a two component mixture model, the likelihood function is given by

$$L(\lambda_1,\lambda_2,p|\underline{x}) = \frac{n!}{(n-r)!}\prod_{j=1}^{r_1} p_1 f_1(x_{1j}) \prod_{j=1}^{r_2} p_2 f_2(x_{2j}) [1-F(t)]^{(n-r)} \tag{10}$$

where

$$r = r_1 + r_2, \qquad F(t) = \sum_{j=1}^{2} p_j F_j(t), \qquad j = 1,2$$

$$r = \begin{cases} R & \text{for case } I \\ d & \text{for case } II \\ n & \text{for case } III \end{cases} \qquad\qquad t = \begin{cases} x_{R:n} & \text{for case } I \\ T & \text{for case } II \end{cases}$$

The likelihood function can be written as

$$L(\lambda_1,\lambda_2,p|\underline{x}) = \frac{n!}{(n-r)!}\prod_{j=1}^{r_1} p_1\lambda_1\beta_1 x_{1j}^{-(\beta_1+1)} e^{\left[x_{1j}^{-\beta_1}+\lambda_1\left(1-e^{x_{1j}^{-\beta_1}}\right)\right]} \prod_{j=1}^{r_2} p_2(2\lambda_2)\beta_2^{\lambda_2} x_{2j}^{-3}\left(\beta_2+\frac{1}{x_{2j}^2}\right)^{-(\lambda_2+1)} [G]^{(n-r)} \tag{11}$$

where

$$G = 1 - \left\{ p_1 e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)} + p_2\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2} \right\}, \qquad p_1 = p,\ p_2 = 1-p.$$

Then, the log likelihood function can be expressed as

$$\begin{aligned} L &= \ln L(\lambda_1,\lambda_2,p|\underline{x}) \\ &= \ln\frac{n!}{(n-r)!} + r_1 \ln p_1 + r_1 \ln\lambda_1 + r_1\ln\beta_1 - (\beta_1+1)\sum_{j=1}^{r_1}\ln x_{1j} \\ &+ \sum_{j=1}^{r1} x_{1j}^{-\beta_1} + \lambda_1\sum_{j=1}^{r_1}\left(1-e^{x_{1j}^{-\beta_1}}\right) + r_2\ln p_2 + r_2\ln(2\lambda_2) + r_2\lambda_2\ln\beta_2 \\ &- 3\sum_{j=1}^{r_2}\ln x_{2j} - (\lambda_2+1)\sum_{j=1}^{r_2}\ln\left(\beta_2+\frac{1}{x_{2j}^2}\right) + (n-r)\ln[G] \end{aligned} \tag{12}$$

Taking derivatives with respect to λ_1, λ_2 and p in Equation (12), we obtain and equate it to zero

$$\frac{\partial \ln L}{\partial \lambda_1} = \frac{r_1}{\lambda_1} + \sum_{j=1}^{r_1}\left(1 - e^{x_{1j}^{-\beta_1}}\right) + \frac{(n-r)}{G}\left(\frac{\partial G}{\partial \lambda_1}\right)$$

$$\frac{\partial \ln L}{\partial \lambda_2} = \frac{r_2}{\lambda_2} + r_2 \ln(\beta_2) - \sum_{j=1}^{r_2} \ln\left(\beta_2 + \frac{1}{x_{2j}^2}\right) + \frac{(n-r)}{G}\left(\frac{\partial G}{\partial \lambda_2}\right)$$

$$\frac{\partial \ln L}{\partial p} = \frac{r_1}{p_1} - \frac{r_2}{p_2} + \frac{(n-r)}{G}\left(\frac{\partial G}{\partial p}\right)$$

where,

$$\frac{\partial G}{\partial \lambda_1} = -p_1 e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)}\left(1 - e^{t^{-\beta_1}}\right),$$

$$\frac{\partial G}{\partial \lambda_2} = p_2 \beta_2^{\lambda_2}\left(\beta_2 + \frac{1}{t^2}\right)^{-\lambda_2}\left[\ln\left(\beta_2 + \frac{1}{t^2}\right) - \ln(\beta_2)\right]$$

$$\frac{\partial G}{\partial p} = -e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)} + \beta_2^{\lambda_2}\left(\beta_2 + \frac{1}{t^2}\right)^{-\lambda_2}$$

It is clear that the normal equations do not have explicit solutions. Therefore, a numerical method, such as the Newton-Raphson method, is used to solve the equations to obtain the maximum likelihood estimates (MLEs). In this section, we compute the observed Fisher information for MLEs for obtained confidence intervals for the parameters. We have the approximation variance-covariance matrix given by

$$\Sigma = \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \lambda_1^2} & -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} & -\frac{\partial^2 \ln L}{\partial \lambda_1 \partial p} \\ -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial \lambda_1} & -\frac{\partial^2 \ln L}{\partial \lambda_2^2} & -\frac{\partial^2 \ln L}{\partial \lambda_2 \partial p} \\ -\frac{\partial^2 \ln L}{\partial p \partial \lambda_1} & -\frac{\partial^2 \ln L}{\partial p \partial \lambda_2} & -\frac{\partial^2 \ln L}{\partial p^2} \end{pmatrix}^{-1}$$

where the elements of the observed Fisher information matrix are, as follows:

$$\frac{\partial^2 \ln L}{\partial \lambda_1^2} = -\frac{r_1}{\lambda_1^2} + \frac{(n-r)}{G}\left\{\frac{\partial^2 G}{\partial \lambda_1^2} - \left(\frac{\partial G}{\partial \lambda_1}\right)^2 \frac{1}{G}\right\}$$

$$\frac{\partial^2 \ln L}{\partial \lambda_2^2} = -\frac{r_2}{\lambda_2^2} + \frac{(n-r)}{G}\left\{\frac{\partial^2 G}{\partial \lambda_2^2} - \left(\frac{\partial G}{\partial \lambda_2}\right)^2 \frac{1}{G}\right\}$$

$$\frac{\partial^2 \ln L}{\partial p^2} = -\frac{r_1}{p_1^2} - \frac{r_2}{p_2^2} - \frac{(n-r)}{G^2}\left(\frac{\partial G}{\partial p}\right)^2$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \partial \lambda_2} = -\frac{(n-r)}{G^2}\left(\frac{\partial G}{\partial \lambda_1}\right)\left(\frac{\partial G}{\partial \lambda_2}\right)$$

$$\frac{\partial^2 \ln L}{\partial \lambda_1 \partial p} = \frac{(n-r)}{G}\left\{\frac{\partial^2 G}{\partial \lambda_1 \partial p} - \left(\frac{\partial G}{\partial \lambda_1}\right)\left(\frac{\partial G}{\partial p}\right)\frac{1}{G}\right\}$$

$$\frac{\partial^2 \ln L}{\partial \lambda_2 \partial p} = \frac{(n-r)}{G}\left\{\frac{\partial^2 G}{\partial \lambda_2 \partial p} - \left(\frac{\partial G}{\partial \lambda_2}\right)\left(\frac{\partial G}{\partial p}\right)\frac{1}{G}\right\}$$

and

$$\frac{\partial^2 G}{\partial \lambda_1^2} = -p_1 \, e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)}\left(1-e^{t^{-\beta_1}}\right)^2$$

$$\frac{\partial^2 G}{\partial \lambda_2^2} = p_2 \beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}\left[-\left[\ln\left(\beta_2\right)\right]^2 + 2\ln\left(\beta_2\right)\ln\left(\beta_2+\frac{1}{t^2}\right) - \left[\ln\left(\beta_2+\frac{1}{t^2}\right)\right]^2\right]$$

$$\frac{\partial^2 G}{\partial \lambda_1 \partial p} = -e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)}\left(1-e^{t^{-\beta_1}}\right)$$

$$\frac{\partial^2 G}{\partial \lambda_2 \partial p} = \beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}\left[\ln\left(\beta_2\right) - \ln\left(\beta_2+\frac{1}{t^2}\right)\right]$$

The above matrix can be obtained by inversion to obtain the estimate of the asymptotic variance-covariance matrix of the MLEs. Hence $100\,(1-\gamma)\,\%$ approximate confidence intervals for λ_1, λ_2 and p are respectively given, as follows:

$$\widehat{\lambda}_1 \pm z_{\gamma/2}\sqrt{\widehat{\Sigma}_{11}}, \quad \widehat{\lambda}_2 \pm z_{\gamma/2}\sqrt{\widehat{\Sigma}_{22}}, \qquad \text{and} \qquad \hat{p} \pm z_{\gamma/2}\sqrt{\widehat{\Sigma}_{33}}$$

where $\widehat{\Sigma}_{11}$,$\widehat{\Sigma}_{22}$ and $\widehat{\Sigma}_{33}$ are the elements on the main diagonal of covariance matrix Σ, and $z_{\gamma/2}$ is the upper $100\gamma th$ percentile of the standard normal distribution.

3 Bayesian Estimation

In this section, we derive Bayes estimators of the parameters λ_1, λ_2 and p of the considered model under Type-II hybrid censoring scheme using various priors under different symmetric and asymmetric loss functions. Assume the prior distributions of the parameters λ_1, λ_2 and p are $\lambda_1 \sim Gamma\,(a_1,b_1)$, $\lambda_2 \sim Gamma\,(a_2,b_2)$, and $p \sim Beta\,(c_1,c_2)$ for the mixing parameters $p_i, i=1,2$, where $p_1 = p$ and $p_2 = 1-p$.

Assuming the independence of the parameters, the joint prior for λ_1, λ_2 and p may be written as

$$\pi\left(\lambda_1,\lambda_2,p\right) = \pi_1\left(\lambda_1\right)\pi_2\left(\lambda_2\right)\pi_3\left(p\right) \tag{13}$$

$$\pi_i\left(\lambda_i\right) \propto \prod_{i=1}^{2}\left[\lambda_i^{a_i-1}e^{-b_i\lambda_i}\right], \qquad \lambda_i > 0,\ a_i,b_i > 0; i=1,2.$$

$$\pi_3\left(p\right) \propto \prod_{i=1}^{2} p_i^{c_i-1}; \qquad c_i, > 0\ ; i=1,2$$

where a_1,a_2,b_1,b_2,c_1 *and* c_2 are the hyperparameters. Particularly, if $a_1=a_2=b_1=b_2=0$ and $c_1=c_2=1$, the case of non-informative improper prior is given by

$$\pi_i\left(\lambda_i\right) \propto \prod_{i=1}^{2}\frac{1}{\lambda_i} \ , \lambda_i > 0; i=1,2.$$

$$\pi_3\left(p\right) = 1, \quad p \sim U\left[0,1\right]$$

Suppose β_1 and β_2 are known, then the likelihood function (11) reduces to

$$\begin{aligned} L\left(\lambda_1,\lambda_2,p|\underline{x}\right) \propto & \sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k\lambda_1^{r_1}\left(2\lambda_2\right)^{r_2}p_1^{r_1+k-m}p_2^{r_2+m}e^{\lambda_1\left[\Sigma_{j=1}^{r_1}\left(1-e^{x_{1j}^{-\beta_1}}\right)+\left((k-m)\left(1-e^{t^{-\beta_1}}\right)\right)\right]} \\ & \times \beta_2^{\lambda_2(r_2+m)}\prod_{j=1}^{r_2}\left(\beta_2+\frac{1}{x_{2j}^2}\right)^{-\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-m\lambda_2} \end{aligned} \tag{14}$$

Therefore, the joint posterior density function of λ_1, λ_2 and p based on informative prior, and the likelihood function (14) can be given by

$$P(\lambda_1,\lambda_2,p|\underline{x}) = K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k p_1^{\delta_1-1}p_2^{\delta_2-1}\lambda_1^{r_1+a_1-1}\lambda_2^{r_2+a_2-1}e^{-\lambda_1\varphi_1}e^{-\lambda_2\varphi_2} \tag{15}$$

where,

$$\delta_1 = r_1+c_1+k-m, \qquad \delta_2 = r_2+c_2+m,$$

$$\varphi_1 = b_1 - \sum_{j=1}^{r_1}\left(1-e^{x_{1j}^{-\beta_1}}\right) - (k-m)\left(1-e^{t^{-\beta_1}}\right),$$

$$\varphi_2 = b_2-(r_2+m)\ln\beta_2 + \sum_{j=1}^{r_2}\ln\left(\beta_2+\frac{1}{x_{2j}^2}\right)+m\ln\left(\beta_2+\frac{1}{t^2}\right)$$

where K is a normalizing constant given by

$$K = \sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2]^{r_2+a_2}}$$

3.1 Bayes estimation under square error loss function (SELF)

A very well-known symmetric loss function is the SELF, which is defined as $L_1\left(\widehat{\theta},\theta\right) = c\left(\widehat{\theta}-\theta\right)^2$. For SELF, Bayes estimator is the mean of posterior density functions which are given by

$$\widehat{\lambda}_{1,SELF} = K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1+1)}{[\varphi_1]^{r_1+a_1+1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2]^{r_2+a_2}}$$

$$\widehat{\lambda}_{2,SELF} = K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2+1)}{[\varphi_2]^{r_2+a_2+1}}$$

$$\hat{p}_{SELF} = K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1+1,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2]^{r_2+a_2}}$$

3.2 Bayes estimator under linear-exponential loss function (LINEX)

Avery useful asymmetric loss function is known as the LINEX loss function. The Bayes estimator of any parameter A is obtained from

$$\hat{A}_{LINEX} = -\frac{1}{q}\ln\left[E\left(e^{-qA}|\underline{x}\right)\right]$$

provided that the above expectation exists and is finite.

Bayes estimations of λ_1, λ_2 and p based on the LINEX loss function are

$$\widehat{\lambda}_{1,LINEX} = -\frac{1}{q}\ln\left[K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1+q]^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2]^{r_2+a_2}}\right]$$

$$\widehat{\lambda}_{2,LINEX} = -\frac{1}{q}\ln\left[K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2+q]^{r_2+a_2}}\right]$$

$$\hat{p}_{LINEX} = -\frac{1}{q}\ln\left[K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\sum_{l=0}^{\infty}\binom{n-r}{k}\binom{k}{m}(-1)^k\frac{(-q)^l}{l!}B(\delta_1+l,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2]^{r_2+a_2}}\right]$$

3.3 Bayes Estimator under General Entropy Loss Function (GELF)

Another commonly asymmetric loss function is called the general entropy loss function. The Bayes estimator of A is obtained as

$$\hat{A}_{GELF} = \left[E\left(A^{-h}|\underline{x}\right)\right]^{-\dfrac{1}{h}}$$

provided that the above expectation exists and is finite.

Bayes estimations of λ_1, λ_2 and p based on the GELF are

$$\widehat{\lambda}_{1,GELF} = \left[K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1-h)}{[\varphi_1]^{r_1+a_1-h}}\frac{\Gamma(r_2+a_2)}{[\varphi_2]^{r_2+a_2}}\right]^{-\dfrac{1}{h}}$$

$$\widehat{\lambda}_{2,GELF} = \left[K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2-h)}{[\varphi_2]^{r_2+a_2-h}}\right]^{-\dfrac{1}{h}}$$

$$\hat{p}_{GELF} = \left[K^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B(\delta_1-h,\delta_2)\frac{\Gamma(r_1+a_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2]^{r_2+a_2}}\right]^{-\dfrac{1}{h}}$$

Note that for value $h=-1$, the general entropy loss function is the same as the squared error loss function.

4 Credible Interval

Let: $g(\lambda|x)$ be the posterior distribution based the respective prior. Feroze and Aslam [32] introduced the credible interval defined as:

$$\int_0^L g(\lambda|x)\,d\lambda = \frac{\gamma}{2}, \qquad \int_U^\infty g(\lambda|x)\,d\lambda = \frac{\gamma}{2}$$

where L and U denote the lower and upper bounds and γ is level of significance.

The $(1-\gamma)100\%$ credible interval of λ_1, λ_2 and p on the based informative prior can be obtained by solving the following equations for L and U

$$K^{-1}\left[\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B_{L^{i_3}}(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1,i_1L\varphi_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2,i_2L\varphi_2)}{[\varphi_2]^{r_2+a_2}}\right] = 1-\frac{\gamma}{2}$$

$$K^{-1}\left[\sum_{k=0}^{n-r}\sum_{m=0}^{k}\binom{n-r}{k}\binom{k}{m}(-1)^k B_{U^{i_3}}(\delta_1,\delta_2)\frac{\Gamma(r_1+a_1,i_1U\varphi_1)}{[\varphi_1]^{r_1+a_1}}\frac{\Gamma(r_2+a_2,i_2U\varphi_2)}{[\varphi_2]^{r_2+a_2}}\right] = \frac{\gamma}{2}$$

where $i_1,\ i_2,i_3=0,1,\ \Gamma(a,z)=\int_0^z x^{a-1}e^{-x}dx$ is the incomplete gamma function, and $B_z(a,b)=\int_0^z x^{a-1}(1-x)^{b-1}dx$ is the incomplete beta function.

5 Bayesian Two-Sample Prediction

In this section, the Bayesian prediction of a future ordered statistic is considered. Take a sample of size n from the population with pdf (8), and take a future sample of size m_1 independent of the informative sample from the same population. An important aspect of prediction about the s^{th} order statistic y_s in the future sample, $1\le s\le m_1$, is that s^{th} order statistic in a sample of size m_1 represents the life length of (m_1-s+1) out of m_1 system. The density function of y_s is given by

$$f_{Y_s}(y_s|\lambda_1,\lambda_2,p) \propto [1-R(y_s)]^{s-1}[R(y_s)]^{m_1-s}f(y_s)$$
$$= \sum_{j_1=0}^{s-1}(-1)^{j_1}\binom{s-1}{j_1}[R(y_s)]^{m_1-s+j_1}f(y_s)$$

where $f(y_s)$ and $R(y_s)$ are given, respectively by Equations (8) and (9), after replacing x by y_s
Using the binomial expansion for $[R(y_s)]^{m_1-s+j_1}$, it follows that

$$f_{Y_s}(y_s|\lambda_1,\lambda_2,p) \propto \sum_{j_1=0}^{s-1}\sum_{j_2=0}^{m_1-s+j_1}\sum_{j_3=0}^{j_2}\binom{s-1}{j_1}\binom{m_1-s+j_1}{j_2}\binom{j_2}{j_3}(-1)^{j_1+j_2}p_1^{\delta_3}p_2^{j_3}e^{\lambda_1\left(1-e^{y_s^{-\beta_1}}\right)\delta_3}$$
$$\times\left(\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{x^2}\right)^{-\lambda_2}\right)^{j_3}f(y_s) \tag{16}$$

The Bayes predictive pdf of y_s given x is defined by

$$f^*(y_s|x) = \int_0^1\int_0^\infty\int_0^\infty f(y_s|\lambda_1,\lambda_2,p)P(\lambda_1,\lambda_2,p|\underline{x})\,d\lambda_1 d\lambda_2 dp \tag{17}$$

where $P(\lambda_1,\lambda_2,p|\underline{x})$ is the joint posterior density function and $f(y_s|\lambda_1,\lambda_2,p)$ is the density function of the s^{th} component in a future sample. Thus, we have

$$f^*(y_s|x) = K^{-1}\sum K^*\left[B(\delta_1+\delta_3+1,\delta_2+j_3)\frac{\Gamma(r_1+a_1+1)}{[\varphi_1^*]^{r_1+a_1+1}}\frac{\Gamma(r_2+a_2)}{[\varphi_2^*]^{r_2+a_2}}+2B(\delta_1+\delta_3,\delta_2+j_3+1)\frac{\Gamma(r_1+a_1)}{[\varphi_1^{**}]^{r_1+a_1}}\frac{\Gamma(r_2+a_2+1)}{[\varphi_2^{**}]^{r_2+a_2+1}}\right]$$
$$= K^{-1}\sum K^{**}[W_1(y_s)+W_2(y_s)] \tag{18}$$

where K is the normalizing constant satisfying

$$\int_0^\infty f^*(y_s|x)\,dy_s = 1,$$

$$\sum = \sum_{j_1=0}^{s-1}\sum_{j_2=0}^{m_1-s+j_1}\sum_{j_3=0}^{j_2}\sum_{k=0}^{n-r}\sum_{m=0}^{k},$$

$$K^* = \binom{n-r}{k}\binom{k}{m}\binom{s-1}{j_1}\binom{m_1-s+j_1}{j_2}\binom{j_2}{j_3}(-1)^{j_1+j_2+k}, \qquad \delta_3 = j_2-j_3,$$

$$K^{**} = K^*\left[\frac{\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3)\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+1)}\right]$$

$$W_1(y_s) = \frac{(\delta_1+\delta_3)(r_1+a_1)}{[\varphi_1^*]^{r_1+a_1+1}[\varphi_2^*]^{r_2+a_2}}, \qquad W_2(y_s) = 2\frac{(\delta_2+j_3)(r_2+a_2)}{[\varphi_1^{**}]^{r_1+a_1}[\varphi_2^{**}]^{r_2+a_2+1}}$$

$$\varphi_1^* = \varphi_1-\left(1-e^{y_s^{-\beta_1}}\right)(\delta_3+1) \qquad \varphi_2^* = \varphi_2-j_3\ln\beta_2+j_3\ln\left(\beta_2+\frac{1}{y_s^2}\right), \quad \varphi_1^{**} = \varphi_1-\left(1-e^{y_s^{-\beta_1}}\right)\delta_3,$$

$$\varphi_2^{**} = \varphi_2-\ln\beta_2(j_3+1)+\ln\left(\beta_2+\frac{1}{y_s^2}\right)(j_3+1)$$

Therefore, using the Bayesian predictive density of y_s, for a given value ν, we obtain

$$Pr[y_s \geq \nu|\underline{x}] = \int_\nu^\infty f^*(y_s|x)\,dy_s \tag{19}$$

A $100\gamma\%$ prediction interval for y_s is given by

$$P\left[L\left(\underline{x}\right) < y_s < U\left(\underline{x}\right)\right] = \gamma \tag{20}$$

where $L\left(\underline{x}\right)$ and $U\left(\underline{x}\right)$ are obtained, respectively, by solving the following two equations

$$P\left[y_s > L\left(\underline{x}\right)\right] = \frac{1+\gamma}{2} \qquad \text{and} \qquad P\left[y_s > U\left(\underline{x}\right)\right] = \frac{1-\gamma}{2} \tag{21}$$

Since Equation (21) cannot be solved analytically, we need to apply suitable numerical techniques for solving non-linear equations.

6 Numerical Application

6.1 Comparison of estimations

In this subsection, the behavior of proposed estimators is investigated using Monte Carlo simulations. The performance of the competitive estimates has been compared on the basis of their mean squared errors (MSE). We have generated a random sample of size $(n = 20, 30\ ,40\&\ 50)$ for fixed values of $\lambda_1 = 1$, $\lambda_2 = 2$ and $p = 0.45$ along with $(\beta_1 = 1.5, \beta_2 = 1)$ and with different choices of (R, T). We carry out Monte Carlo simulation according the following steps:

- Specify the value n, R and T.
- Generate observation u from Uniform$(0,1)$.
- If $u \leq p$ the observation is randomly generated from the first subpopulation and if $u > p$, then the observation is generated from the second subpopulation.
- Apply hybrid sampling procedure and obtain the hybrid Type II censored sample of size r, $R \leq d \leq n$.
- Maximum likelihood estimates and the informative Bayes estimates with respect to the prior where hyper parameters take values as $(a_1 = 0.5, a_2 = 0.3, b_1 = 0.5, b_2 = 0.2, c_1 = 2\ and\ c_2 = 2)$. In case of non-informative prior, we take $(a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0, c_1 = 1\ and\ c_2 = 1)$ and estimates of the parameters are calculated according to Section 2 and Section 3, respectively.
- All results are based on 1000 replication and report the average values and mean squared error (MSE) of the estimates in Tables1-7.

On the basis of simulation study of given tables, we observe the following:

- Tables (1-7) show the performance of Bayes estimators obtained under informative prior has less MSE compared to the non-informative prior for all loss functions.
- For all estimators, it is observed that the value of expected the MSE of each estimator decreases as the sample size increases. Also, MSE of all estimator decreases with the increase in the value of T for fixed R and n. Moreover, for fixed n and T as R increases, the MSE decreases as expected.
- Estimates of the mixing weight parameter tend to converge to the true parametric value by increasing the sample size for all estimators, and this indicates that the MSE is positively skewed.
- For Bayes estimators relative to SELF, GELF and LINEX loss function with positive values of h and q, performance is better than their corresponding MLEs for the parameter of λ_1 and λ_2 . Also, for the mixing proportion p the MSE is better than MLEs.
- Asymmetric loss functions estimators are better at the positive value q and h than with negative values.

All results are obtained using Mathematica 10.0.

Table 1: Average estimates and the corresponding MSE of the parameters λ_1 , λ_2 and p based MLEs

n	T	R	Maximum Likelihood Estimation		
			λ_1	λ_2	p
20	3	12	1.17061 (0.34061)	2.16896 (0.77693)	0.45415 (0.01652)
		16	1.13026 (0.31120)	2.17825 (0.66695)	0.45354 (0.01440)
		18	1.15950 (0.28428)	2.20813 (0.60768)	0.45050 (0.01384)
	6	12	1.13994 (0.27473)	2.22823 (0.65916)	0.45497 (0.01441)
		16	1.13960 (0.26672)	2.20969 (0.61743)	0.44808 (0.01380)
		18	1.15507 (0.26955)	2.17206 (0.56710)	0.45005 (0.01389)
30	3	20	1.09445 (0.15399)	2.12513 (0.37651)	0.44725 (0.01012)
		25	1.08460 (0.14249)	2.14691 (0.37729)	0.44996 (0.00992)
		28	1.06319 (0.12307)	2.15193 (0.37295)	0.44358 (0.00833)
	6	20	1.09974 (0.14155)	2.11870 (0.35868)	0.44835 (0.00896)
		25	1.07375 (0.12074)	2.12373 (0.34258)	0.45025 (0.00842)
		28	1.07847 (0.11706)	2.12507 (0.31305)	0.44853 (0.00805)
40	3	28	1.06058 (0.09111)	2.08651 (0.26176)	0.45476 (0.00752)
		30	1.06483 (0.08540)	2.11700 (0.24662)	0.44382 (0.00727)
		35	1.07356 (0.08597)	2.10065 (0.25442)	0.45327 (0.00719)
	6	28	1.07162 (0.09061)	2.10184 (0.24110)	0.44884 (0.00709)
		30	1.05821 (0.08583)	2.08377 (0.23825)	0.45404 (0.00687)
		35	1.05741 (0.07805)	2.08217 (0.22973)	0.45443 (0.00678)
50	3	38	1.05084 (0.06571)	2.07277 (0.20562)	0.44281 (0.00596)
		45	1.06027 (0.06142)	2.07771 (0.17861)	0.45413 (0.00564)
	6	38	1.05025 (0.06053)	2.08535 (0.18766)	0.44795 (0.00527)
		45	1.04899 (0.05786)	2.08709 (0.17392)	0.45366 (0.00506)

Table 2: Average estimates and corresponding MSE of the parameter λ_1 based on informative prior

n	T	R	SELF $h=-1$	LINEX				GELF		
				$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
20	3	12	1.12432 (0.19795)	1.08022 (0.15718)	1.17553 (0.26360)	1.04122 (0.12964)	1.23677 (0.39075)	1.01505 (0.14481)	1.0887 (0.17715)	0.97662 (0.13402)
		16	1.1280 (0.18530)	1.08622 (0.15094)	1.17537 (0.23517)	1.04883 (0.12666)	1.23043 (0.31321)	1.02360 (0.13952)	1.09380 (0.16733)	0.98739 (0.13001)
		18	1.12035 (0.17523)	1.08084 (0.14246)	1.16578 (0.22639)	1.04552 (0.11961)	1.21969 (0.32229)	1.02158 (0.13322)	1.08788 (0.15863)	0.98759 (0.12478)
	6	12	1.11006 (0.17664)	1.07088 (0.14238)	1.15488 (0.22894)	1.03591 (0.11884)	1.20801 (0.31921)	1.01263 (0.13379)	1.0780 (0.15987)	0.97919 (0.12470)
		16	1.09222 (0.16443)	1.05448 (0.13307)	1.13523 (0.21162)	1.02074 (0.11155)	1.18581 (0.28811)	0.99695 (0.12632)	1.06087 (0.14932)	0.96427 (0.11865)
		18	1.09136 (0.15200)	1.05393 (0.12389)	1.13426 (0.19922)	1.02044 (0.10509)	1.21170 (0.29061)	0.99559 (0.11700)	1.05984 (0.13790)	0.96275 (0.11042)
30	3	20	1.08128 (0.10606)	1.05614 (0.09315)	1.10820 (0.12239)	1.03259 (0.08302)	1.13716 (0.14305)	1.01435 (0.08794)	1.05924 (0.09896)	0.99146 (0.08410)
		25	1.06153 (0.09847)	1.03811 (0.08772)	1.08654 (0.11208)	1.01610 (0.07930)	0.11334 (0.12925)	0.99807 (0.08436)	1.04059 (0.09281)	0.97647 (0.08163)
		28	1.06492 (0.09767)	1.04174 (0.01016)	1.08979 (0.11305)	1.02004 (0.07680)	1.11667 (0.13348)	1.00321 (0.08174)	1.04451 (0.09143)	0.98228 (0.07834)
	6	20	1.08813 (0.10615)	1.06421 (0.09285)	1.11379 (0.12305)	1.04180 (0.08237)	0.14148 (0.14466)	1.02561 (0.08724)	1.06746 (0.09890)	1.00441 (0.08288)
		25	1.06936 (0.09692)	1.04594 (0.08535)	1.09448 (0.11185)	1.02399 (0.07638)	1.12152 (0.13127)	1.00681 (0.08125)	1.04868 (0.09075)	0.98559 (0.07799)
		28	1.06446 (0.09426)	1.04166 (0.08353)	1.08889 (0.10801)	1.02029 (0.07519)	1.11521 (0.12573)	1.00342 (0.07946)	1.04427 (0.08841)	0.98273 (0.07641)
40	3	28	1.05483 (0.08097)	1.03675 (0.07379)	1.07388 (0.08970)	1.01943 (0.06790)	1.09381 (0.10025)	1.00493 (0.07136)	1.03830 (0.07722)	0.98819 (0.06924)
		30	1.05350 (0.07224)	1.03544 (0.06546)	1.07249 (0.08054)	1.01821 (0.05998)	1.09248 (0.09064)	1.00382 (0.06273)	1.03709 (0.06849)	0.98695 (0.06076)
		35	1.04703 (0.06758)	1.03029 (0.06203)	1.06457 (0.07432)	1.01428 (0.05751)	1.08298 (0.08247)	1.00062 (0.05984)	1.03167 (0.06449)	0.98493 (0.05830)
	6	28	1.05145 (0.07261)	1.03497 (0.06640)	1.06873 (0.08010)	1.01921 (0.06128)	1.08687 (0.08908)	1.00639 (0.06383)	1.03652 (0.06920)	0.99118 (0.06188)
		30	1.04926 (0.07051)	1.03270 (0.06456)	1.06661 (0.07773)	1.01688 (0.05969)	1.08484 (0.08645)	1.00373 (0.06226)	1.03417 (0.06727)	0.98837 (0.06050)
		35	1.04700 (0.06916)	1.03043 (0.06326)	1.06438 (0.07635)	1.01459 (0.05846)	1.08266 (0.08510)	1.00142 (0.06087)	1.03190 (0.06590)	0.98605 (0.05912)
50	3	38	1.05615 (0.06392)	1.04112 (0.05886)	1.07174 (0.06986)	1.02671 (0.05470)	1.08796 (0.07668)	1.01446 (0.05655)	1.04259 (0.06119)	1.00042 (0.05492)
		45	1.04545 (0.05537)	1.03217 (0.05165)	1.05922 (0.05976)	1.01934 (0.04852)	1.07352 (0.06489)	1.00835 (0.04986)	1.03315 (0.05322)	0.99586 (0.04868)
	6	38	1.05337 (0.05940)	1.04007 (0.05519)	1.06733 (0.06434)	1.02709 (0.05168)	1.08138 (0.07002)	1.01657 (0.05308)	1.04112 (0.05698)	1.00437 (0.05151)
		45	1.04796 (0.05265)	1.03513 (0.04911)	1.06126 (0.05680)	1.02273 (0.04611)	1.07504 (0.06162)	1.01222 (0.04720)	1.03611 (0.05054)	1.0002 (0.04598)

Table 3: Average estimates of the different estimates and corresponding MSE of the parameter λ_2 based on informative prior

n	T	R	SELF $h=-1$	LINEX				GELF		
				$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
20	3	12	2.20346 (0.58532)	2.07276 (0.41058)	2.3665 (0.97059)	1.96265 (0.32073)	2.48066 (1.30341)	2.02852 (0.45005)	2.1461 (0.53277)	1.96801 (0.42069)
		16	2.19714 (0.54323)	2.07457 (0.40470)	2.34405 (0.78749)	1.96967 (0.32738)	2.47865 (1.25533)	2.03095 (0.43653)	2.14246 (0.50074)	1.97396 (0.41481)
		18	2.18603 (0.51708)	2.06714 (0.37118)	2.33511 (0.88267)	1.96625 (0.29629)	2.46246 (1.26742)	2.02672 (0.40262)	2.13349 (0.47246)	1.97237 (0.37785)
	6	12	2.1822 (0.55382)	2.06565 (0.40801)	2.3236 (0.83195)	1.96617 (0.32794)	2.43959 (1.22250)	2.02624 (0.44489)	2.13071 (0.51146)	1.97316 (0.42106)
		16	2.15732 (0.51953)	2.04306 (0.39174)	2.29435 (0.74672)	1.94520 (0.32083)	2.46627 (1.17725)	2.00239 (0.42333)	2.10617 (0.48152)	1.94966 (0.40351)
		18	2.14232 (0.46287)	2.03237 (0.35026)	2.27351 (0.66667)	1.93775 (0.28922)	2.43762 (1.06717)	1.99140 (0.37908)	2.09248 (0.42930)	1.94007 (0.36051)
30	3	20	2.10887 (0.33120)	2.03117 (0.27346)	2.19552 (0.41959)	1.96085 (0.23755)	2.29327 (0.55275)	1.99967 (0.28708)	2.07283 (0.31369)	1.96250 (0.27816)
		25	2.10830 (0.31419)	2.00944 (0.26067)	2.16497 (0.39866)	1.94279 (0.22821)	2.25767 (0.53361)	1.97886 (0.27686)	2.04856 (0.29917)	1.94355 (0.26971)
		28	2.10482 (0.30923)	2.03395 (0.25739)	2.18363 (0.38814)	1.96961 (0.22447)	2.27250 (0.50886)	2.00584 (0.26923)	2.07203 (0.29356)	1.97242 (0.26067)
	6	20	2.10410 (0.30350)	2.03268 (0.25316)	2.18350 (0.37981)	1.96783 (0.22131)	2.27286 (0.49455)	2.00403 (0.26456)	2.07095 (0.28814)	1.97023 (0.25644)
		25	2.08408 (0.27999)	2.01480 (0.23565)	2.16084 (0.34766)	1.95174 (0.20818)	2.24678 (0.44885)	1.98597 (0.24686)	2.05158 (0.26666)	1.95285 (0.24047)
		28	2.08262 (0.26936)	2.01435 (0.22922)	2.15805 (0.33028)	1.95208 (0.20441)	2.24217 (0.42020)	1.98529 (0.23997)	2.05037 (0.25733)	1.95245 (0.23472)
40	3	28	2.04488 (0.22006)	1.99058 (0.19419)	2.10337 (0.25770)	1.94005 (0.17767)	2.16665 (0.31049)	1.96620 (0.20230)	2.01892 (0.21267)	1.93961 (0.19935)
		30	2.05277 (0.20930)	1.99910 (0.18550)	2.11050 (0.24390)	1.94902 (0.17033)	2.17788 (0.29389)	1.97489 (0.19253)	2.02699 (0.20229)	1.94855 (0.18980)
		35	2.06780 (0.19996)	2.01539 (0.17608)	2.12417 (0.23415)	1.96647 (0.16046)	2.18507 (0.28137)	1.99241 (0.18156)	2.04280 (0.19859)	1.96699 (0.17811)
	6	28	2.06742 (0.20832)	2.01627 (0.18310)	2.12243 (0.24276)	1.96850 (0.16776)	2.18186 (0.29000)	1.99403 (0.18988)	2.04307 (0.20093)	1.96934 (0.18632)
		30	2.05255 (0.19117)	2.00236 (0.16965)	2.10649 (0.22249)	1.95547 (0.15590)	2.16474 (0.26652)	1.97990 (0.17579)	2.02845 (0.18481)	1.95545 (0.17316)
		35	2.06115 (0.19076)	2.01082 (0.16883)	2.11519 (0.22207)	1.96377 (0.15446)	2.17347 (0.26513)	1.98859 (0.17426)	2.03707 (0.18404)	1.96417 (0.17123)
50	3	38	2.04471 (0.17319)	2.0014 (0.15614)	2.09027 (0.19613)	1.96106 (0.14552)	2.13825 (0.22621)	1.98211 (0.16148)	2.02379 (0.16836)	1.96078 (0.15898)
		45	2.04900 (0.16947)	2.00845 (0.15390)	2.09189 (0.19073)	1.97003 (0.14310)	2.13737 (0.21875)	1.99028 (0.15790)	2.02951 (0.16482)	1.97055 (0.15565)
	6	38	2.04375 (0.16501)	2.00411 (0.15014)	2.08568 (0.18536)	1.96652 (0.13987)	2.13011 (0.21224)	1.98628 (0.15412)	2.02466 (0.16062)	1.96697 (0.15202)
		45	2.04561 (0.15364)	2.00616 (0.14008)	2.08729 (0.17232)	1.96873 (0.13085)	2.13143 (0.19706)	1.98824 (0.14355)	2.02655 (0.14952)	1.96898 (0.14172)

Table 4: Average estimates of the different estimates and corresponding MSE of the parameter p based on informative prior

n	T	R	SELF $h=-1$	LINEX				GELF		
				$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
20	3	12	0.46034 (0.00989)	0.45745 (0.00982)	0.46324 (0.00997)	0.45456 (0.00976)	0.46615 (0.01006)	0.43899 (0.01072)	0.45351 (0.01005)	0.43125 (0.01127)
		16	0.45828 (0.00964)	0.45551 (0.00959)	0.46105 (0.00971)	0.45275 (0.00954)	0.46383 (0.00979)	0.43789 (0.01046)	0.45174 (0.00981)	0.43052 (0.01097)
		18	0.46518 (0.00948)	0.46259 (0.00940)	0.46778 (0.00958)	0.46001 (0.00932)	0.47038 (0.00969)	0.44651 (0.00992)	0.45918 (0.00954)	0.43981 (0.01026)
	6	12	0.45899 (0.00948)	0.45648 (0.00943)	0.46151 (0.00955)	0.45397 (0.00938)	0.46403 (0.00963)	0.44065 (0.01014)	0.45309 (0.00962)	0.43408 (0.01056)
		16	0.46153 (0.00919)	0.45901 (0.00912)	0.46406 (0.00927)	0.45649 (0.00906)	0.46659 (0.00936)	0.44325 (0.00972)	0.45565 (0.00928)	0.43670 (0.01009)
		18	0.45945 (0.00900)	0.45695 (0.00895)	0.46196 (0.00907)	0.45445 (0.00890)	0.46447 (0.00914)	0.44125 (0.00967)	0.45359 (0.00912)	0.43474 (0.00910)
30	3	20	0.45598 (0.00750)	0.45387 (0.00747)	0.45801 (0.00754)	0.45177 (0.00748)	0.46021 (0.00759)	0.44082 (0.00797)	0.45107 (0.00760)	0.43546 (0.00824)
		25	0.45268 (0.00711)	0.45069 (0.00710)	0.45468 (0.00714)	0.44870 (0.00708)	0.45668 (0.00717)	0.43829 (0.00762)	0.44801 (0.00723)	0.43323 (0.00790)
		28	0.45826 (0.00681)	0.45643 (0.00677)	0.46009 (0.00685)	0.45461 (0.00674)	0.46192 (0.00689)	0.44532 (0.00709)	0.45405 (0.00686)	0.44078 (0.00727)
	6	20	0.45644 (0.00680)	0.45618 (0.00679)	0.45827 (0.00684)	0.45280 (0.00675)	0.46010 (0.00688)	0.44347 (0.00713)	0.45222 (0.00687)	0.43891 (0.00733)
		25	0.45567 (0.00665)	0.45384 (0.00663)	0.45750 (0.00668)	0.452023 (0.00662)	0.45933 (0.00672)	0.44267 (0.00610)	0.45144 (0.00673)	0.43812 (0.00721)
		28	0.45813 (0.00642)	0.45633 (0.00639)	0.45993 (0.00646)	0.45454 (0.00636)	0.46173 (0.00650)	0.44543 (0.00663)	0.45310 (0.00647)	0.44098 (0.00685)
40	3	28	0.45812 (0.00595)	0.45645 (0.00592)	0.45979 (0.00598)	0.45478 (0.00589)	0.46147 (0.00602)	0.44638 (0.00615)	0.45430 (0.00598)	0.44228 (0.00630)
		30	0.45904 (0.00594)	0.45738 (0.00590)	0.46070 (0.00598)	0.45573 (0.00587)	0.46236 (0.00602)	0.44743 (0.00611)	0.45525 (0.00596)	0.44338 (0.00624)
		35	0.45534 (0.00562)	0.45382 (0.00561)	0.45686 (0.00562)	0.45230 (0.00561)	0.45839 (0.00563)	0.44465 (0.00584)	0.45185 (0.00569)	0.44094 (0.00592)
	6	28	0.45219 (0.00567)	0.45077 (0.00566)	0.45362 (0.00569)	0.44935 (0.00565)	0.45504 (0.00570)	0.44214 (0.00594)	0.44890 (0.00574)	0.43865 (0.00609)
		30	0.45516 (0.00539)	0.45373 (0.00538)	0.45659 (0.00542)	0.45231 (0.00536)	0.45801 (0.00544)	0.44517 (0.00559)	0.45189 (0.00544)	0.44171 (0.00571)
		35	0.45835 (0.00528)	0.45692 (0.00525)	0.45978 (0.00531)	0.45549 (0.00523)	0.46121 (0.00534)	0.44842 (0.00539)	0.45510 (0.00530)	0.44498 (0.00549)
50	3	38	0.45559 (0.00498)	0.45347 (0.00527)	0.45764 (0.00532)	0.45258 (0.00498)	0.45852 (0.00510)	0.44556 (0.00518)	0.45213 (0.00504)	0.44271 (0.00524)
		45	0.45678 (0.00486)	0.45555 (0.00484)	0.45710 (0.00491)	0.45433 (0.00483)	0.45922 (0.00491)	0.44831 (0.00497)	0.45400 (0.00488)	0.44540 (0.00504)
	6	38	0.45314 (0.00458)	0.45198 (0.00457)	0.45431 (0.00459)	0.45082 (0.00457)	0.45549 (0.00461)	0.44496 (0.00473)	0.45043 (0.00461)	0.44218 (0.00482)
		45	0.45216 (0.00444)	0.45099 (0.00444)	0.45333 (0.00445)	0.44982 (0.00443)	0.45450 (0.00447)	0.44399 (0.00461)	0.44948 (0.00448)	0.44118 (0.00470)

Table 5: Average estimates of the different estimates and corresponding MSE of the parameter λ_1 based on non- informative prior

n	T	R	SELF $h=-1$	LINEX				GELF		
				$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
20	3	12	1.12694 (0.26221)	1.07833 (0.19506)	1.22832 (0.37628)	1.03629 (0.15459)	1.25324 (0.56653)	1.01161 (0.18815)	1.08938 (0.23415)	0.97100 (0.17058)
		16	1.14796 (0.24460)	1.10019 (0.19142)	1.20382 (0.32837)	1.05829 (0.15562)	1.27277 (0.48519)	1.03350 (0.17999)	1.11053 (0.21971)	0.99364 (0.16560)
		18	1.13500 (0.23561)	1.08916 (0.18320)	1.18931 (0.32135)	1.04917 (0.14866)	1.25939 (0.50089)	1.02580 (0.17353)	1.09917 (0.21159)	0.98805 (0.15990)
	6	12	1.14092 (0.23598)	1.09509 (0.18332)	1.19549 (0.32635)	1.0551 (0.14875)	1.22685 (0.47107)	1.03277 (0.17160)	1.10540 (0.21128)	0.99547 (0.15698)
		16	1.12581 (0.20911)	1.08403 (0.16949)	1.17352 (0.26729)	1.04684 (0.14168)	1.22950 (0.35855)	1.02322 (0.16102)	1.09206 (0.19037)	0.98710 (0.15066)
		18	1.11006 (0.19518)	1.06815 (0.15607)	1.15905 (0.25948)	1.03117 (0.12984)	1.24410 (0.43030)	1.00696 (0.14668)	1.07616 (0.17609)	0.97152 (0.13665)
30	3	20	1.08550 (0.15557)	1.05652 (0.12731)	1.11816 (0.20930)	1.03003 (0.10875)	1.18892 (0.21211)	1.01216 (0.12350)	1.06142 (0.14365)	0.98685 (0.11522)
		25	1.09680 (0.13161)	1.06969 (0.11371)	1.12613 (0.15475)	1.04449 (0.09981)	1.15810 (0.18503)	1.02698 (0.10730)	1.07378 (0.12233)	1.00315 (0.10163)
		28	1.09704 (0.12167)	1.07109 (0.10589)	1.25040 (0.18891)	1.04692 (0.09351)	1.15470 (0.16766)	1.03019 (0.09921)	1.07494 (0.11309)	1.00749 (0.09396)
	6	20	1.09761 (0.13995)	1.07144 (0.12041)	1.12602 (0.16564)	1.04716 (0.10534)	1.15719 (0.20033)	1.03132 (0.11455)	1.07570 (0.13039)	1.00881 (0.10834)
		25	1.09194 (0.12529)	1.06615 (0.10755)	1.11994 (0.14933)	1.04221 (0.09413)	1.15079 (0.18413)	1.02594 (0.10165)	1.07013 (0.11633)	1.00353 (0.09599)
		28	1.07756 (0.11214)	1.05287 (0.09816)	1.10417 (0.13029)	1.02983 (0.08734)	1.13306 (0.15425)	1.01277 (0.09327)	1.05614 (0.10482)	0.99080 (0.08909)
40	3	28	1.05801 (0.08291)	1.03914 (0.00754)	1.07775 (0.09207)	1.02136 (0.06922)	1.09863 (0.10321)	1.00650 (0.07272)	1.04097 (0.07890)	0.98892 (0.07060)
		30	1.04541 (0.07986)	1.02695 (0.07279)	1.06481 (0.08852)	1.00938 (0.06709)	1.08526 (0.09909)	0.99442 (0.07064)	1.02857 (0.07617)	0.97709 (0.06883)
		35	1.04402 (0.07680)	1.02637 (0.07016)	1.06258 (0.08493)	1.00955 (0.06480)	1.08213 (0.09482)	0.99536 (0.06804)	1.02792 (0.07332)	0.97889 (0.06628)
	6	28	1.06053 (0.08016)	1.04292 (0.07266)	1.07907 (0.08927)	1.02613 (0.06652)	1.09863 (0.10034)	1.01295 (0.06949)	1.04477 (0.07606)	0.99690 (0.06704)
		30	1.06082 (0.07755)	1.04326 (0.07039)	1.07923 (0.08616)	1.02656 (0.06455)	1.09864 (0.09656)	1.01330 (0.06715)	1.04520 (0.07370)	0.99735 (0.06487)
		35	1.04881 (0.07434)	1.03180 (0.06801)	1.06667 (0.08205)	1.01557 (0.06285)	1.08547 (0.09141)	1.00217 (0.06555)	1.03336 (0.07089)	0.98643 (0.06367)
50	3	38	1.05622 (0.06806)	1.04102 (0.06269)	1.07172 (0.07444)	1.0272 (0.05824)	1.08764 (0.08181)	1.01521 (0.06039)	1.0428 (0.06512)	1.00155 (0.05854)
		45	1.04606 (0.06394)	1.03228 (0.05941)	1.06037 (0.06928)	1.01900 (0.05559)	1.07526 (0.07552)	1.00793 (0.05749)	1.03342 (0.06145)	0.99509 (0.05602)
	6	38	1.04736 (0.05931)	1.03385 (0.05501)	1.06123 (0.06432)	1.02069 (0.05139)	1.07577 (0.07023)	1.01037 (0.05298)	1.03495 (0.05682)	0.99778 (0.05145)
		45	1.04843 (0.05538)	1.03488 (0.05132)	1.06249 (0.06016)	1.02182 (0.04792)	1.07710 (0.06578)	1.01081 (0.04932)	1.03596 (0.05303)	0.99813 (0.04797)

Table 6: Average values of the different estimates and corresponding MSE of the parameter λ_2 based on non- informative prior

n	T	R	SELF $h=-1$	LINEX				GELF		
				$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
20	3	12	2.21754 (0.68125)	2.07977 (0.46392)	2.39262 (1.19512)	1.96494 (0.35534)	2.54036 (1.61228)	2.03571 (0.52254)	2.15797 (0.62033)	1.97271 (0.48647)
		16	2.20911 (0.60343)	2.08165 (0.43729)	2.36394 (0.90952)	1.97343 (0.34602)	2.53723 (1.51044)	2.03885 (0.47810)	2.15312 (0.55468)	1.98040 (0.45087)
		18	2.20443 (0.58533)	2.07995 (0.43028)	2.35595 (0.86771)	1.97436 (0.34429)	2.51448 (1.48673)	2.03844 (0.46522)	2.14971 (0.53844)	1.98176 (0.43938)
	6	12	2.21886 (0.62176)	2.09481 (0.44670)	2.37088 (0.95347)	1.98979 (0.34972)	2.55471 (1.54667)	2.05627 (0.48929)	2.16521 (0.57103)	2.00089 (0.45870)
		16	2.20639 (0.59189)	2.08487 (0.43550)	2.35377 (0.87436)	1.98145 (0.34745)	2.51270 (1.42927)	2.04480 (0.47211)	2.15306 (0.54619)	1.98976 (0.44816)
		18	2.22729 (0.57259)	2.10382 (0.41389)	2.37754 (0.86576)	1.99894 (0.32546)	2.50151 (1.38405)	2.06425 (0.44786)	2.17347 (0.52445)	2.00874 (0.41981)
30	3	20	2.12283 (0.37264)	2.04324 (0.30425)	2.21176 (0.47641)	1.97132 (0.26069)	2.31235 (0.63326)	2.01221 (0.32204)	2.08632 (0.35293)	1.97456 (0.31103)
		25	2.13638 (0.33237)	2.05905 (0.27037)	2.22271 (0.42633)	1.98910 (0.23084)	2.32033 (0.56836)	2.02928 (0.28313)	2.10097 (0.31326)	1.99296 (0.27225)
		28	2.11144 (0.32534)	2.03890 (0.26956)	2.19252 (0.41272)	1.97320 (0.23434)	2.28580 (0.57189)	2.01045 (0.28255)	2.07799 (0.30862)	1.97633 (0.27330)
	6	20	2.12126 (0.34113)	2.04758 (0.28010)	2.20357 (0.43362)	1.98093 (0.24084)	2.29691 (0.57491)	2.01952 (0.29417)	2.08756 (0.32210)	1.98516 (0.28359)
		25	2.11256 (0.30164)	2.03939 (0.24955)	2.19404 (0.38094)	1.97307 (0.21680)	2.28591 (0.50045)	2.01013 (0.26125)	2.07863 (0.28568)	1.97552 (0.25287)
		28	2.11461 (0.29162)	2.04334 (0.24168)	2.19378 (0.36701)	1.97859 (0.20988)	2.28285 (0.48042)	2.01511 (0.25165)	2.08164 (0.27594)	1.98152 (0.24311)
40	3	28	2.08969 (0.25495)	2.03253 (0.22091)	2.15144 (0.30301)	1.97927 (0.19764)	2.21848 (0.36917)	2.00891 (0.22901)	2.06272 (0.24454)	1.98110 (0.22275)
		30	2.10163 (0.24832)	2.04397 (0.21638)	2.16393 (0.29727)	1.99039 (0.18947)	2.23161 (0.36429)	2.02032 (0.22007)	2.07472 (0.23739)	1.99282 (0.21377)
		35	2.09737 (0.23745)	2.04244 (0.20576)	2.15662 (0.28147)	1.99130 (0.18389)	2.22086 (0.34124)	2.01979 (0.21201)	2.07165 (0.22757)	1.99363 (0.20636)
	6	28	2.09368 (0.23860)	2.04040 (0.20739)	2.15116 (0.28182)	1.99078 (0.18566)	2.21349 (0.34050)	2.01864 (0.21351)	2.06878 (0.22892)	1.99337 (0.20780)
		30	2.07054 (0.21825)	2.01856 (0.19227)	2.12652 (0.25492)	1.97009 (0.17480)	2.18711 (0.30516)	1.99622 (0.19840)	2.04588 (0.21034)	1.97121 (0.19441)
		35	2.08461 (0.22276)	2.03227 (0.19425)	2.14098 (0.26249)	1.98347 (0.17466)	2.20201 (0.31651)	2.01043 (0.20008)	2.05999 (0.21392)	1.98546 (0.19512)
50	3	38	2.07103 (0.20341)	2.02797 (0.18407)	2.11869 (0.23053)	1.9847 (0.16783)	2.16839 (0.26640)	2.00863 (0.18985)	2.04975 (0.19850)	1.98716 (0.18515)
		45	2.06594 (0.19958)	2.02382 (0.17960)	2.11058 (0.22614)	1.98399 (0.16516)	2.15803 (0.26063)	2.00574 (0.18416)	2.04595 (0.19360)	1.98551 (0.18072)
	6	38	2.09025 (0.18267)	2.04811 (0.16250)	2.13485 (0.20909)	2.00827 (0.14776)	2.18226 (0.24317)	2.03049 (0.16523)	2.07043 (0.17608)	2.01051 (0.16126)
		45	2.06175 (0.17707)	2.02088 (0.15988)	2.10500 (0.20014)	1.98219 (0.14765)	2.15092 (0.23025)	2.00304 (0.16359)	2.04225 (0.17178)	1.98333 (0.16071)

Table 7: Average values of the different estimates and corresponding MSE of the parameter p based on non-informative prior

n	T	R	SELF $h=-1$	LINEX				GELF		
				$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
20	3	12	0.45702 (0.01287)	0.45386 (0.01281)	0.46018 (0.01295)	0.45071 (0.01276)	0.46336 (0.01305)	0.43303 (0.01429)	0.44938 (0.01319)	0.42423 (0.01514)
		16	0.45211 (0.01185)	0.44910 (0.01181)	0.45514 (0.01190)	0.44609 (0.01179)	0.45817 (0.01196)	0.429187 (0.01329)	0.44480 (0.01218)	0.42083 (0.01410)
		18	0.44833 (0.01092)	0.44553 (0.01091)	0.45113 (0.01094)	0.44273 (0.01092)	0.45395 (0.01098)	0.42707 (0.01231)	0.44152 (0.01126)	0.41936 (0.01305)
	6	12	0.45636 (0.01073)	0.45364 (0.01068)	0.45909 (0.01079)	0.45093 (0.01064)	0.46182 (0.01087)	0.43615 (0.01169)	0.44988 (0.01094)	0.42885 (0.01225)
		16	0.45318 (0.01019)	0.45046 (0.01016)	0.45591 (0.01024)	0.44774 (0.01014)	0.45865 (0.01029)	0.43283 (0.01126)	0.44666 (0.01044)	0.42548 (0.01187)
		18	0.45809 (0.00998)	0.45537 (0.00993)	0.46081 (0.01005)	0.45266 (0.00988)	0.46354 (0.01013)	0.43805 (0.01080)	0.45166 (0.01015)	0.43083 (0.01131)
30	3	20	0.45325 (0.00899)	0.45098 (0.00897)	0.45551 (0.00901)	0.44873 (0.00896)	0.45779 (0.00905)	0.43664 (0.00974)	0.44788 (0.00916)	0.43367 (0.01025)
		25	0.45436 (0.00819)	0.45224 (0.00817)	0.45650 (0.00823)	0.45011 (0.00815)	0.45863 (0.00827)	0.43894 (0.00876)	0.44937 (0.00832)	0.43349 (0.00908)
		28	0.45481 (0.00767)	0.45287 (0.00764)	0.45675 (0.00770)	0.45094 (0.00763)	0.45869 (0.00774)	0.44089 (0.00813)	0.45029 (0.00777)	0.43599 (0.00839)
	6	20	0.45779 (0.00803)	0.45586 (0.00799)	0.45972 (0.00808)	0.45393 (0.00796)	0.46166 (0.00813)	0.44398 (0.00841)	0.45331 (0.00811)	0.43913 (0.00864)
		25	0.45395 (0.00742)	0.45202 (0.00740)	0.45588 (0.00745)	0.45010 (0.00738)	0.45781 (0.00749)	0.44008 (0.00789)	0.44945 (0.00753)	0.43520 (0.00815)
		28	0.45614 (0.00758)	0.45424 (0.00755)	0.45804 (0.00762)	0.45234 (0.00753)	0.45995 (0.00766)	0.44252 (0.00799)	0.45172 (0.00767)	0.43773 (0.00823)
40	3	28	0.45251 (0.00674)	0.45077 (0.00673)	0.45426 (0.00647)	0.44903 (0.00672)	0.45599 (0.00679)	0.43987 (0.00715)	0.44837 (0.00683)	0.43556 (0.00736)
		30	0.45246 (0.00659)	0.45072 (0.00658)	0.45420 (0.00662)	0.44899 (0.00657)	0.45595 (0.00664)	0.44006 (0.00698)	0.44842 (0.00668)	0.43572 (0.00719)
		35	0.45754 (0.00645)	0.45595 (0.00643)	0.45914 (0.00649)	0.45436 (0.00640)	0.46074 (0.00653)	0.44632 (0.00660)	0.45388 (0.00650)	0.44241 (0.00682)
	6	28	0.45224 (0.00626)	0.45075 (0.00624)	0.45374 (0.00627)	0.44925 (0.00624)	0.45524 (0.00629)	0.44163 (0.00656)	0.44876 (0.00633)	0.43794 (0.00673)
		30	0.45201 (0.00585)	0.45051 (0.00584)	0.45350 (0.00586)	0.44902 (0.00583)	0.45410 (0.00588)	0.44142 (0.00615)	0.44855 (0.00592)	0.43774 (0.00631)
		35	0.45167 (0.00567)	0.45018 (0.00567)	0.45317 (0.00569)	0.44870 (0.00566)	0.45466 (0.00570)	0.44111 (0.00597)	0.44822 (0.00575)	0.43744 (0.00613)
50	3	38	0.45295 (0.00545)	0.45165 (0.00543)	0.45471 (0.00547)	0.45016 (0.00544)	0.45601 (0.00547)	0.44274 (0.00573)	0.44933 (0.00559)	0.43951 (0.00581)
		45	0.45071 (0.00506)	0.44944 (0.00505)	0.45198 (0.05061)	0.44817 (0.00549)	0.45325 (0.00508)	0.44178 (0.00529)	0.44778 (0.00512)	0.43870 (0.00541)
	6	38	0.45162 (0.00504)	0.45040 (0.00504)	0.45284 (0.00505)	0.44919 (0.00503)	0.45405 (0.00507)	0.44309 (0.00525)	0.44882 (0.00510)	0.44016 (0.00535)
		45	0.44909 (0.00486)	0.44787 (0.00486)	0.45030 (0.00486)	0.44666 (0.00486)	0.45152 (0.00487)	0.44054 (0.00519)	0.44628 (0.00492)	0.43759 (0.00522)

6.2 Example

In this subsection, we consider the following simulated hybrid Type-II censored sample of size $(n = 30)$ from the mixture of the ICICR models with parameters $p = 0.40$, $\lambda_1 = 2, \lambda_2 = 3, \beta_1 = 1.2, \beta_2 = 2.5$. For the set value $R = 25$ and $T = 4$, according to Type II hybrid censored, it was found that $r = 27$. The simulated hybrid Type II censored sample is given, as follows:

$$1.02915, 1.76915, 1.93382, 2.25868, 2.30983, 2.36497, 3.1418, 3.27032, 3.8834, 3.89353 \qquad r_1 = 10$$

$$0.497222, 0.609207, 0.682019, 0.748575, 0.751229, 0.834523, 0.908811, 1.08412, 1.08675, 1.30115, 1.65474, 2.0688, 2.14596, 2.19979, 2.46552, 2.78366, 3.19005 \qquad r_2 = 17$$

Based on the above simulated data, we present some results to compare the performance of the classical estimators, such as the MLEs and Bayesian approaches for different choices of prior parameters. We have also computed approximate 95% confidence intervals (ACI) of the parameters and Credible intervals (CRI). The 95% Predictive interval for the future observation y_s are obtained by numerically solving Equation (21) . All the results are summarized in Tables 8-12.

Table 8: Estimated value of λ_1, λ_2 and p based on informative prior

Parameter	MLEs	Loss Function							
		SELF	LINEX				GELF		
		$h=-1$	$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
λ_1	0.41122	2.16871	2.07885	2.26938	1.99798	2.38326	2.03731	2.15529	1.9927
λ_2	2.28381	3.11245	2.98153	3.25976	2.86416	3.42723	2.97911	3.06823	2.93419
p	3.16358	0.4205	0.41864	0.42237	0.41678	0.42424	0.40647	0.41595	0.40153

Table 9: Estimated value of λ_1, λ_2 and p based on non-informative prior

Parameter	Loss Function							
	SELF	LINEX				GELF		
	$h=-1$	$q=0.5$	$q=-0.5$	$q=1$	$q=-1$	$h=0.5$	$h=-0.5$	$h=1$
λ_1	2.27681	2.1741	2.39313	2.08249	2.52638	2.13301	2.22932	2.08413
λ_2	3.17076	3.0328	3.32668	2.90956	3.50483	3.03256	3.12493	2.98599
p	0.41543	0.41345	0.41741	0.41148	0.41940	0.40028	0.41052	0.39493

Table 10: 95% Interval estimates of λ_1, λ_2 and p

Method	λ_1	λ_2	p
	(Lower,Upper) Length	(Lower,Upper) Length	(Lower,Upper) Length
ACI	(0.98859,3.57903) 2.59044	(1.66561,4.66155) 2.99594	(0.22814,0.59431) 0.36617
CRI-Informative prior	(1.12921,3.52861) 2.3994	(1.82717,4.73114) 2.90397	(0.25647,0.59253) 0.33606
CRI-nonInformative prior	(1.16639,3.73744) 2.57105	(1.85161,4.83542) 2.98381	(0.24687,0.59303) 0.34616

Table 11: Two sample prediction intervals for Y_S in case of informative prior

95% predictive intervals for Y_S		
s	(Lower,Upper)	Length
1	(0.06541,0.24523)	0.17982
5	(0.09235,0.37009)	0.27773
10	(0.15177,0.75598)	0.60421

Table 12: Two sample prediction intervals for Y_S in case of non-informative prior

95% predictive intervals for Y_S		
s	(Lower,Upper)	Length
1	(0.06754,0.25655)	0.18901
5	(0.09333,0.38607)	0.29274
10	(0.15265,0.76628)	0.61363

7 Conclusion

The present paper addressed Type-II hybrid censored data for the mixture of the ICICRD. The maximum likelihood and Bayes estimators using symmetric and asymmetric loss function assuming informative and non-informative priors of the parameters have been derived. A comparison between the performance of the estimators for parameters has been conducted via Monte Carlo simulation .We have also computed 95% asymptotic confidence interval and credible interval estimates under the respective approaches. Table 9 showed that the length of credible intervals was smaller than the asymptotic confidence intervals and length of the intervals under informative prior was shorter than that under non-informative prior. Moreover, the predictive intervals for future observations could not be evaluated analytically, so we have applied numerical methods to obtain the limits. Furthermore, the predictive intervals increased as s increased.

Conflict of Interest

The authors declare that they have no conflict of interest.

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