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A New Generalization of the Moyal Distribution

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Abstract: In this article, a three parameter generalization of Moyal distribution is obtained to provide a more flexible model relative to the behaviour of hazard rate functions. Various statistical properties such as density, hazard rate functions, quantile function, mode, moments, incomplete moments, moment generating functions, mean deviation, Lorenz, Bonferroni and Zenga curves, Renyi and continuous entropies and distribution of rth order statistics have been derived. Maximum likelihood estimation has been used to estimate the parameters of the generalized Moyal distribution. Further confidence intervals are also obtained. Finally applicability of the proposed model to the real data is analyzed. A comparison has also been made with some existing distributions.

Keywords: Moyal distribution, Topp Leone distribution, moment generating function, entropy and maximum likelihood estimation.

1 Introduction

Statistical distributions are used to model the life of an item and investigate its important properties. Proper distribution provides useful information that results in sound conclusions and decisions. When there is a need for more flexible distributions, several researchers use the new one with more generalization. Let G be any valid cumulative distribution function defined on the real line. The last decade has seen various approaches for generating new distributions based on G. All these approaches can be put in the form

$$F(x) = H(G(x)), \tag{1}$$

where $H : [0, 1] \to [0, 1]$ and F is a valid cumulative distribution function. Thus, for every G, one can use (1) to generate a new distribution. The first approach of the form (1) in recent years was the exponentiated G distributions due to Mudholkar and Srivastava [1], Gupta and Kundu [2], Nassar and Eissa [3], [4] and others. The second approach was beta-G distributions due to Eugene et al. [5], Jones [6], Nadarajah and Kotz [7], [8], Cordeiro and Lemonte [9], Cordeiro et al. [10], Nassar and Nada [11], [12], [13], Nassar and Elmasry [14] and Mahmoud et al. [15], followed by Gamma-G distributions due to Zografos and Balakrishnan [[16]. Jones [17], Cordeiro et al. [18], [19], Cordeiro and de Castro [20], Elbatal and Elgarhy [21], Nassar [22] and others introduced Kumaraswamy – G distributions that serve survival analysis and marketing research.

Recently, applying new generators for continuous distributions has become more interesting. This methodology can improve the goodness of fit and define tail properties. One of the most important generators is the Topp-Leone (TL) distribution which was proposed by Topp and Leone [23] for empirical data with J-shaped histograms, such as powered band tool and automatic calculating machine failures.

In the present paper, we introduce a three parameter model(i.e. the Topp Leone Moyal (TLMo) distribution, to extend the Moyal distribution. The TLMo distribution is convenient for modeling comfortable upside-down bathtub

shaped failure rates and is a competitive model to the Moyal, half-normal, beta normal, skew normal and Gumbel distributions.

The Moyal distribution was proposed by J. E. Moyal [24] as an approximation for the Landau distribution. It was also shown that it remains valid taking into account quantum resonance effects and details of atomic structure of the absorber.

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Let Z be a random variable following the Moyal standard probability density function (pdf) given by

$$g(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(z+e^{-z})} \quad , \qquad -\infty < z < \infty. \tag{2}$$

A location parameter μ and a scale factor σ can be introduced to define the random variable $Z = \frac{x-\mu}{\sigma}$ having a Moyal distribution, say Mo(μ, σ), given by

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}[(\frac{x-\mu}{\sigma})+e^{-(\frac{x-\mu}{\sigma})}]} \quad , \qquad -\infty < x, \mu < \infty, \ \sigma>0. \tag{3}$$

The cumulative distribution function (cdf) corresponding to (3) depends on the lower incomplete gamma function $\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt$, and is given by

$$G(x) = 1 - \frac{\gamma\left(\frac{1}{2}, \ \frac{1}{2} e^{-(\frac{x-\mu}{\sigma})}\right)}{\Gamma\left(\frac{1}{2}\right)} \quad , \qquad -\infty < x, \mu < \infty, \ \sigma>0. \tag{4}$$

The cumulative distribution function of the standard Moyal distribution (2) is

$$G(z) = 1 - \frac{\gamma\left(\frac{1}{2}, \ \frac{1}{2} e^{-z}\right)}{\Gamma\left(\frac{1}{2}\right)} \ , \qquad -\infty < z < \infty \tag{5}$$

The only generalization of the Moyal distribution was the four – parameter Beta Moyal distribution proposed by Cordeiro et al. [10] to extend the Moyal model for its importance and usefulness in many practical situations.

2 The Topp-Leone Moyal Distribution

In this section, we introduce the Topp-Leone Moyal (TLMo) distribution. Some reliability functions corresponding to the TLMo distribution are also discussed. Consider the Topp-Leone generated family of distributions proposed by Al-Shomrani et al. [25], with its cumulative distribution function (cdf) and probability density function (pdf) given by,

$$F_{TLG}(x) = [G(x)]^{\alpha}[2 - G(x)]^{\alpha}, \qquad \alpha > 0. \tag{6}$$

$$f_{TLG}(x) = 2\alpha g(x)\overline{G}(x)[G(x)]^{\alpha-1}[2 - G(x)]^{\alpha-1}, \qquad \alpha > 0. \tag{7}$$

where $G(x)$ is the baseline distribution function, $\overline{G}(x) = 1 - G(x)$ and $g(x) = \partial G(x)/\partial x$ is the baseline density function.
Inserting Equation (4) in Equation (6), we obtain a new distribution, the so-called Topp-Leone Moyal (TLMo) distribution with cdf given by

$$F(x;\alpha,\mu,\sigma) = \left\{1 - \left[\frac{\gamma\left(\frac{1}{2}, \ 1/2\, e^{-(\frac{x-\mu}{\sigma})}\right)}{\Gamma\left(\frac{1}{2}\right)}\right]^2\right\}^{\alpha} = \left\{1 - \left[erf\left(\frac{\sqrt{e^{-(\frac{x-\mu}{\sigma})}}}{\sqrt{2}}\right)\right]^2\right\}^{\alpha}, \ -\infty < x, \mu < \infty, \ \sigma>0. \tag{8}$$

where $\gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi} erf\left(\sqrt{x}\right)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and the error function $erf(x)$ is de-fined by $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.
The pdf corresponding to Equation (8) is given by

$$f(x;\alpha,\mu,\sigma) = \frac{e^{-\frac{1}{2}e^{-(\frac{x-\mu}{\sigma})}} \sqrt{e^{-(\frac{x-\mu}{\sigma})}} \sqrt{\frac{2}{\pi}} \alpha erf\left(\frac{\sqrt{e^{-(\frac{x-\mu}{\sigma})}}}{\sqrt{2}}\right) \left\{1 - \left[erf\left(\frac{\sqrt{e^{-(\frac{x-\mu}{\sigma})}}}{\sqrt{2}}\right)\right]^2\right\}^{\alpha-1}}{\sigma} \tag{9}$$

where $-\infty < x, \mu < \infty$, $\sigma > 0$ *and* $\alpha > 0$. *For* $\mu = 0$ *and* $\sigma = 1$, *we obtain* the standard TLMo cdf given by

$$F(x) = \left\{1 - \frac{\gamma^2\left(\frac{1}{2}, \ \frac{e^{-x}}{2}\right)}{\pi}\right\}^{\alpha} = \left\{1 - \left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2\right\}^{\alpha} \tag{10}$$

and the standard TLMo density function given by

$$\begin{aligned} f(x) &= e^{-\frac{1}{2}e^{-x}}\sqrt{e^{-x}}\sqrt{\frac{2}{\pi}}\alpha erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\left\{1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{2}\right\}^{\alpha-1} \\ &= \frac{\sqrt{2}\alpha}{\pi}\mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}\gamma\left(\frac{1}{2},\ \frac{e^{-x}}{2}\right)\left\{1-\frac{\gamma^{2}\left(\frac{1}{2},\ \frac{e^{-x}}{2}\right)}{\pi}\right\}^{\alpha-1} \end{aligned} \tag{11}$$

Plots of the density function (9) for selected values of the TLMo distribution are given in Figure 1. It is noticeable that these plots show great flexibility of the TLMo for different values of the shape parameter α, including the special case in Equation (11). It is also observed that when the value of μ changes from -ve to +ve value, $f(x)$ is displaced to the +ve side of x-axis. The pdf given in (9) shows that $f(x)$ can be symmetric and asymmetric.

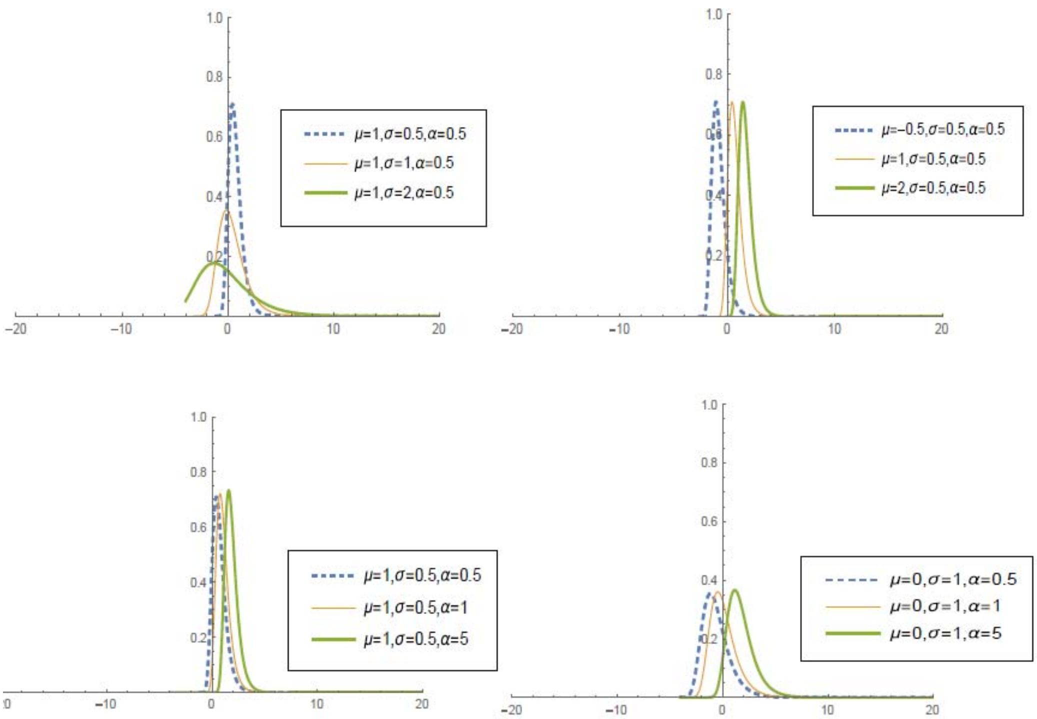

Fig. 1: Plots of the pdf of TLMo distribution for some parameter values.

We define the hazard function of the TLMo distribution as follows:

$$h(x) = \frac{f(x)}{S(x)},$$

where $S(x)$ is the survival (reliability) function of the TLMo distribution,

$$S(x) = 1 - F(x) = 1 - \left\{1 - \left[erf\left(\frac{\sqrt{e^{-\left(\frac{x-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^{2}\right\}^{\alpha}.$$

Then the hazard function of the TLMo distribution (9) is given by

$$h(x)=\frac{e^{-\frac{1}{2}e^{-(\frac{x-\mu}{\sigma})}}\sqrt{e^{-(\frac{x-\mu}{\sigma})}}\sqrt{\frac{2}{\pi}}\alpha erf\left(\frac{\sqrt{e^{-(\frac{x-\mu}{\sigma})}}}{\sqrt{2}}\right)\left\{1-\left[erf\left(\frac{\sqrt{e^{-(\frac{x-\mu}{\sigma})}}}{\sqrt{2}}\right)\right]^{2}\right\}^{\alpha-1}}{\sigma\left(1-\left\{1-\left[erf\left(\frac{\sqrt{e^{-(\frac{x-\mu}{\sigma})}}}{\sqrt{2}}\right)\right]^{2}\right\}^{\alpha}\right)}. \tag{12}$$

and we define the hazard function of the standard TLMo distribution as follows:

$$h(x)=\frac{e^{-\frac{1}{2}e^{-x}}\sqrt{e^{-x}}\sqrt{\frac{2}{\pi}}\alpha erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\left\{1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{2}\right\}^{\alpha-1}}{\left(1-\left\{1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^{2}\right\}^{\alpha}\right)} \tag{13}$$

Plots of the hazard function (12) of the TLMo distribution (9) are given in Figure 2 for selected values of the parameters.

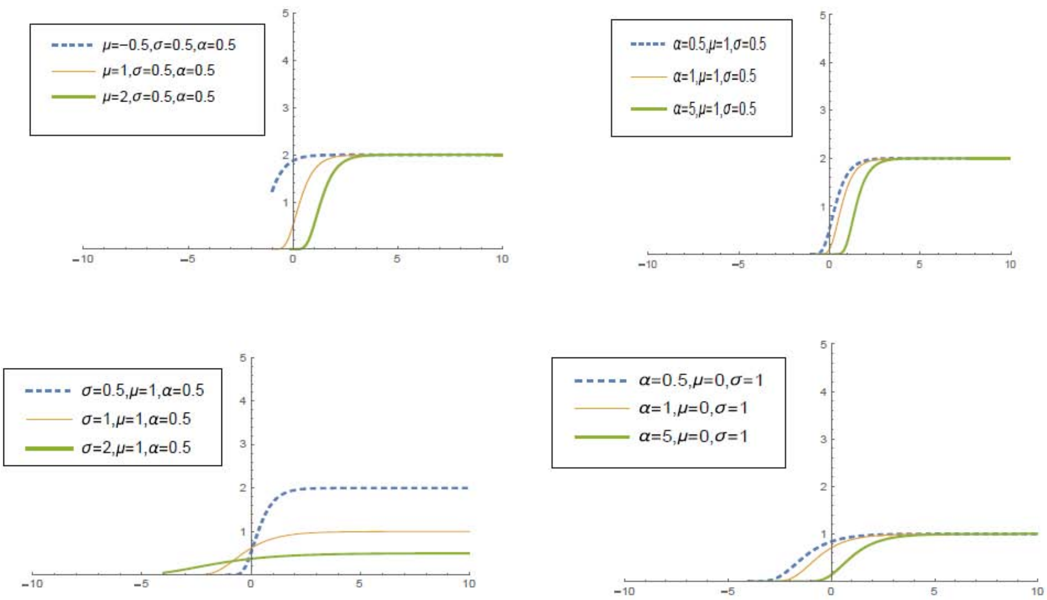

Fig. 2: Plots of the hazard function of TLMo distribution for some parameter values.

3 Properties of the TLMo Distribution

If X is a random variable with probability density function (pdf) (9), we write $X\ :TLMo(\alpha,\mu,\sigma)$. Without loss of generality, for simplicity, we will take $\mu=0, and\ \sigma=1$.

In this section, we discuss some statistical properties of the proposed distribution such as quantile function, mode, n^{th} moment, moment generating functions, mean deviation, incomplete moment, Lorenz, Bonferroni and Zenga curves, Rényi entropy and continuous entropy.

3.1 Quantile function

Theorem 1. *Let X be a random variable following $TLMo$ distribution and let $u \in (0,1)$. Then the quantile function is given by the approximate value*

$$x \approx \; ln\left(\frac{1}{2\left[erf^{-1}\left(\sqrt{\frac{1-u}{\alpha}}\right)\right]^2}\right) \tag{14}$$

Proof. Since $F(x)$ is continuous and strictly increasing, then the quantile function $x = F^{-1}(u),\; u \in (0,1)$ can be straightforward computed by inverting Equation (10) to obtain

$$F(x) = u$$

If $|z|<1$ and $\alpha>0$ is real non-integer, we have the series representation

$$(1-z)^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha}{k}(-z)^k \tag{15}$$

Using the expansion (15) in Equation (10) the cdf of TLMo is written as

$$F(x) = \sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{-1}{\pi}\gamma^2\left(\frac{1}{2},\frac{1}{2}e^{-x}\right)\right)^k = u \tag{16}$$

The summation on the left – hand side converges absolutely for $\left|\frac{1}{\pi}\gamma^2(\frac{1}{2},\frac{1}{2}e^{-x})\right| < 1$. Using the approximation technique, the second approximation is given as follows

$$1-\frac{\alpha}{\pi}\gamma^2\left(\frac{1}{2},\frac{1}{2}e^{-x}\right) \approx u \quad \text{or} \quad \gamma\left(\frac{1}{2},\frac{1}{2}e^{-x}\right) \approx \sqrt{\frac{\pi}{\alpha}(1-u)}$$

i.e.

$$\sqrt{\pi}erf(\frac{\sqrt{e^{-x}}}{\sqrt{2}}) \approx \sqrt{\frac{\pi}{\alpha}(1-u)}$$

Then

$$e^{-x} \approx 2\left(erf^{-1}\left(\sqrt{\frac{1-u}{\alpha}}\right)\right)^2$$

Hence,

$$x \approx -ln\left(2\left[erf^{-1}\left(\sqrt{\frac{1-u}{\alpha}}\right)\right]^2\right)$$

Therefore, an approximate quantile function of order u of the $TLMo$ distribution is the solution of Equation (14).
Putting $u = 0.5$ in Equation (14) we get the median of $TLMo$ distribution.
The TLMo distribution is easily simulated from $F(x)$ in Equation (8) using the approximate form of the quantile function in Equation (14).

3.2 Mode

One of the most important features for the distribution is the mode. The mode of the $TLMo$ is deduced by differentiating the pdf (11)and equating to zero.

i.e.

$$\frac{df(x)}{dx}=0=2\frac{\alpha}{\pi}\mathrm{e}^{-\mathrm{e}^{-x}-x}(\alpha-1)\left(erf\left[\frac{\sqrt{\mathrm{e}^{-x}}}{\sqrt{2}}\right]\right)^{2}\left(1-\left(erf\left[\frac{\sqrt{\mathrm{e}^{-x}}}{\sqrt{2}}\right]\right)^{2}\right)^{\alpha-2}$$
$$+\frac{\alpha}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\mathrm{e}^{-x}}{2}}\sqrt{\mathrm{e}^{-x}}\left\{\left(\mathrm{e}^{-x}-1\right)erf\left[\frac{\sqrt{\mathrm{e}^{-x}}}{\sqrt{2}}\right]-\frac{\sqrt{2}}{\sqrt{\pi}}\mathrm{e}^{-\frac{\mathrm{e}^{-x}}{2}-\frac{x}{2}}\right\}\left(1-\left(erf[\frac{\sqrt{\mathrm{e}^{-x}}}{\sqrt{2}}]\right)^{2}\right)^{\alpha-1}$$

However, we cannot obtain an explicit form so we calculate the mode numerically for different values of α.

Table 1: Mode for some chosen different values of α.

The values of α	Mode
α=0.1	−2.90202
α= 0.2	−2.13679
α= 0.3	−1.69495
α= 0.4	−1.39069
α= 0.5	−1.15927
α= 0.6	−0.972358
α= 0.7	−0.815446
α= 0.8	−0.680139
α= 0.9	−0.561155
α= 1	−0.454946
α= 2	0.240793
α= 5	1.15774
α= 10	1.85097
α= 20	2.54414
α= 50	3.46044
α= 100	4.15359

3.3 Moments

Theorem 2. *If X follows the $TLMo$ distribution given by the pdf (11), the n^{th} moment of X is given by*

$$\mu_n^{'}=2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{n}v_{r,m,n,k}(\alpha)\Gamma_r(m+k), \tag{17}$$

where

$$\left.\begin{array}{l} v_{r,m,n,k}(\alpha)=\binom{n}{r}\binom{\alpha}{k}(-1)^{k+n+1}\frac{k(ln(2))^{n-r}}{\pi^k}c_{m,2k-1}, \\ \text{and } c_{m,2k-1} \text{ is defined in detail in the following proof.}\end{array}\right\} \tag{18}$$

Proof. The n^{th} moment of the $TLMo$ distribution is given as follows

$$\mu_n'=E(x^n)=\int_{-\infty}^{\infty}x^n f(x)\,dx\,.$$

Using the series expansion (15) in Equation (11), we obtain the pdf of $TLMo$ as follows

$$f(x)=2\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(\frac{-1}{\pi}\right)^{k}k\frac{-e^{\frac{-e^{-x}}{2}}e^{\frac{-x}{2}}}{\sqrt{2}}\gamma^{2k-1}\left(\frac{1}{2},\frac{e^{-x}}{2}\right). \tag{19}$$

Consequently,

$$\mu_n' = 2\sum_{k=1}^{\infty}\binom{\alpha}{k}(-1)^k\frac{k}{\pi^k}\int_{-\infty}^{\infty}x^n\frac{-e^{\frac{-e^{-x}}{2}}e^{\frac{-x}{2}}}{\sqrt{2}}\gamma^{2k-1}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx.$$

Setting $u = \frac{e^{-x}}{2}$, μ_n' reduces to

$$\mu_n' = 2\sum_{k=1}^{\infty}\binom{\alpha}{k}(-1)^{k+n+1}\frac{k}{\pi^k}\int_0^{\infty}u^{\frac{-1}{2}}(ln(2u))^n e^{-u}\gamma^{2k-1}\left(\frac{1}{2},u\right)du.$$

Using the binomial expansion in the last equation, we can obtain

$$\mu_n' = 2\sum_{k=1}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\binom{\alpha}{k}(-1)^{k+n+1}\frac{k(ln(2))^{n-r}}{\pi^k}\int_0^{\infty}u^{\frac{-1}{2}}(ln(u))^r e^{-u}\gamma^{2k-1}\left(\frac{1}{2},u\right)du. \tag{20}$$

Let

$$I_{r,k} = \int_0^{\infty}u^{\frac{-1}{2}}(ln(u))^r e^{-u}\gamma^{2k-1}\left(\frac{1}{2},u\right)du.$$

Using the series expansion

$$\gamma(\alpha,x) = x^{\alpha}\sum_{m=0}^{\infty}\frac{(-x)^m}{(\alpha+m)m!,}$$

in the last equation ,we can obtain

$$I_{r,k} = \int_0^{\infty}u^{\frac{-1}{2}}(ln(u))^r e^{-u}\left[u^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-u)^m}{\left(\frac{1}{2}+m\right)m!}\right]^{2k-1}du.$$

Using the identity of a power series raised to an integer, namely

$$\left(\sum_{k=0}^{\infty}a_k x^k\right)^n = \sum_{k=0}^{\infty}c_{k,n}x^k$$

(see Gradshteyn and Ryzhik, [26], p.14 Section 0.314), where $c_{0,n} = a_0^n$ and

$$c_{k,n} = (ka_0)^{-1}\sum_{l=1}^{k}(nl-k+l)a_l c_{k-l,n}.$$

Hence,

$$I_{r,k} = \int_0^{\infty}u^{k-1}(ln(u))^r e^{-u}\sum_{m=0}^{\infty}c_{m,2k-1}u^m du,$$

where

$$c_{m,2k-1} = m^{-1}\sum_{l=1}^{m}\frac{(-1)^l(2kl-m)}{(2l+1)\,l!}c_{m-l,2k-1} \text{ for } m = 1,2,\ldots \text{ and } c_{0,2k-1} = 2^{2k-1}, k = 1,2,\ldots$$

Then Equation (20) can be written as follows

$$\mu_n' = 2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\binom{\alpha}{k}(-1)^{k+n+1}\frac{k(ln(2))^{n-r}}{\pi^k}c_{m,2k-1}J(r) \tag{21}$$

where

$$J(r) = \int_0^{\infty}u^{m+k-1}(ln(u))^r e^{-u}du.$$

This integral $J(r)$ in (21) can be calculated from the result given by Gradshteyn and Ryzhik, [26], (p. 578 Section 4.358 integral 5). From the definition of

$$\Gamma_r(p) = \frac{\partial^r\Gamma(p)}{\partial p^r},$$

we have

$$J(r) = \int_0^{\infty}u^{m+k-1}(ln(u))^r e^{-u}du = \Gamma_r(m+k).$$

This yields the n^{th} moment given in Equation (17).
Putting $n = 1$ in Equation (17), we easily obtain the mean of TLMo distribution.

3.4 Moment generating function

Theorem 3. *If X follows the TLMo distribution given by the pdf (11), the moment generating function (mgf) of X is given by*

$$\left.\begin{array}{l} M_x(t) = 2^{1-t}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} w_{k,m}(\alpha)\,\Gamma(m+k-t), \\ \quad where \quad w_{k,m}(\alpha) = \binom{\alpha}{k}\frac{k}{\pi^k}(-1)^{k+1}c_{m,2k-1}, \\ \quad and\, c_{m,2k-1}\ is\, defined\, in\, detail\, in\, the\, proof\, of\, Theorem\, 2\ Section\,(3.3) \end{array}\right\} \tag{22}$$

Proof. The mgf of $TLMo$ distribution is defined by

$$M_x(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Using Equation (19), the mgf of TLMo distribution reduces to

$$M_x(t) = 2\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k}{\pi^k}(-1)^k\int_{-\infty}^{\infty} e^{tx}\frac{-e^{\frac{-e^{-x}}{2}}e^{\frac{-x}{2}}}{\sqrt{2}}\gamma^{2k-1}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx$$

Again setting $u = \frac{e^{-x}}{2}$, we have

$$M_x(t) = 2^{1-t}\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k}{\pi^k}(-1)^{k+1}\int_0^{\infty} u^{-t-\frac{1}{2}}e^{-u}\gamma^{2k-1}\left(\frac{1}{2},u\right)du$$

Let $A_{t,k} = \int_0^{\infty} u^{-t-\frac{1}{2}}e^{-u}\gamma^{2k-1}\left(\frac{1}{2},u\right)du$
Following similar steps of Theorem 2, $M_x(t)$ takes the form

$$M_x(t) = 2^{1-t}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\binom{\alpha}{k}\frac{k}{\pi^k}(-1)^{k+1}c_{m,2k-1}\int_0^{\infty} u^{m+k-t-1}e^{-u}du$$

which yields the result (22).

3.5 Mean deviations

Theorem 4. *Let X follow TLMo distribution given by the pdf (11). The mean deviation of X about the mean μ_1' and the median M are defined, respectively, by*

$$\delta_1(x) = 2\mu_1' F\left(\mu_1'\right) - 2T\left(\mu_1'\right)$$

and

$$\delta_2(x) = \mu_1' - 2T(M)$$

where $T(q)$ is given by

$$T(q) = 2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\Gamma_r\left(m+k,\frac{e^{-q}}{2}\right), \tag{23}$$

and $v_{r,m,1,k}$ is defined previously in Equation (18) for $n = 1$.

Proof. The mean deviations of X about the mean and the median are given by

$$\delta_1(x) = \int_{-\infty}^{\infty}\left|x-\mu_1'\right| f(x)\,dx = 2\mu_1' F\left(\mu_1'\right) - 2T\left(\mu_1'\right)$$

and

$$\delta_2(x) = \int_{-\infty}^{\infty}|x-M|\,f(x)\,dx = \mu_1' - 2T(M)\ ,$$

where $F(M) = \frac{1}{2}$, $F\left(\mu_1'\right)$ can be easily calculated from Equation (10) and $T(q) = \int_{-\infty}^{q} xf(x)\,dx$.

Using Equation (19), we write $T(q)$ in the following form

$$T(q)=2\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k}{\pi^k}(-1)^k\int_{-\infty}^{q}x\frac{-e^{\frac{-e^{-x}}{2}}e^{\frac{-x}{2}}}{\sqrt{2}}\gamma^{2k-1}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx.$$

Again setting $u=\frac{e^{-x}}{2}$, we have

$$T(q)=2\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k}{\pi^k}(-1)^k\int_{\frac{e^{-q}}{2}}^{\infty}u^{\frac{-1}{2}}ln(2u)\,e^{-u}\gamma^{2k-1}\left(\frac{1}{2},u\right)du.$$

Following similar steps of Theorem 2, $T(q)$ takes the form

$$T(q)=2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1}\binom{\alpha}{k}\binom{1}{r}(-1)^k\frac{k(ln(2))^{1-r}}{\pi^k}c_{m,2k-1}\int_{\frac{e^{-q}}{2}}^{\infty}u^{m+k-1}(ln(u))^{r}e^{-u}du.$$

From the last equation

$$\int_{\frac{e^{-q}}{2}}^{\infty}u^{m+k-1}(ln(u))^{r}e^{-u}du=\frac{\partial^r}{\partial(m+k)^r}\int_{\frac{e^{-q}}{2}}^{\infty}u^{m+k-1}e^{-u}du=\frac{\partial^r}{\partial(m+k)^r}\Gamma\left(m+k,\frac{e^{-q}}{2}\right)=\Gamma_r(m+k,\frac{e^{-q}}{2}).$$

This yields the $T(q)$ given in Equation (23).
Hence, the measures $\delta_1(x)$ and $\delta_2(x)$ are immediately defined from (23).

*Remark.*Regarding the paper " The Beta Moyal: A Useful Skew Distribution" by Cordeiro et al. [10], the authors need to point out that in Equation (20) calculated by Maple, the integral in $T(q)$ p.179 defined by $\int_{\frac{1}{2}e^{-q}}^{\infty}u^{m+r+\frac{k+1}{2}-1}log(u)e^{-u}du$ gives the solution

$$\left[1+\left(m+r+\frac{k+1}{2}\right)(q-ln(1/2))\right]\left(m+r+\frac{k+1}{2}\right)^{-2}\left(\frac{1}{2}e^{-q}\right)^{m+r+\frac{k+1}{2}}$$

under the condition $\left(m+r+\frac{k+1}{2}\right)<0$ which is stated in a wrong form neglecting the condition which is a must.

3.6 Incomplete moments

Theorem 5.*If X follows the TLMo distribution defined in Equation (11), the n^{th} incomplete moment is given by*

$$m_n(z)=2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{n}v_{r,m,n,k}(\alpha)\Gamma_r\left(m+k,\frac{e^{-z}}{2}\right),\tag{24}$$

where $v_{r,m,n,k}(\alpha)$ is defined in Equation (18).

*Proof.*The n^{th} incomplete moments denoted as $m_n(z)$ can be obtained as follows:

$$m_n(z)=\int_{-\infty}^{z}x^n f(x)dx$$

From Equation (19), we have

$$m_n(z)=2\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k}{\pi^k}(-1)^k\int_{-\infty}^{z}x^n\frac{-e^{\frac{-e^{-x}}{2}}e^{\frac{-x}{2}}}{\sqrt{2}}\gamma^{2k-1}\left(\frac{1}{2},\frac{e^{-x}}{2}\right)dx.$$

Using the transformation $u=\frac{e^{-x}}{2}$, we have

$$m_n(z)=2\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k}{\pi^k}(-1)^{k+n+1}\int_{\frac{e^{-z}}{2}}^{\infty}u^{\frac{-1}{2}}(ln(2u))^{n}e^{-u}\gamma^{2k-1}\left(\frac{1}{2},u\right)du.$$

Following similar steps of Theorem 2, $m_n(z)$ takes the form

$$m_n(z)=2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\binom{\alpha}{k}(-1)^{k+n+1}\frac{k(ln(2))^{n-r}}{\pi^k}c_{m,2k-1}\int_{\frac{e^{-z}}{2}}^{\infty}u^{m+k-1}(ln(u))^{r}e^{-u}du.$$

which is similar to the integral in Section 3.5
Therefore, this reduces to the result in Equation (24).

3.7 Lorenz, Bonferroni and Zenga curves

The Lorenz, Bonferroni and Zenga curves are important applications for the first incomplete moments. These curves are useful in many fields such as insurance, medicine, demography, reliability and economics.

The Lorenz, Bonferroni and Zenga curves are obtained, respectively, as follows:

$$L(F(z)) = \frac{1}{E(z)} \int_{-\infty}^{z} xf(x)\,dx = \frac{m_1(z)}{\mu_1'}, \qquad B(F(z)) = \frac{L(F(z))}{F(z)}$$

and

$$A(z) = 1 - \frac{M^-(z)}{M^+(z)}$$

where

$$M^-(z) = \frac{1}{F(z)} \int_{-\infty}^{z} xf(x)\,dx\,, \quad M^+(z) = \frac{1}{1-F(z)} \int_{z}^{\infty} xf(x)\,dx$$

Therefore, using Equation (17) and (24), we obtain the Lorenz curve as follows

$$L(F(z)) = \frac{\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\Gamma_r\left(m+k\,,\,\frac{e^{-z}}{2}\right)}{\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\Gamma_r(m+k)}, \tag{25}$$

From Equations (25) and (10), we write the Bonferroni curve as

$$B(F(z)) = \frac{\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\Gamma_r\left(m+k,\frac{e^{-z}}{2}\right)}{\left[1-\frac{\gamma^2\left(\frac{1}{2}\,,\,\frac{e^{-z}}{2}\right)}{\pi}\right]^{\alpha}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\Gamma_r(m+k)}\,, \tag{26}$$

Hence, the Zenga curve can be obtained as follows

$$A(z) = 1 - \left\{\left(\left[1-\frac{\gamma^2\left(\frac{1}{2}\,,\,\frac{e^{-z}}{2}\right)}{\pi}\right]^{-\alpha} - 1\right)\frac{\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\Gamma_r\left(m+k,\frac{e^{-z}}{2}\right)}{\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\gamma_r\left(m+k,\frac{e^{-z}}{2}\right)}\right\}, \tag{27}$$

where

$$M^-(z) = \frac{2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\Gamma_r\left(m+k,\frac{e^{-z}}{2}\right)}{\left[1-\frac{\gamma^2\left(\frac{1}{2}\,,\,\frac{e^{-z}}{2}\right)}{\pi}\right]^{\alpha}}\,,$$

$$M^+(z) = \frac{2\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{1} v_{r,m,1,k}(\alpha)\,\gamma_r\left(m+k,\frac{e^{-z}}{2}\right)}{\left\{1-\left[1-\frac{\gamma^2\left(\frac{1}{2}\,,\,\frac{e^{-z}}{2}\right)}{\pi}\right]^{\alpha}\right\}}\,,$$

and

$$\gamma_r\left(m+k,\frac{e^{-z}}{2}\right) = \frac{\partial^r}{\partial(m+k)^r}\gamma\left(m+k,\frac{e^{-z}}{2}\right) = \frac{\partial^r}{\partial(m+k)^r}\int_0^{\frac{e^{-z}}{2}} u^{m+k-1}e^{-u}du = \int_0^{\frac{e^{-z}}{2}} u^{m+k-1}(ln(u))^{\,r}e^{-u}du.$$

3.8 Rényi entropy

The Rényi entropy has various applications in different areas such as statistics and ecology as an indication of diversity. In quantum information, it can be used as a measure of entanglement. The Rényi entropy of order ξ, where $\xi \geq 0,\ and\ \xi \neq 1$, is defined as

$$J_R(\xi) = \frac{1}{1-\xi} log(I(\xi))\,,$$

where

$$I(\xi) = \int f^{\xi}(x)\,dx\,.$$

Using this notion, we deduce the Rényi entropy of a random variable following the TLMo pdf (11), in Theorem 6.

Theorem 6.*Let X be a continuous random variable following the TLMo distribution given by Equation (11). The Rényi entropy of X is given by*

$$J_R(\xi) = (1-\xi)^{-1}\left\{\xi log\left(\frac{2\alpha}{\pi}\right) + log\left[\sum_{k,j,m=0}^{\infty}\sum_{r=0}^{j}\binom{\xi(\alpha-1)}{k}\binom{\xi+2k}{j}\binom{j}{r}c_{m,r}\frac{(-1)^{k+j+r}}{\pi^k}\xi^{-\left(m+\frac{\xi+r}{2}\right)}\Gamma(m+\frac{\xi+r}{2})\right]\right\} \tag{28}$$

*Proof.*Substituting the pdf of TLMo (11) in the definition of Rényi entropy given above, we have

$$I(\xi) = 2^{\frac{\xi}{2}}\left(\frac{\alpha}{\pi}\right)^{\xi}\int_{-\infty}^{\infty} e^{-\frac{\xi}{2}(e^{-x}+x)}\gamma^{\xi}\left(\frac{1}{2},\ \frac{e^{-x}}{2}\right)\left\{1-\frac{\gamma^2\left(\frac{1}{2},\ \frac{e^{-x}}{2}\right)}{\pi}\right\}^{\xi(\alpha-1)} dx$$

Using the series expansion (15) in the last equation, we obtain

$$I(\xi) = 2^{\frac{\xi}{2}}\left(\frac{\alpha}{\pi}\right)^{\xi}\sum_{k=0}^{\infty}\binom{\xi(\alpha-1)}{k}\left(\frac{-1}{\pi}\right)^{k}\int_{-\infty}^{\infty} e^{-\frac{\xi}{2}(e^{-x}+x)}\gamma^{\xi+2k}\left(\frac{1}{2},\ \frac{e^{-x}}{2}\right)dx$$

Again using the expansion (15) and then applying the binomial expansion, we obtain

$$I(\xi) = 2^{\frac{\xi}{2}}\left(\frac{\alpha}{\pi}\right)^{\xi}\sum_{k,j=0}^{\infty}\sum_{r=0}^{j}\binom{\xi(\alpha-1)}{k}\binom{\xi+2k}{j}\binom{j}{r}\frac{(-1)^{k+j+r}}{\pi^k}\int_{-\infty}^{\infty} e^{-\frac{\xi}{2}(e^{-x}+x)}\gamma^{r}\left(\frac{1}{2},\ \frac{e^{-x}}{2}\right)dx$$

Using the transformation $u = \frac{e^{-x}}{2}$, we have

$$I(\xi) = \left(\frac{2\alpha}{\pi}\right)^{\xi}\sum_{k,j=0}^{\infty}\sum_{r=0}^{j}\binom{\xi(\alpha-1)}{k}\binom{\xi+2k}{j}\binom{j}{r}\frac{(-1)^{k+j+r}}{\pi^k}\int_{0}^{\infty} u^{\frac{\xi}{2}-1}e^{-\xi u}\gamma^{r}\left(\frac{1}{2},u\right)du$$

Following similar steps of Theorem 2, we have

$$I(\xi) = \left(\frac{2\alpha}{\pi}\right)^{\xi}\sum_{k,j,m=0}^{\infty}\sum_{r=0}^{j}\binom{\xi(\alpha-1)}{k}\binom{\xi+2k}{j}\binom{j}{r}c_{m,r}\frac{(-1)^{k+j+r}}{\pi^k}\int_{0}^{\infty} u^{m+\frac{\xi+r}{2}-1}e^{-\xi u}du$$

where $c_{m,r}$ is defined in Section 3.3. The integral in the last equation can be easily calculated. Hence,

$$I(\xi) = \left(\frac{2\alpha}{\pi}\right)^{\xi}\sum_{k,j,m=0}^{\infty}\sum_{r=0}^{j}\binom{\xi(\alpha-1)}{k}\binom{\xi+2k}{j}\binom{j}{r}c_{m,r}\frac{(-1)^{k+j+r}}{\pi^k}\xi^{-\left(m+\frac{\xi+r}{2}\right)}\Gamma(m+\frac{\xi+r}{2})$$

Finally, the Rényi entropy can be expressed as in Equation (28).

3.9 Continuous entropy

various information theory 's key results and principles can be extended using the Continuous Entropy defined by Marsh [27] based on the following

$$h(x) = \int -ln(f(x))\,f(x)\,dx$$

Theorem 7.*Let X be a continuous random variable following the TLMo distribution given by Equation (11). The Continuous entropy of X is given by*

$$h(x) = \frac{1}{2}\left(M_x(-1)+\mu_1'\right) - \left[\frac{1}{2}ln\left(\frac{2}{\pi}\right) + ln(\alpha)\right] - \sum_{k=0}^{\infty}\frac{\binom{\alpha}{k+1}(-1)^{k+1}}{2k+2} + \frac{\alpha-1}{\alpha} \tag{29}$$

*Proof.*Substituting the pdf of TLMo (11) in the definition of the continuous entropy given above, we have

$$h(x)=\frac{1}{2}\,I_1+\frac{1}{2}\,I_2-\frac{1}{2}ln\left(\frac{2}{\pi}\right)I_3-ln(\alpha)\;I_3-I_4\;-(\alpha-1)\;I_5 \tag{30}$$

$$I_1=\int_{-\infty}^{\infty}e^{-x}f(x)dx=M_x(-1) \tag{i}$$

$$I_2=\int_{-\infty}^{\infty}xf(x)dx=\mu_1' \tag{ii}$$

$$I_3=\int_{-\infty}^{\infty}f(x)dx=1 \tag{iii}$$

Also,

$$I_4=\int_{-\infty}^{\infty}ln\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]f(x)\;dx$$

$$=\int_{-\infty}^{\infty}ln\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]\;\mathrm{e}^{-\frac{1}{2}\mathrm{e}^{-x}}\sqrt{\mathrm{e}^{-x}}\sqrt{\frac{2}{\pi}}\alpha erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\;\left\{1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2\right\}^{\alpha-1}\;dx$$

Take the transformation
Let $v=erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)$, we have

$$\int_{-\infty}^{\infty}ln\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]f(x)\;dx\;=-2\alpha\int_{1}^{0}ln(v)\;v\,(1-v^2)^{\alpha-1}dv$$

$$=-2\alpha\sum_{k=0}^{\infty}\binom{\alpha-1}{k}(-1)^k\int_{1}^{0}ln(v)\;v^{2k+1}dv\,,$$

Integration by parts yields

$$I_4=\int_{-\infty}^{\infty}ln\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]f(x)\;dx\;=\sum_{k=0}^{\infty}\frac{\binom{\alpha}{k+1}(-1)^{k+1}}{2k+2} \tag{iv}$$

Now the last integral will give

$$I_5=\int_{-\infty}^{\infty}ln\left(1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2\right)f(x)\,dx$$

$$=\int_{-\infty}^{\infty}ln\left(1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2\right)\mathrm{e}^{-\frac{1}{2}\mathrm{e}^{-x}}\sqrt{\mathrm{e}^{-x}}\sqrt{\frac{2}{\pi}}\alpha erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\;\left\{1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2\right\}^{\alpha-1}\;dx$$

Let

$$z=1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2$$

$$\int_{-\infty}^{\infty}ln\left(1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2\right)f(x)\,dx=\alpha\int_{0}^{1}ln(z)\;z^{\alpha-1}dz$$

Therefore,

$$I_5=\int_{-\infty}^{\infty}ln\left(1-\left[erf\left(\frac{\sqrt{e^{-x}}}{\sqrt{2}}\right)\right]^2\right)f(x)\,dx=-\frac{1}{\alpha} \tag{v}$$

Substituting (i), (ii), (iii), (iv) and (v) in Equation (30), the Continuous entropy can be expressed as the result (29).

4 Order Statistics

Order statistics are very important in probability and statistics. Let $X_{1:m} \leq X_{2:m} \leq \cdots \leq X_{m:m}$ be the ordered sample of size m from a continuous population with $pdf\ f(x)$ and $cdf\ F(x)$. The pdf of $X_{k:m}$, the k^{th} order statistic is given by

$$f_{X_{k:m}}(x) = \frac{m!}{(k-1)!\,(m-k)!} f(x)\,[F(x)]^{k-1}[1-F(x)]^{m-k} \quad ;k=1,2,\ldots,m.$$

Then , the pdf of the k^{th}order $TLMo$ random variable $X_{k:m}$can be obtained using Equations (10) and (11) in the last equation to give

$$f_{X_{k:m}}(x) = \frac{m!\,\sqrt{2}\alpha}{(k-1)!\,(m-k)!\pi} \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)} \gamma\left(\frac{1}{2}\,,\,\frac{e^{-x}}{2}\right) \left(1-\frac{\gamma^2\left(\frac{1}{2}\,,\,\frac{e^{-x}}{2}\right)}{\pi}\right)^{\alpha k-1} \left(1-\left[1-\frac{\gamma^2\left(\frac{1}{2}\,,\,\frac{e^{-x}}{2}\right)}{\pi}\right]^{\alpha}\right)^{m-k}$$

Using the binomial expansion, we obtain

$$f_{X_{k:m}}(x) = \frac{m!\,\sqrt{2}\alpha}{(k-1)!\,(m-k)!\pi} \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)} \gamma\left(\frac{1}{2}\,,\,\frac{e^{-x}}{2}\right) \sum_{i=0}^{\infty} \binom{m-k}{i} (-1)^i \left[1-\frac{\gamma^2\left(\frac{1}{2}\,,\,\frac{e^{-x}}{2}\right)}{\pi}\right]^{\alpha(k+i)-1}$$

The series expansion (15) yields the pdf the k^{th} order $TLMo$ random variable $X_{k:m}$ as follows

$$f_{X_{k:m}}(x) = \frac{m!\,\sqrt{2}\alpha}{(k-1)!\,(m-k)!} \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)} \sum_{i,j=0}^{\infty} \binom{m-k}{i} \binom{\alpha(k+i)-1}{j} \frac{(-1)^{i+j}}{\pi^{j+1}} \gamma^{2j+1}\left(\frac{1}{2}\,,\,\frac{e^{-x}}{2}\right) \tag{31}$$

Also, the n^{th} moment for the k^{th} order statistic with pdf $f_{X_{k:m}}(x)$ is given by

$$\mu_{k:m}^{(n)} = \int_{-\infty}^{\infty} x^n\, f_{k:m}(x)dx$$

Then , the n^{th} moment for the k^{th} order $TLMo$ random variable $X_{k:m}$ can be obtained using Equation (31)

$$\mu_{k:m}^{(n)} = \int_{-\infty}^{\infty} x^n \frac{m!\,\sqrt{2}\alpha\, \mathrm{e}^{-\frac{1}{2}(\mathrm{e}^{-x}+x)}}{(k-1)!\,(m-k)!} \sum_{i,j=0}^{\infty} \binom{m-k}{i} \binom{\alpha(k+i)-1}{j} \frac{(-1)^{i+j}}{\pi^{j+1}} \gamma^{2j+1}\left(\frac{1}{2}\,,\,\frac{e^{-x}}{2}\right) dx$$

Again, using $u = \frac{e^{-x}}{2}$, we have

$$\mu_{k:m}^{(n)} = 2\frac{m!\alpha}{(k-1)!\,(m-k)!} \sum_{i,j=0}^{\infty} \binom{m-k}{i} \binom{\alpha(k+i)-1}{j} \frac{(-1)^{i+j+n}}{\pi^{j+1}} \int_0^{\infty} [ln(2u)]^n u^{\frac{-1}{2}} e^{-u} \gamma^{2j+1}(\frac{1}{2},u)du$$

which yields

$$\mu_{k:m}^{(n)} = 2\frac{m!\alpha}{(k-1)!\,(m-k)!} \sum_{i,j=0}^{\infty} \sum_{r=0}^{n} \binom{m-k}{i} \binom{\alpha(k+i)-1}{j} \binom{n}{r} \frac{(-1)^{i+j+n}}{\pi^{j+1}} [ln(2)\,]^{n-r} \int_0^{\infty} u^{\frac{-1}{2}} e^{-u} [ln(u)\,]^r \gamma^{2j+1}\left(\frac{1}{2},u\right) du \tag{32}$$

Set

$$B_{r,j} = \int_0^{\infty} u^{\frac{-1}{2}} e^{-u} [ln(u)\,]^r \gamma^{2j+1}(\frac{1}{2},u)du$$

Using the series expansion

$$\gamma(\alpha,x) = x^{\alpha} \sum_{m=0}^{\infty} \frac{(-x)^m}{(\alpha+m)\,m!}$$

and the identity of a power series raised to an integer, namely

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_{k,n} x^k$$

in the last equation, we have

$$B_{r,j}=\sum_{s=0}^{\infty}c_{s,2j+1}\int_{0}^{\infty}u^{j+s}e^{-u}[ln(u)]^{r}du$$

where $c_{s,2j+1}$ is defined in Section 3.3. The integral in $B_{r,j}$ can be calculated from the result given by Gradshteyn and Ryzhik. [26], (p. 578, Section 4.358, integral 5). From the definition of $\Gamma_r(p)=\frac{\partial^r\Gamma(p)}{\partial p^r}$, we have

$$B_{r,j}=\sum_{s=0}^{\infty}c_{s,2j+1}\Gamma_r(j+s+1)$$

This yields the n^{th} moment of $X_{k:m}$ given by

$$\mu_{k:m}^{(n)}=2\frac{m!\alpha}{(k-1)!(m-k)!}\sum_{i,j,s=0}^{\infty}\sum_{r=0}^{n}\binom{m-k}{i}\binom{\alpha(k+i)-1}{j}\binom{n}{r}\frac{(-1)^{i+j+n}}{\pi^{j+1}}[ln(2)]^{n-r}c_{s,2j+1}\Gamma_r(j+s+1) \tag{33}$$

5 Estimation of Parameters

In this section, we derive the maximum likelihood estimators (MLEs) and the observed information matrix of the $TLMo$ distribution. Suppose $X_1,X_2,\ldots,X_n$ be a random sample from the $TLMo$ distribution, then the log-likelihood function is given by

$$\begin{aligned} l=&n\left[\left(ln(\alpha)+\frac{1}{2}ln(2)-ln(\sigma)-\frac{1}{2}ln(\pi)\right)\right]-\frac{1}{2}\sum_{i=1}^{n}\mathrm{e}^{-\left(\frac{x_i-\mu}{\sigma}\right)}-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)\\ &+\sum_{i=1}^{n}ln\left[erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]+(\alpha-1)\sum_{i=1}^{n}ln\left(1-\left[erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^2\right)\end{aligned} \tag{34}$$

Then

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}ln\left(1-\left[erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^2\right) \tag{35}$$

$$\frac{\partial l}{\partial\mu}=\frac{n}{2\sigma}-\frac{1}{2}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{\sigma}+\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2\pi}\sigma erf\left[\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right]}-(\alpha-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{\frac{2}{\pi}}erf\left[\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right]}{\sigma\left(1-\left[erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^2\right)} \tag{36}$$

$$\begin{aligned}\frac{\partial l}{\partial\sigma}=&-\frac{n}{\sigma}+\frac{1}{2}\sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^2}-\frac{1}{2}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}(x_i-\mu)}{\sigma^2}+\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}(x_i-\mu)}{\sqrt{2\pi}\sigma^2 erf\left[\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right]}\\ &-(\alpha-1)\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\sqrt{\frac{2}{\pi}}erf\left[\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right](x_i-\mu)}{\sigma^2\left(1-\left[erf\left(\frac{\sqrt{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}}{\sqrt{2}}\right)\right]^2\right)}\end{aligned} \tag{37}$$

The MLEs $(\widehat{\alpha},\widehat{\mu},\widehat{\sigma})$ of the parameters (α,μ,σ) are obtained by solving the system of nonlinear equations (35) ,(36) and (37). These equations cannot be solved analytically, but they require using numerical techniques such as Newton-Raphson method.

For interval estimation and testing of hypotheses of the parameters (α,μ,σ) , we require the 3×3 unit observed information matrix

$$J(\psi)=\begin{pmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\mu} & \frac{\partial^2 l}{\partial\alpha\partial\sigma} \\ \frac{\partial^2 l}{\partial\mu\partial\alpha} & \frac{\partial^2 l}{\partial\mu^2} & \frac{\partial^2 l}{\partial\mu\partial\sigma} \\ \frac{\partial^2 l}{\partial\sigma\partial\alpha} & \frac{\partial^2 l}{\partial\sigma\partial\mu} & \frac{\partial^2 l}{\partial\sigma^2} \end{pmatrix}$$

whose elements are given by the following.

$$\frac{\partial^2 l}{\partial\alpha^2}=-\frac{n}{\alpha}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\mu}=\frac{\partial^2 l}{\partial\mu\partial\alpha}=-\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}\sqrt{\frac{2}{\pi}}B}{\sigma\left(1-[B]^2\right)}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\sigma}=\frac{\partial^2 l}{\partial\sigma\partial\alpha}=-\sum_{i=1}^{n}\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}\sqrt{\frac{2}{\pi}}B(x_i-\mu)}{\sigma^2\left(1-[B]^2\right)}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\mu^2}=&-\frac{1}{2}\sum_{i=1}^{n}\frac{A}{\sigma}+\sum_{i=1}^{n}\left(-\frac{\mathrm{e}^{-A}A}{2\pi\sigma^2[B]^2}+\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}(1-A)}{2\sqrt{2\pi}\sigma^2 B}\right)\\ &+(\alpha-1)\sum_{i=1}^{n}\left(-\frac{2\mathrm{e}^{-A}A[B]^2}{\pi\sigma^2\left(1-[B]^2\right)^2}-\frac{\mathrm{e}^{-A}A}{\pi\sigma^2\left(1-[B]^2\right)}-\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}B(1-A)}{\sqrt{2\pi}\sigma^2\left(1-[B]^2\right)}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\mu\partial\sigma}=\frac{\partial^2 l}{\partial\sigma\partial\mu}=&-\frac{n}{2\sigma^2}-\frac{1}{2}\sum_{i=1}^{n}\left(-\frac{A}{\sigma^2}+\frac{A\,(x_i-\mu)}{\sigma^3}\right)+\sum_{i=1}^{n}\left(\frac{-\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}}{\sqrt{2\pi}\sigma^2 B}-\frac{\mathrm{e}^{-A}A\,(x_i-\mu)}{2\pi\sigma^3 B}+\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}\,(x_i-\mu)\,(1-A)}{2\sqrt{2\pi}\sigma^3 B}\right)\\ &+(\alpha-1)\sum_{i=1}^{n}\left(\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}\sqrt{\frac{2}{\pi}}B}{\sigma^2\left(1-[B]^2\right)}-\frac{2\mathrm{e}^{-A}A[B]^2\,(x_i-\mu)}{\pi\sigma^3\left(1-[B]^2\right)^2}-\frac{\mathrm{e}^{-A}A\,(x_i-\mu)}{\pi\sigma^3\left(1-[B]^2\right)}-\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}B\,(x_i-\mu)\,(1-A)}{\sqrt{2\pi}\sigma^3\left(1-[B]^2\right)}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\sigma^2}=&\frac{n}{\sigma^2}-\sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma^3}-\frac{1}{2}\sum_{i=1}^{n}\left(-\frac{2A\,(x_i-\mu)}{\sigma^3}+\frac{A(x_i-\mu)^2}{\sigma^4}\right)\\ &+\sum_{i=1}^{n}\left(\frac{-\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}\sqrt{\frac{2}{\pi}}\,(x_i-\mu)}{\sigma^2 B}-\frac{\mathrm{e}^{-A}A(x_i-\mu)^2}{2\pi\sigma^4[B]^2}+\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}(x_i-\mu)^2(1-A)}{2\sqrt{2\pi}\sigma^4 B}\right)\\ &+(\alpha-1)\sum_{i=1}^{n}\left(\frac{2\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}\sqrt{\frac{2}{\pi}}B\,(x_i-\mu)}{\sigma^3\left(1-[B]^2\right)}-\frac{2\mathrm{e}^{-A}A[B]^2(x_i-\mu)^2}{\pi\sigma^4\left(1-[B]^2\right)^2}-\frac{\mathrm{e}^{-A}A(x_i-\mu)^2}{\pi\sigma^4\left(1-[B]^2\right)}-\frac{\mathrm{e}^{-\frac{1}{2}A}\sqrt{A}\,B(x_i-\mu)^2(1-A)}{\sqrt{2\pi}\sigma^4\left(1-[B]^2\right)}\right)\end{aligned}$$

where

$$A=e^{-(\frac{x_i-\mu}{\sigma})},\qquad B=erf\left(\frac{\sqrt{e^{-(\frac{x_i-\mu}{\sigma})}}}{\sqrt{2}}\right)$$

6 Applications

In this section, we use three real data sets. The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients given by Lee and Wang [28] as: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The second data set consists of 63 observations of the gauge lengths of 10 mm from Kundu & Raqab.[29], given as follows: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139,3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435,3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225,4.395, 5.020.

The third data set consists of 106 observations representing that in addition to physical, chemical and biological factors Some widely used pesticides are also harmful to humans. One of the commonly used pesticides is clorpirifos, which harms some animals as well. A study conducted in Chile by Dra. Fernanda Cavieres of the University of Valparaíso, Chile, established that clorpirifos likely causes congenital malformations, which can be avoided by folic acid. One of the variables that was measured in this study was the fetal height of the mouse (in millimeters). The data are :3.4, 3.4, 3.5, 3.5, 3.5, 3.5, 3.5, 3.6, 3.6, 3.6, 3.6, 3.6, 3.7 , 3.7, 3.7, 3.7, 3.7, 3.8, 3.8, 3.8, 3.8, 3.8, 3.8, 3.8, 3.8, 3.8, 3.8, 3.8, 3.9, 3.9, 3.9, 3.9, 3.9, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.0, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.1, 4.2, 4.2, 4.2, 4.2, 4.2, 4.2, 4.2, 4.2, 4.2, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.3, 4.4, 4.4, 4.4, 4.5, 4.5, 4.5, 4.5, 4.5, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 4.6, 4.7, 4.7, 4.7, 4.7, 4.8. This data was used by N. Balakrishnan et al. [30].

We use these data sets to compare the fit of the new distribution, TLMo distribution with Beta Moyal distribution (BMo) and Moyal distribution (Mo). First, we obtain the maximum likelihood estimates (MLEs) for the unknown parameters of each distribution and then compare the results of different criteria , such as AIC (Akaike information criterion), AICC (corrected Akaike information criterion), CAIC (consistent Akaike information criterion) and BIC (Bayesian information criterion), where,

$$AIC = 2K - 2l$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$CAIC = \frac{2kn}{n-k-1} - 2l,$$

$$BIC = k\log n - 2l,$$

l denotes the log – likelihood function evaluated at MLEs, k is the number of parameters and n is the sample size.

The best model corresponds to the lowest AIC, AICC, CAIC and BIC values

Table 2: MLEs for TLMo, BMo, Mo models and the statistics AIC, AICC, CAIC, BIC for the first data set

Model	$\widehat{\alpha}$	$\hat{a}$	$\hat{b}$	$\widehat{\mu}$	$\widehat{\sigma}$	$-\ell$	AIC	AICC	CAIC	BIC
$TLMo$	18.9858	—-	—-	-7.86458	5.41925	416.046	838.092	838.2855	838.2855	838.4136
BMo	—-	1.71863	2.46764	-4.82708	8.5297	421.183	850.366	850.691	850.691	850.7948
Mo	—-	—-	—-	4.72749	3.20053	423.276	850.552	850.648	850.648	850.7664

Table 3: MLEs for TLMo, BMo, Mo models and the statistics AIC, AICC, CAIC, BIC for the second data set

Model	$\widehat{\alpha}$	$\hat{a}$	$\hat{b}$	$\widehat{\mu}$	$\widehat{\sigma}$	$-\ell$	AIC	AICC	CAIC	BIC
$TLMo$	0.749306	—-	—-	3.15792	0.498941	56.4824	118.9648	119.372	119.372	118.363
BMo	—-	0.580778	0.500737	-6.8271	2.86742	242.637	493.274	493.96	493.96	492.471
Mo	—-	—-	—-	2.66284	0.332443	1743.96	3491.92	3492.12	3492.12	3491.52

Table 4: MLEs for TLMo, BMo, Mo models and the statistics AIC, AICC, CAIC, BIC for the third data set

Model	$\widehat{\alpha}$	$\hat{a}$	$\hat{b}$	$\widehat{\mu}$	$\widehat{\sigma}$	$-\ell$	AIC	AICC	CAIC	BIC
TLMo	4.72753	—-	—-	3.57965	0.304893	37.263	80.526	80.761	80.761	80.601
BMo	—-	0.402933	0.676748	−9.59398	6.62878	487.836	983.672	984.068	984.068	983.773
Mo	—-	—-	—-	3.49064	0.0134665	2789	5582	5582.116	5882.116	5882.05

7 Conclusion

The present paper introduces a new distribution , i.e. the Topp-Leone Moyal (TLMo) distribution which is a new extension of the Moyal distribution. Different properties of the new distribution , including the density, hazard rate functions, quantile function, mode, n^{th} moment, moment generating functions, mean deviation, incomplete moment, Lorenz, Bonferroni and Zenga curves, Rényi entropy and continuous entropy and the moments of order statistics. The parameters of the new distribution are estimated using the maximum likelihood approach and the information matrix is derived. Three real data sets are applied to demonstrate that the Topp-Leone Moyal (TLMo) distribution can provide a better fit than the Moyal (Mo) and beta Moyal (BMo) distributions.

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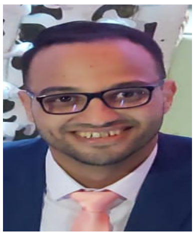

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