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Some Generalizations of Opial Type Inequalities for Conformable Fractional Integrals

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Abstract: In the present paper, some new versions of Opial type inequalities for conformable fractional integral are given by using convex functions.

Keywords: Opial inequality, Hölder's inequality, conformable fractional integrals, convex function.

1 Introduction

In 1960, Opial established the following integral inequality[1]:

Theorem 1. *Let $y(s) \in C^{(1)}[0,b]$ be such that $y(0) = y(b) = 0$, and $y(s) > 0$ in $(0,b)$. Then, the following inequality holds*

$$\int_0^b |y(s)y'(s)|\,ds \le \frac{b}{4}\int_0^b (y'(s))^2\,ds. \tag{1}$$

The constant $b/4$ is best possible.

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Throughout the last twenty years a large number of papers addressed simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [2]-[16].

The Opial inequality is of great interest in differential and difference equations as well as other areas mathematics. The present paper aims to establish some generalizations of Opial type inequalities for conformable integral. Then, conformable Opial type inequalities are given. The paper is outlined as follows. In Section 2 we give the definitions of conformable derivatives and integral. It also involves several useful notations conformable integral used our main results. In Section 3 the main result is presented.

2 Definitions and properties of conformable fractional derivative and integral

The following definitions and theorems with respect to conformable fractional derivative and integral were denoted in (see, [17]-[24]).

Definition 1(Conformable fractional derivative). *Given a function $g : [0,\infty) \to R$. Then the "conformable fractional derivative" of g of order α is defined by*

$$D_\alpha(g)(s) = \lim_{\varepsilon\to 0}\frac{g\left(s+\varepsilon s^{1-\alpha}\right) - g(s)}{\varepsilon} \tag{2}$$

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for all $s>0,\ \alpha\in(0,1)$. *If* g *is* α*-differentiable in some* $(0,a)$, $\alpha>0$, $\lim\limits_{t\to0^+} g^{(\alpha)}(s)$ *exists. Then, define*

$$g^{(\alpha)}(0)=\lim_{t\to0^+} g^{(\alpha)}(s). \tag{3}$$

We can write $g^{(\alpha)}(s)$ *for* $D_\alpha(g)(s)$ *to denote the conformable fractional derivatives of* g *of order* α*. In addition, if the conformable fractional derivative of* g *of order* α *exists, we simply say* g *is* α*-differentiable.*

Theorem 2. *Let* $\alpha\in(0,1]$ *and* h,g *be* α*-differentiable at a point* $s>0$*. Then*

i. $D_\alpha(ah+bg)=aD_\alpha(h)+bD_\alpha(g)$, for all $a,b\in R$,
ii. $D_\alpha(\lambda)=0$, for all constant functions $g(s)=\lambda$,
iii. $D_\alpha(hg)=fD_\alpha(g)+gD_\alpha(h)$,
iv. $D_\alpha(h\circ g)=h'(g(s))D_\alpha(g)(s)$ for h differentiable at $g(s)$,
v. $D_\alpha\left(\frac{h}{g}\right)=\frac{gD_\alpha(h)-hD_\alpha(g)}{g^2}$.

If g is differentiable, then

$$D_\alpha(g)(s)=s^{1-\alpha}\frac{dg}{ds}(s). \tag{4}$$

We mention that the conformable fractional derivative is a local operator.

Definition 2(Conformable fractional integral). *Let* $\alpha\in(0,1]$ *and* $0\le a<b$*. A function* $g:[a,b]\to R$ *is* α*-fractional integrable on* $[a,b]$ *if the integral*

$$\int_a^b g(s)\,d_\alpha s:=\int_a^b g(s)s^{\alpha-1}ds \tag{5}$$

exists and is finite. All α*-fractional integrable on* $[a,b]$ *is indicated by* $L^1_\alpha([a,b])$.

Remark.

$$I^a_\alpha(g)(s)=I^a_1\left(s^{\alpha-1}g\right)=\int_a^t\frac{g(s)}{s^{1-\alpha}}ds,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

Theorem 3. *Let* $g:(a,b)\to R$ *be differentiable and* $0<\alpha\le1$*. Then, for all* $s>a$ *we have*

$$I^a_\alpha D_\alpha g(s)=g(s)-g(a). \tag{6}$$

Theorem 4(Integration by parts). *Let* $h,g:[a,b]\to R$ *be two functions such that* hg *is differentiable. Then*

$$\int_a^b h(s)D_\alpha(g)(s)\,d_\alpha s=hg|_a^b-\int_a^b g(s)D_\alpha(h)(s)\,d_\alpha s. \tag{7}$$

Theorem 5. *Assume that* $g:[a,\infty)\to R$ *such that* $g^{(\alpha)}(t)$ *is continuous and* $\alpha\in(n,n+1]$*. Then, for all* $t>a$ *we have*

$$D_\alpha(I^a_\alpha g(s))=g(s).$$

We can give the Hölder's inequality in conformable integral as follows:

Lemma 1. *Let* $h,g\in C[a,b]$, $p,q>1$ *with* $\frac1p+\frac1q=1$,*. Then,*

$$\int_a^b |h(s)g(s)|\,d_\alpha s\le\left(\int_a^b |h(s)|^p\,d_\alpha s\right)^{\frac1p}\left(\int_a^b |g(s)|^q\,d_\alpha s\right)^{\frac1q}.$$

Remark. If we take $p=q=2$ in Lemma 1 , we have the Cauchy-Schwartz inequality for conformable integral.

Theorem 6(Jensen Inequality). *Let* $\alpha\in(0,1]$, $a,b,c,d\in[0,\infty)$*. If* $h:R\to R$ *and* $f:R\to(c,d)$ *are nonnegative continuous functions with* $\int_a^b p(t)d_\alpha t>0$*, and* $F:(c,d)\to R$ *is continuous and convex function. Then, we have*

$$F\left(\frac{\int_a^b h(s)f(s)d_\alpha s}{\int_a^b h(s)d_\alpha s}\right)\le\frac{\int_a^b h(s)F(f(s))\,d_\alpha s}{\int_a^b h(s)d_\alpha s}.$$

Opial inequality can be represented for conformable fractional integral forms as follows [13]:

Theorem 7. *Let $\alpha \in (0,1]$, $f:[a,b] \to R$ be an α- fractional differentiable function, and $f(a)=0$. Then, we have the following inequality*

$$\int_a^b |f(s)|\,|D_\alpha f(s)|\,d_\alpha s \le \frac{b^\alpha - a^\alpha}{2\alpha}\int_a^b (D_\alpha f(s))^2 d_\alpha s. \tag{8}$$

Theorem 8. *Let $\alpha \in (0,1]$, $f:[a,b] \to R$ be an α- fractional differentiable function, and $f(a)=0$. Then, we have the following inequality*

$$\int_a^b |f(s)|^p\,|D_\alpha f(s)|^q\,d_\alpha s \le \frac{q}{p+q}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^p \int_a^b (D_\alpha f(s))^{p+q} d_\alpha s \qquad \text{for } p \ge 0,\ q \ge 1.$$

For more details about the non-local versus local fractional operators, see [25]-[29] and the references. The following section presents the main results:

3 Opial type inequalities for conformable fractional integral

Theorem 9. *Let $\alpha \in (0,1]$, $f:[0,b] \to R$ be an α-fractional differentiable function. Then, we have the following inequality*

$$\int_0^b |f(t)|\,|D_\alpha f(t)|\,d_\alpha t \le \frac{\gamma}{2\alpha}\left(\int_0^b |D_\alpha f(s)|^2 d_\alpha s\right) + \beta\int_0^b |D_\alpha f(s)|\,d_\alpha s \tag{9}$$

where $\beta = \max\{|f(0)|,|f(b)|\}$ *and* $\gamma = \dfrac{b^\alpha + |b^\alpha - 2\tau^\alpha|}{2}$ *for* $\tau \in [0,b]$.

Proof. We consider

$$y(t) = \int_0^t |D_\alpha f(s)|\,d_\alpha s \quad \text{and} \quad z(t) = \int_t^b |D_\alpha f(s)|\,d_\alpha s$$

such that $D_\alpha y(t) = |D_\alpha f(t)|$, $D_\alpha z(t) = -|D_\alpha f(t)|$. Therefore, we get

$$\begin{aligned} |f(t)| &\le |f(t)-f(0)| + |f(0)| \\ &\le \int_0^t |D_\alpha f(s)|\,d_\alpha s + |f(0)| \\ &= y(t) + |f(0)|, \end{aligned}$$

and similarly

$$|u(t)| \le z(t) + |u(b)|.$$

Let $\tau \in [0,b]$. Then, using (7) and the Cauchy-Schwarz inequality for conformable integral, it follows that

$$\begin{aligned} \int_0^\tau |f(t)|\,|D_\alpha f(t)|\,d_\alpha t &\le \int_0^\tau [y(t)+|f(0)|]\,D_\alpha y(t) d_\alpha t \\ &= \int_0^\tau y(t) D_\alpha y(t) d_\alpha t + |f(0)| \int_0^\tau D_\alpha y(t) d_\alpha t \\ &= \frac{1}{2} y^2(\tau) + |f(0)|\,y(\tau) \\ &\le \frac{\tau^\alpha}{2\alpha}\left(\int_0^\tau |D_\alpha f(s)|^2 d_\alpha s\right) + |f(0)|\int_0^\tau |D_\alpha f(s)|\,d_\alpha s. \end{aligned} \tag{10}$$

Similarly, we get

$$\int_\tau^b |f(t)|\,|D_\alpha f(t)|\,d_\alpha t \le \frac{b^\alpha - \tau^\alpha}{2\alpha}\left(\int_\tau^b |D_\alpha f(s)|^2 d_\alpha s\right) + |f(b)|\int_\tau^b |D_\alpha f(s)|\,d_\alpha s. \tag{11}$$

Putting $\nu(\tau) = \max\{\tau^\alpha, b^\alpha - \tau^\alpha\}$ and $\beta = \max\{|f(0)|,|f(b)|\}$ and adding the inequalities (10) and (11), we have

$$\int_0^b |f(t)|\,|D_\alpha f(t)|\,d_\alpha t \le \frac{\nu(\tau)}{2\alpha}\left(\int_0^b |D_\alpha f(s)|^2 d_\alpha s\right) + \beta\int_0^b |D_\alpha f(s)|\,d_\alpha s$$

for $\tau \in [0,b]$. This completes the proof.

*Remark.*If we take $\beta = 0$ in Theorem 9, it follows that $|f(0)| = |f(b)| = 0$ and $\tau = 0\,(\text{or } \tau = b)$. Then, we have the following inequality

$$\int_0^b |f(t)|\,|D_\alpha f(t)|\,d_\alpha t \le \frac{b^\alpha}{2\alpha}\left(\int_0^b |D_\alpha f(s)|^2\,d_\alpha s\right) \tag{12}$$

which Sarikaya and Bilisik prove in [13].

*Remark.*If we take $\beta = 0$ in Theorem 9, it follows that $|f(0)| = |f(b)| = 0$ and $\tau = \frac{b}{2^{\frac{1}{\alpha}}}$. Then we have the following inequality

$$\int_0^b |f(t)|\,|D_\alpha f(t)|\,d_\alpha t \le \frac{h^\alpha}{4\alpha}\left(\int_0^b |D_\alpha f(s)|^2\,d_\alpha s\right)$$

which Sarikaya and Budak prove in [11].

Theorem 10.*Let $\alpha \in (0,1]$, $f : [0,b] \to R$ be an α-fractional differentiable function with $f(0) = 0$. Let p and q be positive and continuous on $[0,b]$, $\int_0^b \frac{d_\alpha s}{p(s)} < \infty$, and q nonincreasing. Then, we have the following inequality*

$$\int_0^b q(t)\,|f(t)|\,|D_\alpha f(t)|\,d_\alpha t \le \frac{1}{2}\left(\int_0^b \frac{d_\alpha s}{p(s)}\right)\left(\int_0^b p(s)q(s)\,|D_\alpha f(s)|^2\,d_\alpha s\right). \tag{13}$$

*Proof.*Let us consider,

$$y(t) = \int_0^t \sqrt{q(s)}\,|D_\alpha f(s)|\,d_\alpha s.$$

Then we have $D_\alpha y(t) = \sqrt{q(t)}\,|D_\alpha f(t)|$. Using the nonincreasing q function, it follows that

$$\begin{aligned} |u(t)| &\le \int_0^t |D_\alpha f(s)|\,d_\alpha s \\ &\le \int_0^t \sqrt{\frac{q(s)}{q(t)}}\,|D_\alpha f(s)|\,d_\alpha s = \frac{y(t)}{\sqrt{q(t)}}. \end{aligned}$$

Using the Cauchy-Schwarz inequality for conformable integral, we obtain

$$\begin{aligned} \int_0^b q(t)\,|f(t)|\,|D_\alpha f(t)|\,d_\alpha t &\le \int_0^b q(t)\frac{y(t)}{\sqrt{q(t)}}\frac{D_\alpha y(t)}{\sqrt{q(t)}}d_\alpha t \\ &= \frac{y^2(b)}{2} = \left[\int_0^b \frac{1}{\sqrt{p(s)}}\sqrt{p(s)q(s)}\,|D_\alpha f(s)|\,d_\alpha s\right]^2 \\ &\le \frac{1}{2}\left(\int_0^b \frac{d_\alpha s}{p(s)}\right)\left(\int_0^b p(s)q(s)\,|D_\alpha f(s)|^2\,d_\alpha s\right) \end{aligned}$$

which completes the proof.

Corollary 1.*Based on the hypotheses of Theorem 10, if we choose $p(s) = 1$, we have*

$$\int_0^b q(t)\,|f(t)|\,|D_\alpha f(t)|\,d_\alpha t \le \frac{b^\alpha}{2\alpha}\left(\int_0^b q(s)\,|D_\alpha f(s)|^2\,d_\alpha s\right). \tag{14}$$

*Remark.*If we take $q(s) = 1$ in Corollary 1, the inequality (14) reduces to the inequality (12).

Theorem 11.*Let $\alpha \in (0,1]$, f be a convex function on $[0,\infty)$ with $f(0) = 0$. Let $g : [a,b] \to R$ be a α-fractional differentiable function with $g(a) = 0$. Then, we have the inequality*

$$\int_a^b f'(|g(t)|)\,|D_\alpha g(t)|\,d_\alpha t \le f\left(\int_a^b |D_\alpha g(t)|\,d_\alpha t\right). \tag{15}$$

Proof. We consider

$$y(t)=\int_a^t |D_\alpha g(s)|\, d_\alpha s,\ t\in[a,b]$$

So $D_\alpha y(t)=|D_\alpha g(t)|$, and $y(t)\geq |g(t)|$. Thus, it follows that

$$\begin{aligned}\int_a^b f'(|g(t)|)\,|D_\alpha g(t)|\, d_\alpha t &\leq \int_a^b f'(y(t))\, D_\alpha y(t) d_\alpha t\\ &=\int_a^b D_\alpha f(y(t))\, d_\alpha t = f(y(b))\end{aligned}$$

which completes the proof of the inequality (15).

Corollary 2. *Let $\alpha\in(0,1]$, $g:[a,b]\to R$ be a α-fractional differentiable function with $u(a)=0$. Then, we have the inequality*

$$\int_a^b |g(t)|^p\, |D_\alpha g(t)|\, d_\alpha t \leq \frac{1}{p+1}\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^p \int_a^b |D_\alpha g(t)|^{p+1} d_\alpha t.$$

Proof. For $f(t)=t^{p+1}$, $p\geq 0$, the inequality (15) reduces to

$$\int_a^b |g(t)|^p\, |D_\alpha g(t)|\, d_\alpha t \leq \frac{1}{p+1}\left(\int_a^b |D_\alpha g(t)|\, d_\alpha t\right)^{p+1}$$

Hence, from Hölder's inequality with indices $p+1$ and $\frac{p+1}{p}$, it follows that

$$\int_a^b |g(t)|^p\, |D_\alpha g(t)|\, d_\alpha t \leq \frac{1}{p+1}\left(\frac{b^\alpha-a^\alpha}{\alpha}\right)^p \left(\int_a^b |D_\alpha g(t)|^{p+1} d_\alpha t\right).$$

Theorem 12. *Let $\alpha\in(0,1]$, f,g be a convex function on $[0,\infty)$ with $f(0)=0$. Let $p(t)\geq 0$, $D_\alpha p(t)>0$, $t\in[a,b]$ with $p(a)=0$. Furthermore, let $h:[a,b]\to R$ be a α-fractional differentiable function with $h(a)=0$. Then, we have the inequality*

$$\int_a^b D_\alpha p(t) g\left(\frac{|D_\alpha h(t)|}{D_\alpha p(t)}\right) f'\left(p(t) g\left(\frac{|h(t)|}{p(t)}\right)\right) d_\alpha t \leq f\left(\int_a^b |D_\alpha h(t)|\, d_\alpha t\right). \tag{16}$$

Proof. Let

$$y(t)=\int_a^t |D_\alpha h(s)|\, d_\alpha s,\ t\in[a,b]$$

So $D_\alpha y(t)=|D_\alpha h(t)|$, and $y(t)\geq |h(t)|$. Thus, from Jensen's inequality, it follows that

$$g\left(\frac{|h(t)|}{p(t)}\right)\leq g\left(\frac{y(t)}{p(t)}\right)\leq g\left(\frac{\int_a^t D_\alpha p(s)\frac{|D_\alpha h(s)|}{D_\alpha p(s)} d_\alpha s}{\int_a^t D_\alpha p(s) d_\alpha s}\right)\leq \frac{1}{p(t)}\int_a^t D_\alpha p(s) g\left(\frac{D_\alpha y(s)}{D_\alpha p(s)}\right) d_\alpha s. \tag{17}$$

Thus, using (17), we obtain

$$\begin{aligned}\int_a^b D_\alpha p(t) g\left(\frac{|D_\alpha h(t)|}{D_\alpha p(t)}\right) f'\left(p(t) g\left(\frac{|h(t)|}{p(t)}\right)\right) d_\alpha t &\leq \int_a^b D_\alpha p(t) g\left(\frac{y(t)}{D_\alpha p(t)}\right) f'\left(\int_a^t D_\alpha p(s) g\left(\frac{D_\alpha y(s)}{D_\alpha p(s)}\right) d_\alpha s\right) d_\alpha t\\ &=\int_a^b D_\alpha \left[f\left(\int_a^t D_\alpha p(s) g\left(\frac{D_\alpha y(s)}{D_\alpha p(s)}\right) d_\alpha s\right)\right] d_\alpha t\\ &= f\left(\int_a^b D_\alpha p(s) g\left(\frac{D_\alpha y(s)}{D_\alpha p(s)}\right) d_\alpha s\right)\\ &= f\left(\int_a^b D_\alpha p(t) g\left(\frac{|D_\alpha h(t)|}{D_\alpha p(t)}\right) d_\alpha t\right),\end{aligned}$$

which completes the proof of the inequality (16).

Remark. If we take $f(t)=t^2$ and $g(t)=t$ in Theorem 12 and use the Cauchy-Schwarz inequality for conformable integral in right integral, the inequality (16) reduces to the inequality (8).

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