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Eigenparameter Dependent Inverse Sturm-Liouville Problems

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The author studies the inverse scattering problem for a boundary value problem of a generalized one dimensional Schrödinger type with a discontinuous coefficient and eigenparameter dependent boundary condition. The solutions of the considered eigenvalue equation is presented and its scattering function that satisfies some properties is induced. The discrete spectrum is studied and the resolvent of the considered problem is given. The scattering data are determined and hence the inverse scattering problem is formulated and completely solved.

Keywords: Eigeparameter, Schrodinger equation, inverse, scattering, eigenvalue, the discrete.

1 Introduction

Consider the boundary value problem corresponding to the generalized form of the one-dimensional Schrödinger equation

$$-y'' + q(x)y = \lambda\rho(x)\gamma, \qquad x \in [0, \infty), \tag{1.1}$$

and the boundary condition:

$$(\alpha_1 + \lambda\beta_1)y(0) = (\alpha_2 + \lambda\beta_2)y'(0), \tag{1.2}$$

where λ is a complex number and $\delta = (\alpha_1\beta_2 - \alpha_2\beta_1) > 0$ is real number. We assume that the function q is real and satisfies the condition.

$$\int_0^\infty x\,|q(x)|\,dx < \infty, \tag{1.3}$$

which is assumed to hold through out the paper. The function ρ is defined as:

$$\rho(x) = \left\{ \begin{array}{ll} \gamma^2, & .0 \le x \le a \\ I, & .a < x < \infty\ ,\ \gamma \ne 1,\ \gamma \in R^+ \end{array} \right\}. \tag{1.4}$$

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It is worthnoting that a differential equation is given in the direct problems and a particular solution is sought from certain functions. In the inverse problem a solution is given and a particular differential equation is sought from a given class of equations.

The modern trend in quantum scattering theorems is to solve the inverse problem by the so-called scattering data. These scattering data are defined as the collection of quantities $\{S(s); \lambda_1, \lambda_2, \cdots, \lambda_n, M_1, M_2, \cdots, M_n\}$. Here $S(s)$ is the scattering function of the problem and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues and $M_1, M_2, \cdots, M_n$ are called the normalization coefficients.

It is well known (see [1], pp. 146-152) that the boundary value problems with spectral parameter in the boundary condition have many interesting applications in mathematical physics.

It may be pointed out that the direct and inverse problem is considered previously in different cases. In case $\rho = 1$ the direct and inverse problem of (1.1) with $y'(0) - hy(0) = 0$, (see [2], [3]) has been solved earlier by the so called spectral distribution function while the problem (1.1) with $y(0) = 0$ has been studied in the works [3], [4] by the inverse scattering method. Furthermore, the inverse scattering problem of one-dimensional Schrödinger's eigenvalue problem with a discontinuous coefficient was studied when $y(0) = 0$ and $y'(0) = 0$ [5], [6], [7]. It should be mentioned that the spectrum of the boundary value problem (1.1)-(1.2) has been previously investigated in [8] when $\rho(x) > 0$ and the boundary condition $y(0) = 0$ holds. The appearance of the eigenvalue in the boundary condition also causes the Sturm-Liouville operator to lose its self-adjointness in $L^2[0, 1]$. In fact, the eigenfunctions do not form a basis in $L^2[0, 1]$ unless one of the eigenfunctions is removed from the set. It is possible, however, to introduce an associated operator in $L^2[0, 1]$ that is self-adjoint. The analysis of the applications, completeness, and expansion theory of the regular direct problem is found in the work of Walter [9] and Fulton [10]. Andersent [11] studied inverse eigenvalue problems with discontinuous cofficients. Glazman [12] investigated direct methods for the qualitative spectral analysis of singular differential operator. Binding et al. [13] presented asymptotic, oscillation, and comparison results.

Uniqueness results for the inverse spectral problem for Sturm-Liouville equations of this type have been studied recently by Browne and Sleeman [14], [15], and by Binding et al. [4]. Browne and Sleeman [15] discuss uniqueness for nodal spectral data. In Browne and Sleeman [14], they establish uniqueness when $p = r = 1$, given one spectrum and a sequence of norm constants. A more general result is that of Binding et al. [16], which establishes uniqueness results for the cases when (i) two spectra are prescribed, (ii) one spectrum and a sequence of norm constants are prescribed, and (iii) one spectrum with even p, q, and r prescribed. Case (ii) is addressed again in Binding et al. [4], where Binding et al. [4] construct a mapping from the eigenparameter dependent Sturm-Liouville problem to the regulr Sturm-Liouville and apply the known results for the inverse Sturm-Liouville problem. In this paper I have extended the previous results by considering (1.1) with $(\alpha_1 +$

$\lambda\beta_1)y(0) = (\alpha_2 + \lambda\beta_2)y'(0)$ and solve the inverse problem by the inverse scattering method. The organization of this paper is as follows:

We collect in section 2 certain solutions of equation (1.1) from [2], [17] which we shall use subsequently. Moreover, we defined the so called scattering function and we give its asymptotic behavior. In section 3 we shall study of the discrete spectrum of the boundary value problem (1.1)-(1.2) obtained, we obtain the resolvent of (1.1)-(1.2). As will be seen in section 4, the collection of quantities $\left\{S(s) : -\tau_n^2 i\ M_n,\ n = \overline{1,m}\right\}$ is called the scattering data of (1.1)-(1.2), where $S(s)$ is the scattering function and the numbers $-\tau_1^2, -\tau_2^2, \cdots, -\tau_n^2$ are eigenvalues. The numbers $M_1, M_2, \cdots, M_m$ are called the normalization coefficients of (1.1-(1.2). Finally we will be aimed at solving the inverse scattering problem of (1.1)-(1.2). The inverse scattering problem is being extended to a more general case when $\rho(x) = 1$. In terms of the scattering data, the potential function $q(x)$ and weigh function $\rho(x)$ and $\alpha_1, \alpha_2, \beta_1$ and β_2 are defined uniquely. This is a straightforward consequence of Theorem 4.7.

2 Some Solutions of the Equation (1.1)

From condition (1.3) it is evident that equation (1.1) reduces asymptotically to the simpler equation $y'' = \lambda\rho y\ as\ x \to \infty$ this permits us a complete investigation of the properties of solution to equation (1.1)

Let $\varphi\ (x,\ s)$ and $\Psi(x, s)$ denote the solutions to the equation (1.1) on the interval $(0, a)$ that satisfy the initial conditions:

$$\begin{array}{ll} \varphi(0, s) = \propto_2 + s^2\beta_2, & \varphi'(0, s) = \alpha_1 + s^2\beta_1, \\ \psi(0, s) = \beta_2, & \psi'(0, s) = \beta_1, \end{array} \tag{2.1}$$

where $\lambda^{\frac{1}{2}} = s = \sigma + i\tau$ such that $0 \leq \arg s > \pi$.

Lemma 2.1. *The solution $\varphi\ (x,\ s)$ and $\Psi(x, s)$ of the equation (1.1) on the interval $[0, a]$ may be expressed in the form*

$$\varphi(x, s) = (\alpha_2 + s^2\beta_2)\cos\gamma x + \frac{\alpha_1 + s^2\beta_1}{s\gamma}\sin s\gamma x + \frac{1}{s\gamma}\int_0^x \sin s\gamma(x - t)q(t)\varphi(t, s)dt, \tag{2.2}$$

and

$$\psi(x, s) = \beta_2\cos s\gamma x + \frac{\beta_1}{s\gamma}\sin s\gamma x \ + \frac{1}{s\gamma}\int_0^x \sin s\gamma(x - t)q(t)\psi(t, s)dt,$$

For the proof, see [17].

On the interval (a, ∞) equation (1.1) becomes $-y'' + q(x)y = s^2 y$. Then, for any s from the closed upper half plane, the above equation has solution $F(x, s)$ of the form [2].

$$F(x, s) = e^{isx} + \int_s^\infty k(x, t)e^{ist}dt \qquad a < x < \infty \tag{2.3}$$

where the kernel $K(x,t)$ satisfies the inequality

$$|K(x,t)| \leq \frac{1}{2}\sigma\left\{\frac{x+t}{2}\right\}\exp\left\{\sigma_1(x)-\sigma_1(\frac{x+t}{2})\right\},$$

and the condition

$$\frac{dk(x,x)}{dx} = -\frac{1}{2}q(x). \tag{2.4}$$

Here

$$\sigma(x) = \int_x^\infty |q(t)|\,dt, \quad \text{and} \quad \sigma_1(x) = \int_x^\infty \sigma(t)dt.$$

The solution $F(x,s)$ is an analytic function of s in the upper half plane $\tau \geq 0$ and is continuous on the real line. The following estimates hold.

$$|F(x,s)| \leq \exp\left[-\operatorname{Im} sx + \sigma_1\right], \tag{2.5}$$

and

$$|F'_x(x,s) - is\exp(isx)| \leq \sigma(x)\exp\left\{-\operatorname{Im} sx + \sigma_1(x)\right\}. \tag{2.6}$$

Theorem 2.2. *The identity*

$$\frac{2is\varphi(x,s)}{\Delta(s)} = F(x,-s) - S(s)F(x,s), \tag{2.7}$$

holds for all real values $s \neq 0$*, where*

$$\Delta(s) = (\alpha_2 + s^2\beta_2)F'(0,s) - (\alpha_1 + s^2\beta_1)F(0,s),$$

and

$$S(s) = \frac{\Delta(-s)}{\Delta(s)} = \overline{S(s)} = [S(-s)]^{-1}.$$

Proof. Since the functions $F(x,s)$ and $F(x,-s)$ form a fundamental system of solutions to equation (1.1) for all real $s \neq 0$, thus

$$\varphi(x,s) = C_1(s)F(x,s) + C_2(s)F(x,-s).$$

Condition (2.1) yields

$$C_1(s) = \frac{\Delta(-s)}{2is} \quad \text{and} \quad C_2(s) = \frac{\Delta(s)}{2is},$$

where

$$\Delta(s) = (\alpha_2 + s^2\beta_2)F^{/}(0,s) - (\alpha_1 + s^2\beta_1)F(0,s).$$

Hence $\varphi(x,s) = \dfrac{1}{2is}\left[\Delta(s)F(x,s) - \Delta(-s)F(x,s)\right]$. Since $q(x)$ is real, $\Delta(-s) = \overline{\Delta(s)}$ and hence $\Delta(s) \neq 0$ for all real $s \neq 0$. Therefore

$$\frac{2is\varphi(x,s)}{\Delta(s)} = F(x,-s) - S(s)F(x,s),$$

with

$$S(s) = \frac{\Delta(-s)}{\Delta(s)} = \left[.\frac{\overline{\Delta(s)}}{\Delta(-s)} \right] = \left[\frac{\Delta(s)}{\Delta(-s)} \right]^{-1},$$

as claimed. □

Definition 2.3. *The function $S(s)$ is called the scattering function of equation (1.1) with initial conditions (2.1).*

Theorem 2.4. *For large real $s \neq 0, |s| \to \infty$ the following asymptotical formula*

$$S(s) - S_0(s) = 0(\frac{1}{S^2}), \tag{2.8}$$

holds, where

$$\begin{aligned} S_0(s) &= e^{-2isa}[(\beta_2 - is\beta_1)\frac{\sin s\gamma a}{\gamma} - (is\beta_1 + \beta_2)\cos s\gamma a] \\ &\quad \times \left[(\beta_2 + is\beta_1)\frac{\sin s\gamma a}{\gamma} + (is\beta_2 - \beta_1)\cos s\gamma a \right]^{-1}. \end{aligned} \tag{2.9}$$

Proof. It is clear that $\varphi(x,s)$ and $\psi(x,s)$ from a fundamental system of solutions of equation (1.1) on the interval $[0,a]$, so

$$F(x,s) = d_1(s)\varphi(x,s) + d_2(s)\psi(x,s).$$

Now, condition (2.1) yields

$$d_1(s) = \frac{\beta_2 F'(0,s) - \beta_1 F(0,s)}{\delta}, \quad \text{and} \quad d_2 = \frac{\Delta(s)}{\delta}.$$

Thus, we have

$$F(x,s) = \frac{\beta_2 F'(0,S) - \beta_1 F(0,S)}{\delta}\varphi(x,s) + \frac{\Delta(s)}{\delta}\psi(x,s). \tag{2.10}$$

Formula (10) gives

$$\Delta(s) = F'(a,s)\varphi(a,s) - \varphi'(a,s)F(a,s).$$

Taking into account (2.2) and (2.4) we obtain

$$\Delta(s) = e^{isa}\left[(\beta_2 + is\beta_1)\frac{\sin vsa}{a} + (is\beta_1 - \beta_2)\cos vsa \right] + O(\frac{1}{S^2}).$$

Thus we conclude that

$$\begin{aligned} S(s) &= \Delta(-s)[\Delta(s)]^{-1} \\ &= e^{-2isa}[(\beta_2 - is\beta_1)\frac{\sin s\gamma a}{\gamma} - (is\beta_1 + \beta_2)\cos s\gamma a] \\ &\quad \times \left[(\beta_2 + is\beta_1)\frac{\sin \gamma as}{\gamma} + (is\beta_1 - \beta_2)\cos \gamma sa \right]^{-1} + O\,(\frac{1}{s^2}) \\ &= S_0(s) + O\,(\frac{1}{s^2}) \end{aligned}$$

where S_0 is derived by (2.9). $\square$

3 The Discrete Spectrum and the Resolvent of (1.1)-(1.2)

This section is devoted to study the discrete spectrum and to obtained the resovent of (1.1)-(1.2). We have the following lemma from [3]:

Lemma 3.1. *The necessary and sufficient conditions that $\lambda \neq 0$ be an eigenvalue of (1.1)-(1.2) are that*

$$\lambda = s^2, \quad \tau > 0 \text{ and } \Delta(s) = (\alpha_2 + s^2\beta_2)F'(0,s) - (\alpha_1 + s^2\beta_1)F(0,s) = 0.$$

Theorem 3.2. *The problem (1.1)- (1.2) does not have an eigenvalue on the positive semi-axis. The set of eigenvalues is no more than countable, and its limit point can lie on the half-axis $\lambda \geq 0$, the eigenvalues are on the imaginary axis of s-plane and are all simple, and bounded.*

Proof. In the sequel, we show that the function $\Delta(s) \neq 0$ for real values $\lambda = s^2 \neq 0$. Suppose the contrary. Let $s_0 \in [0,\infty)$, $s_0 \neq 0$ such that $\Delta(s_0) = 0$. Then, we have $(\alpha_2 + s_0^2\beta_2)F'(0,-s_0) - (\alpha_1 + s_0^2\beta_1)F(0,s_0) = 0$. Since

$$\begin{aligned} 2is_0 &= W[F(0,-s_0), F(0,-S_0)] = F(0,-s_0)F'(0,s_0) - F'(0,-s_0)F(0,s_0) \\ &= \frac{(\alpha_1 + s_0^2\beta_1)}{\alpha_2 + s_0^2\beta_2}F(0,-s_0)F(0,s_0) - \frac{(\alpha_1 + s_0^2\beta_1)}{(\alpha_2 + s_0^2\beta_2)}F(0,-s_0)F(0,s_0) \\ &= 0, \end{aligned}$$

the assumption leads to a contradiction $s_0 \neq 0$.

Therefore we conclude that equations (1.1)-(1.2) does not have positive eigenvalue and this problem has not a singular spectrum. Since the function $\Delta(s)$ is an analytical function in the upper half plane $\tau > 0$, its zeros form an at most countable set having 0 as the only possible limit point. Here, we show that the eigenvalues lie on the imaginary axis of s-plane. Let $s(s = 0$ or $lms > 0$ be one of the zeros of $\Delta(s)$. Since.

$$W\left[\varphi(x,s), F(x,s)\right] = \Delta(s) = 0,$$

we have

$$F(x,s) = c\,\varphi(x,s) \text{ and } \lim_{x\to 0} F(x,s) = c,$$

and

$$F(x,s) = \frac{\beta_2 F'(0,s) - \beta_1 F(0,s)}{\delta}\varphi(x,s). \tag{3.1}$$

Formula (3.1) leads to

$$\lim_{x\to 0} W.\left[F(x,s_1), \overline{F(x,s_2)}\right] = 0, \tag{3.2}$$

for two arbitrary zeros s_1 and s_2 of $\Delta(s)$. Since $q(x)$ is real function, we have.

$$W\left[F(x,s_1),\overline{F(x,s_2)}\right]_0^{-\infty}=(s_1^2-s_2^{-2})\int_0^\infty F(x,s_1)\overline{F(x,s_2)}dx.$$

Using (3.1) and the estimate (2.5) and (2.6) we obtain.

$$(s_1^2-s_2^{-2})\left[\frac{1}{\delta}\left|\beta_2F'(0,s_1)-\beta_1F(0,s_1)\right|^2+\int_0^\infty F(x,s_1)\overline{F(x,s_2)}\right]=0.$$

Taking, in particular, $s_1=s_2$ to have $s_1+\bar{s_2}=0$,i.e. $s_1=i\,\tau_1$. Then, the zero of the function $\Delta(s)$ can lie only on the imaginary axis. Also, From [2] and the estimate equation (2.5) and equation (3.2) we find that the zeros of $\Delta(s)$ are simple. Finally, since

$$\Delta(s)=e.^{isa}\left[(\beta_2+is\beta_1)\frac{\sin s\gamma a}{\gamma}+(is\beta_1-\beta_2)\cos s\gamma a\right]+O\,(\frac{1}{s^2})\neq 0,$$

for sufficiently large s, the number $\lambda=s^2$ cannot be an eigenvalue of equations (1.1)-(1.2) and we conclude that these eigenvalues are a bounded set. The theorem is proved. □

Let $H=L_2(0,\alpha;\;\rho(x))x\;C$ be the Hilbert space with scalar product

$$(F,G)_H=\int_0^\infty\left\{f(x)\overline{g(x)}\right\}dx+\frac{1}{\delta}f_1g_1,$$

where

$$F=\begin{pmatrix}f(x)\\ f_1\end{pmatrix},\qquad G=\begin{pmatrix}g(x)\\ g_1\end{pmatrix}\in H.$$

Let $D(L)$ be the set of functions $f(x)\in H$, where $f(x)$ is absolutely continuous on every finite interval $[0,a]$.

$$-f^{//}(x)+q(x)f(x)\in L_2(0,\infty,C^2)\;\text{ and }\;f_1=B_2f^{/}(0)-B_1f(0)$$

defines the operator

$$LF=\begin{bmatrix}-F^{//}(x)+q(x)f(x)\\ -(\alpha_2f^{//}(0)-\alpha_1f(0))\end{bmatrix},\;F\in D(L).$$

Lemma 3.3. *All numbers $\lambda=s^2$ and $\tau>0$, and $\tau\Delta(s)\neq 0$ belong to the resolvent set of the problems (1.1)-(1.2). If $\Delta(s)\neq 0$ then the resolvent for $y''+g(x)y_2-\lambda\rho y=\rho f$, $(\alpha_1+s^2\beta_1)y(0)=(\alpha_2+s\beta_2)y(0)+f_1$, $x\in[0,\infty)$ $f(x)\in L_2(0,\infty;\rho(x)$ is an integral operator*

$$G_s(\rho f)=\int_0^\infty G(x,t;s)\rho(t)f(t)dt+\frac{f_1}{\Delta(\lambda)}F(x,\lambda),$$

where $G(x,t,s)$ is defined by

$$G(x,t;s)=\frac{-1}{\Delta(s)}\begin{cases}\varphi(x,s)F(t,s), & t\leq x\\ F(x,s)\varphi(t,s), & t\geq x\end{cases}\tag{3.3}$$

called the kernel resolvent for the problem (1.1)-(1.2).

This lemma can be proved by using variation of parameter method and taking into account the condition (1.2). It follows from variation of parameters that the general solution of the nonhomogenous equation $y'' + q(x)y - \lambda\rho y = \rho f, \quad x \in [0, \infty)$ is

$$y(x, s) = C_1 F(x, s) + C_2\varphi(x, s) + \int_0^\infty G(x, t; s)\rho(t) f/t) dt,$$

where $G(x, t, s)$ is defined by (3.3). Since, for values of s in the upper half-plane, the functions $F(x, s) \in L_2(0, \infty, \rho(x))$ and $\varphi(x, s) \notin L_2(0, \infty, \rho(x))$ are the solutions of the homogenous equation (1.1), $c_1 F(x, s) + c_2\varphi(x, s)$ is a solution of (1.1) that belongs to $L_2(0, \infty; \rho(x))$ only for those values s in upper half plane when $c_2 = 0,\ c_1 = \dfrac{F_1}{\Delta(S)}$. So, $y(x, s)$ is the unique solution that belongs to $L_2(0, \infty, \rho(x))xc$ of equation (1.1). It can be directly checked that the function $y(x, s)$ satisfies condition (1.2).

4 Eigenfunction Expansion and Formulation of Inverse Scattering Problem of (1.1)-(1.2)

In this section, we obtain the expansion of eigenfunctions of (1.1)-(1.2) by Titchmaris methods [18] and the results of [19]. Moreover we give the inverse scattering problem (1.1)-(1.2) for this purpose since the scattering data of (1.1)-(1.2) are known, thus we can construct the fundamental equation for unique kernel $k(x, t)$ of formula (2.3).

Lemma 4.1. *Assume that the function $f(t)$ is finite and has continuous derivative in $L_2(0, \infty, \rho(x))$ and satisfies the boundary condition (1.2). Then*

$$\frac{f_1 F(x, s)}{\Delta(s)} + \int_0^\infty G(x, t\,; s)\rho(t) f(t) dt$$
$$= -\frac{f(x)}{s^2} + \frac{1}{s^2}\int_0^\infty G(x, t.; s)h(t)dt + \frac{F(x, s)(\alpha_2 f^{/}(0) - \alpha_1 f(0)\}}{s^2\Delta(s)}$$

where $h(t) = -f''(t) + q(t)f(t)$. Furthermore, if $\tau > 0$ and $s \to \infty$, then

$$\int_0^\infty G(x, t\,; s)\ \rho(t) f(t) dt + \frac{f_1 F(x, s)}{\Delta(s)} = \frac{-f(x)}{s^2} + 0(\frac{1}{s^2}). \tag{4.1}$$

Proof. Since $\varphi(x, s)$ and $F(x, s)$ satisfy equation (1.1), using equation (3.3) we find that

$$\int_0^\infty G(x, t, s)\rho(t) f(t) dt = \frac{-1}{s^2\Delta(s)}[F(x, s)\int_0^x [\varphi''(t, s) - q(t)\varphi(t, s)]\, f(t) dt$$
$$+\varphi(x, s)\int_0^\infty [F''(t, s) - q(t) f(t, s)]\, f(t) dt].$$

Integrating this identity by parts we get the first part of the lemma.

From equations (2.2), (2.5) and (3.3) we find that $G(x,t,s)h(t) = O(1.1)$ as $\tau > 0$ and $|s| \to \infty$. Hence equation (4.1) follows directly. $\square$

The following lemma is from [19]:

Lemma 4.2. $\overline{G(x,t;s)} = G(x,t;s)$.

In view of these lemmas we prove the following expansion theorem.

Theorem 4.3. *If the function $f(x)$ satisfies the conditions of lemma 4.1, then its expansion by eigenfunctions of (1.1)-(1.2) can be written in the form*

$$f(x) = \frac{1}{2\pi}\int_0^\infty\int_0^\infty U(x,s)U(t,-s)\rho(t)f(t)dtds + \sum_{n=1}^{m}\int_0^\infty U(x,i\tau_n)U(t,it)\rho(t)f(t)dt$$
$$+\frac{1}{\pi}\int \frac{\varphi(x,s)}{|\Delta(s)|^2}f_1 ds + \sum_{n=1}^{\infty}\frac{F(x,\tau_n)}{\Delta(i\tau_n)}f_1, \tag{4.2}$$

where $U(x,i\,\tau_n) = M_n F(x,i\,\tau_n)$ and

$$M_n^2 = \frac{2i\tau_n\delta}{\Delta(i\tau_n)\,[\beta_2 F'(0,i\tau_n) - \beta_1 F(0,i\tau_n])}.$$

Proof. Suppose that $f(x)$ satisfied the conditions of lemma 4.1, then equation (4.1) holds. Multiplying both sides of equation (4.1) by $\dfrac{k}{\pi i}$ and integrating over the semi-circle $|s| = r$ with respect to s in the upper half plane of s. Evidently, the integral, $\int_0^\infty G(x,t\,;s)\rho(t)f(t)dt$ is a holomorphic function except at the zeroes $\{i\,\tau_1,\cdots,i\,\tau_n\}$ of the function $\Delta(s)$. Then use [19] to obtain

$$\begin{aligned} f(x) \;=\; & \frac{1}{\pi i}\int_0^\infty s\int_0^\infty \left[G(x,t;s+i\,0) - G(x,t;s-i\,0)\right]\rho(t)f(t)dtds \\ & -\sum_{n=1}^{m}\operatorname{Re}s\left[2s\int_0^\infty G(x,\tau;s)\rho(t)f(t)dt\right]_{s=i\tau_n} - \sum_{n=1}^{m}\operatorname{Re}s\frac{f_1F(x,\lambda)}{\Delta(s)}]_{s=i\tau_n} \\ & +\frac{1}{2\pi}\int_0^\infty\left[\frac{F(x,s+i\,0)}{\Delta(s+i\,0)} - \frac{F(x,s-i\,0)}{\Delta(s-i\,0)}\right]f_1 ds. \end{aligned} \tag{4.3}$$

Let us compute the first term in the right hand side of equation (4.3). In order to compute this term, we determine $G(x,t;s-i\,0)$ and then use Lemma 4.2 to obtain $G(x,t;s+i\,0)$. Substituting from equation (2.10) into equation (3.3), we get

$$G(x,t;s) = -\frac{1}{\Delta(s)}\frac{(\beta_2F'(0,S) - \beta_1F(0,S))}{\delta}\varphi(x,s)\varphi(t,s) - R(x,t,s), \tag{4.4}$$

where

$$R(x,t,s)=\begin{cases}\psi(x,s)\varphi(t,s) & t\le x\\ \varphi(x,s)\psi(t,s) & t\ge x\end{cases}$$

In view of equation (4.4) we get

$$G(x,t;s+i\,0)-G(x,t;s-i\,0)=\frac{2i\varphi(x,s)\varphi(t,s)}{\Delta(s)\Delta(-s)}. \tag{4.5}$$

Taking into account formula (2.7) then equation (4.5) can be rewritten on the form

$$G(x,t;s+i\,0)-G(x,t;s-i\,0)=\frac{U(x,s)U(t,s)}{-2is},$$

where

$$U(x,s)=\frac{2is\varphi(x,s)}{\Delta(s)}=F(x_1-s)-S(s)F(x,s).$$

Therefore

$$\frac{1}{2ni}\int_0^\infty\int_0^\infty s\left[G(x,t;s+io)-G(x,t;s-io)\right]\rho(t)f(t)dtds$$
$$=\frac{1}{2n}\int_0^\infty\int_0^\infty U(x,s)U(t,-s)\rho(t)f(t)dtds \tag{4.6}$$

Here, we compute the second term in the right hand side of (4.3). Since

$$\int_0^\infty G(x,t;s)\rho(t)f(t)dt=\frac{-(\beta_2F'(0,s)-\beta_1F(0,s))}{\Delta(s)\delta}\varphi(x,s)\int_0^\infty\varphi(t,s)\rho(t)f(t)dt$$
$$-\psi(x,s)\int_0^x\varphi(t,s)\rho(t)f(t)dt-\varphi(x,s)\int_x^\infty\psi(t,s)\rho(t)f(t)dt$$

Evidently, the function $\varphi(x,s)$ and $\psi(x,s)$ are analytic and hence

$$\sum_{n=1}^{m}\operatorname{Re}s\left[-\int_0^x\psi(x,s)\varphi(t,s)\rho(t)f(t)dt-\int_x^\infty\varphi(x,s)\psi(t,s)\rho(t)f(t)dt\right]_{s=i\;\tau_n=0}=0$$

Therefore, we have

$$\begin{aligned}I&=-\sum_{n=1}^{\infty}\operatorname{Re}s\left[2s\int_0^\infty G(x,t;s)\rho(t)f(t)dt\right]_{s=i\;\tau_n}\\&=-\sum_{n=1}^{\infty}-\operatorname{Re}s\;\left[[\frac{2s(\beta_2F'(0,s)-\beta_2F(0,s))}{\Delta(s)\delta}]\varphi(x,s)\int_0^\infty\varphi(t,s)\rho(t)f(t)dt\right]_{s=i\tau_n}\\&=-\sum_{n=1}^{m}\frac{2i\,\tau_n\left[\beta_2F'(0,i\,\tau_n)-\beta_1F(0,i\,\tau_n)\right]}{\delta\Delta(i\tau_n)}\varphi(x,i\,\tau_n)\int_0^\infty\varphi(t,i\,\tau_n)\rho(t)f(t)dt.\end{aligned}$$

Setting

$$M_n^2 = \frac{2i\tau_n\delta}{\Delta(i\tau_n)\left[\beta_2 F'(0,i\tau_n) - \beta_1 F(0,i\,\tau_n)\right]}$$

and using (3.1) we find that:

$$\begin{aligned} I &= -\sum_{n=1}^{\infty} M_n^2 \int_0^{\infty} F(x,i\tau_n)F(t,i\tau_n)\rho(t)f(t)dt \\ &= -\sum_{n=1}^{m} \int_0^{\infty} U(x,i\tau_n)U(t,i\tau_n)\rho(t)f(t)dt, \end{aligned} \tag{4.7}$$

where $U(x,i\tau_n) = M_n F(x,i\tau_n)$ is the normalization eigenfunctions and the numbers $M_n, n=\overline{1,m}$ are called normalization coefficients. Hence, we obtain from equations (4.3) ,(4.6) and (4.7) the following expansion of $f(x)$ by eigenfunctions of equations (1.1)-(1.2)

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{\infty} U(x,s)U(t,-s)\rho(t)f(t)dtds \\ &+ \sum_{n=1}^{\infty}\int_0^{\infty} U(x,i\tau_n)U(t,i\tau_n)\rho(\tau)f(t)dt + \frac{1}{\pi}\int \frac{\varphi(x,s)}{|\Delta(s)|^2} f_1 ds + \sum_{0}^{\infty} \frac{f(x,i\tau_n)}{\Delta(i\tau_n)} f_1, \end{aligned}$$

and

$$\begin{aligned} f_1 &= \frac{\delta}{\pi}\int_0^{\infty}\int_0^{\infty} \frac{\varphi(t,s)}{|\Delta(s)|^2}\rho(t)f(t)dtds + \frac{\delta}{\pi}\int \frac{f_1}{|\Delta(s)|^2} ds \\ &+ \sum_{z=1}^{\rho} M^2 \int_0^{\infty} F(t,s)f(t)\left[\beta_2 F'(0,i\tau_n) - \beta_1 F(0,i\tau_n)\right] dt \\ &+ \sum_{z=1}^{\rho} \frac{\beta_2 F'(0,i\tau_n) - \beta_1 F(0,i\tau_n)}{\Delta(iz_n)} f_1. \end{aligned}$$

This is the required result and thus the theorem is proved. □

Definition 4.4. *The collection of quantities* $\left\{S(s), -\tau_n^2, M_n,\ n=\overline{1,m}\right\}$ *is called the scattering data of the problem (1.1)-(1.2).*

Here the inverse scattering problem of the problem (1.1)-(1.2) can be stated as follows: Knowing the scattering data of the problem (1.1)-(1.2), can we reconstruct equation (1.1) and (1.2), that is, can we find the potential function q, the density function $\rho(x)$, $\alpha_1, \alpha_2, \beta_1$ and β_2?

In this section we give an answer to this question for solving the inverse problem (1.1)-(1.2). For this purpose, since scattering data of the problem (1.1)-(1.2) are known, we can construct the fundamental equation for unique kernel $k(x,t)$ of formula (2.3). Using the methods of the [5] and [14] we can prove the following theorem:

Theorem 4.5. *The kernel of formula (2.3) satisfies the fundamental equation.*

$$F(x,t)+k(x,t)+\int_0^\infty k(x,y)f(y+t)dy=0, \qquad 0<x<y<\infty, \tag{4.7}$$

where

$$F(x)=\frac{1}{2\pi}\int_0^\infty [S_0(s)-S(s)]\exp(isx)ds+\sum_{n=1}^\infty M_n^2\exp(-\tau_n x), \tag{4.8}$$

and $S_0(s)$ is defined by (2.9).

It should be mentioned that for construction, the fundamental equation, it is sufficient to know the function $F(x)$ in its turn, to find $F(x)$ it is sufficient to know the scattering data. Equation (4.8) plays an important role in the solution of the inverse scattering problem of (1.1)-(1.2) on $(a,\ \infty)$. If equation (4.8) has a unique solution $K(x,t)$ then the potential $q(x)$ can be found from (2.4). As we have just mentioned previously, we prove that the fundamental (4.8) has a unique solution $K(x,t)$ as $a<x<\infty$

Theorem 4.6. *For every fixed $a<x<\infty$ the fundamental equation (4.8) has a unique solution in $L_2(x,\infty)$*

To prove this theorem, it is sufficient to show that the homogenous equations

$$f(t)+\int_s^\infty f(y)F(y+t)dy=0$$

has only the zero solution in $L_2(x,\infty)$ [5].

Furthermore, in this section we have constructed the fundamental equation for the kernel $k(x,t)$ of (2.3) and showed that it has a unique solution at once. Finally, we establish the uniqueness theorem.

Theorem 4.7. *Assume that the condition (1.3) and the formula (1.4) holds. If the scattering data $\left\{S(s);\ -\tau_n^2, M_n, n=\overline{1,m}\right\}$ are known then the function q an ρ are defined uniquely.*

Proof. Consider $S_0(s,a,\nu,\alpha_1,\alpha_2,\beta_1,\beta_2)$ as the scattering function of the problem $y''=s^2\nu^2y, \quad (\alpha_1+s\beta_1)y'(0)-(\alpha_2+s\beta_2)y(0)=0$, then $S_0(s,a,\alpha_1,\alpha_2,\beta_1,\beta_2)$ can be defined by formula (2.9). Taking $\nu=\nu',\ \ \alpha_1=\alpha_1',\ \ \alpha_2=\alpha_2'\ \ \beta_1=\beta_1',\ and\ \ \beta_2=\beta_2'$, thus it is easily seen that $\lim_{s\to\infty} S_0(s,a_1,\alpha_1,\alpha_2,\beta_1,\beta_2)S_0^{-1}(s,a,\alpha_1',\alpha_2',\beta_1',\beta_2')$ does not exist and therefore by $S_0(s,a,\alpha_i,\beta_i)$. The numbers a,ν and α_i,β_i are defined uniquely. Upon using theorem 2.4 we find that $\lim_{s\to\infty} S(s)S_0^{-1}(s,a,\nu,\alpha_i,\beta_i)=1$, where $S(s_0)$ is the scattering function of (1.1)-(1.2). Hence , by using $S(s)$ the numbers a,α_i,β_i are defined uniquely and then the density function ρ is reconstructed uniquely. Now, we have already obtained fundamental equation (4.8) in theorem 4.6 and proved that this equation has a unique solution $K(x,t)$ such that

$$q(x) = -2\frac{d}{dx}K(x,x)$$

holds as $a \prec x \prec \infty$.

Thus the potential function q is defined uniquely as $a < x < \infty$ and hence equation (1.1) can be reconstructed on this interval. Since the functions $F(a,s)$ and $F'(a,s)$ are already defined thus we have the collection of quantities

$$\left\{S(s); -\tau_n^2,\ M_n;\ \ F(a,s);\ \ F'(a,s),\ \ n = \overline{1,m}\right\}.$$

The problem now is using these data to define q when $0 \leq x \leq a$. For this purpose, we construct Weyl's Function [3], [16], [20] for equation on $[0,a]$ by two spectra. Thus taking the following boundary value problems

$$-y'' + q(x)y = y^2\gamma^2 y \tag{4.9}$$

$$y'(0) - \theta y(0) = 0; \qquad y'(a) + HY(a) = 0, \tag{4.10}$$

and

$$-y'' + q(x)y = s^2\gamma^2 y \tag{4.11}$$

$$y'(0) - \theta' y(0) = 0; \quad y'(a) + Hy(a) = 0, \tag{4.12}$$

where $\theta = \dfrac{\alpha_1 + \lambda\beta_{11}}{\alpha_2 + \lambda\beta_2}$, $\theta' = \dfrac{\alpha_1' + \lambda\beta_1'}{\alpha_2' + \lambda\beta_2'}$ and H are real numbers such that $\theta \neq \theta'$.

Denote by $\varphi(x,s)$ and $Z(x,s)$ the solutions of equation (4.9) with the initial conditions $\varphi(0,s) = 1$, $\varphi'(0,s) = \theta$ and $Z(0,s) = 1$, and $Z'(0,S) = \theta'$respectively. Then

$$M(s) = -\frac{Z'(a,s) + HZ(a,s)}{\varphi'(a,s) + H\varphi(a,s)},$$

which is called Weyl's function of the problems (4.9)-(4.10) and (4.11)-(4.12).

Since

$$\varphi(x,s) = \frac{1}{2is}\left[\Delta(s)F(x,-s) - \Delta(-s)F(x,s)\right],$$

and

$$Z(x,s) = \frac{1}{2is}\left[\Delta_1(s)F(x,s) - \Delta_1(-s)F(x,s)\right],$$

where

$$\Delta_1(s) = F'(0,s) - h_1 F(0,s),$$

$$M(s) = \frac{S_1(s)F'(a,s) - F'(a,-s) + H\left[S_1(s)F(a,s) - F(a,-s)\right]}{F'(A,-s) - S(s)F'(a,s) + H\left[f(a,-s) - S(s)F(a,s)\right]},$$

with

$$S_1(s) = \frac{\Delta_1(-s)}{\Delta_1(s)}.$$

The function $M(s)$ is moromorphic such that its poles and zeros coincide with eigenvalues of the problems (4.11)-(4.12), respectively.

Since the functions $S(s)$, $F(a,s)$, and $F'(a,s)$ are defined, $M(s)$ is uniquely defined by this way.

We set up the function

$$\sigma(s) = \lim_{\varsigma \to 0} \frac{1}{a} \int_0^s \operatorname{Im} M(s + i\varsigma) ds$$

Hence the function q is uniquely defined by two spectra on $[0, a]$ from the work of Gelfand-Levitan-Gasymov-Seleeman and William Rundell [14], [20], [21], [22]. Finally, we conclude that equation (1.1) can be reconstructed on the interval $[0, \infty)$ and this completes the proof of the theorem. □

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