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Insertion of A Contra -$\alpha-$Continuous Function between Two Comparable Real-Valued Functions

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Abstract: A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a contra-$\alpha-$continuous function between two comparable real-valued functions.

Keywords: Insertion, Strong binary relation, Semi-open set, Preopen set, $\alpha-$open set, Lower cut set

1 Introduction

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [4]. A subset A of a topological space (X,τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [20], while the concept of a , locally dense, set was introduced by H.H. Corson and E. Michael [4].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [17]. A subset A of a topological space (X,τ) is called *semi-open* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X,τ) is called $\alpha-$*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called $\alpha-$*closed* if its complement is $\alpha-$open or equivalently if A is union of a closed and a nowhere dense set.

We have a set is $\alpha-$open if and only if it is semi-open and preopen.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [19].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [24] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra-continuity. S. Jafari and T. Noiri in [12, 13] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha-$continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 23].

Hence, a real-valued function f defined on a topological space X is called *contra-*$\alpha-$*continuous* (resp. *contra-semi*$-$*continuous* , *contra-precontinuous*) if the preimage of every open subset of $\mathbb{R}$ is $\alpha-$closed (resp. *semi*$-$closed , preclosed) in X[6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-$\alpha-$continuous function between two comparable real-valued functions.

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If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a $c\alpha-$*property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-$\alpha-$continuous function also has property P. If P_1 and P_2 are $c\alpha-$property, the following terminology is used:(i) A space X has the *weak* $c\alpha-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-$\alpha-$continuous function h such that $g \leq h \leq f$.(ii) A space X has the $c\alpha-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a contra-$\alpha-$continuous function h such that $g < h < f$.(iii) A space X has the *weakly* $c\alpha-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 , f has property P_2 and $f - g$ has property P_2, then there exists a contra-$\alpha-$continuous function h such that $g < h < f$.

In this paper, is given a sufficient condition for the weak $c\alpha-$insertion property. Also for a space with the weak $c\alpha-$insertion property, we give a necessary and sufficient condition for the space to have the $c\alpha-$insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the weak insertion of a contra-continuous function has also recently considered by the authors in [21].

2 The Main Result

Before giving a sufficient condition for insertability of a contra-$\alpha-$continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all $\alpha-$open, $\alpha-$closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.
In [7, 18, 22], A^Λ is called the *kernel* of A.

We define the subsets $\alpha(A^\Lambda), \alpha(A^V), p(A^\Lambda), p(A^V), s(A^\Lambda)$ and $s(A^V)$ as follows:

$\alpha(A^\Lambda) = \cap\{O : O \supseteq A, O \in \alpha O(X, \tau)\}$
$\alpha(A^V) = \cup\{F : F \subseteq A, F \in \alpha C(X, \tau)\}$,
$p(A^\Lambda) = \cap\{O : O \supseteq A, O \in pO(X, \tau)\}$,
$p(A^V) = \cup\{F : F \subseteq A, F \in pC(X, \tau)\}$,
$s(A^\Lambda) = \cap\{O : O \supseteq A, O \in sO(X, \tau)\}$ and
$s(A^V) = \cup\{F : F \subseteq A, F \in sC(X, \tau)\}$.

$\alpha(A^\Lambda)$ (resp. $p(A^\Lambda)$, $s(A^\Lambda)$) is called the $\alpha-$*kernel* (resp. *prekernel*, *semi*$-$*kernel*) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.
2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.
3) If $A\ \rho\ B$, then $\alpha(A^\Lambda) \subseteq B$ and $A \subseteq \alpha(B^V)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which $\alpha-$kernel sets are $\alpha-$open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a contra-$\alpha-$continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [15] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = \alpha(H(t_2)^V) \setminus \alpha(H(t_1)^\Lambda)$. Hence $h^{-1}(t_1,t_2)$ is $\alpha-$closed in X, i.e., h is a contra-$alpha-$continuous function on X. ■

The above proof used the technique of theorem 1 in [14].

Theorem 2.2. Let P_1 and P_2 be $c\alpha-$property and X be a space that satisfies the weak $c\alpha-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2.The space X has the $c\alpha-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g,3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g,3^{-n+1})$ are completely separated by contra-$\alpha-$continuous functions.

Proof. Assume that X has the weak $c\alpha-$insertion property for (P_1,P_2). Let g and f be functions such that $g < f, g$ has property P_1 and f has property P_2. By hypothesis there exist lower cut sets $A(f-g,3^{-n+1})$ and there exists a sequence (D_n) such that $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and such that for each $n, X \setminus D_n$ and $A(f-g,3^{-n+1})$ are completely separated by contra-$\alpha-$continuous functions. Let k_n be a contra-$\alpha-$continuous function such that $k_n = 0$ on $A(f-g,3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function k on X be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the properties of contra-$\alpha-$continuous functions, the function k is a contra-$\alpha-$continuous function. Since $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if x is in $A(f-g,3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If x is any point in X, then $x \notin A(f-g,1)$ or for some n,

$$x \in A(f-g,3^{-n+1}) - A(f-g,3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are $c\alpha-$properties, then g_1 has property P_1 and f_1 has property P_2. Since X has the weak $c\alpha-$insertion property for (P_1,P_2), then there exists a contra-$\alpha-$continuous function such that $g_1 \leq h \leq f_1$. Thus $g < h < f$, it follows that X satisfies the $c\alpha-$insertion property for (P_1,P_2). (The technique of this proof is by Katětov[14]).

Conversely, let g and f be functions on X such that g has property P_1, f has property P_2 and $g < f$. By hypothesis, there exists a contra-$\alpha-$continuous function such that $g < h < f$. We follow an idea contained in Lane [16]. Since the constant function 0 has property P_1, since $f - h$ has property P_2, and since X has the $c\alpha-$insertion property for (P_1,P_2), then there exists a contra-$\alpha-$continuous function such that $0 < k < f - h$. Let $A(f-g,3^{-n+1})$ be any lower cut set for $f-g$ and let $D_n = \{x \in X : k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \varnothing$. Since

$$A(f-g,3^{-n+1}) \subseteq \{x \in X : (f-g)(x) \leq 3^{-n+1}\} \subseteq \{x \in X : k(x) \leq 3^{-n+1}\}$$

and since $\{x \in X : k(x) \leq 3^{-n+1}\}$ and $\{x \in X : k(x) \geq 3^{-n+2}\} = X \setminus D_n$ are completely separated by contra-$\alpha-$continuous functions $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$, it follows that for each $n, A(f-g,3^{-n+1})$ and $X \setminus D_n$ are completely separated by contra-$\alpha-$continuous functions.■

3 Applications

The abbreviations $c\alpha c$, cpc and csc are used for contra-$\alpha-$continuous, contra-precontinuous and contra-$semi-$continuous, respectively.

Before stating the consequences of theorems 2.1, 2.2, we suppose that X is a topological space whose $\alpha-$kernel sets are $\alpha-$open.

Corollary 3.1. If for each pair of disjoint preopen (resp. $semi-$open) sets G_1, G_2 of X , there exist $\alpha-$closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the weak $c\alpha-$insertion property for (cpc, cpc) (resp. (csc, csc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and $g \leq f$.If a binary relation ρ is defined by $A \rho B$ in case $p(A^\Lambda) \subseteq p(B^V)$ (resp. $s(A^\Lambda) \subseteq s(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preopen (resp. $semi-$open) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. $semi-$closed) set, it follows that $p(A(f,t_1)^\Lambda) \subseteq p(A(g,t_2)^V)$ (resp. $s(A(f,t_1)^\Lambda) \subseteq s(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint preopen (resp. $semi-$open) sets G_1, G_2, there exist $\alpha-$closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then every contra-precontinuous (resp. contra-$semi-$continuous) function is contra-$\alpha-$continuous.

Proof. Let f be a real-valued contra-precontinuous (resp. contra-$semi-$continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-$\alpha-$continuous function h such that $g = h = f$.■

Corollary 3.3. If for each pair of disjoint preopen (resp. *semi*−open) sets G_1, G_2 of X , there exist $\alpha-$closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the $c\alpha-$insertion property for (cpc, cpc) (resp. (csc, csc)).
Proof. Let g and f be real-valued functions defined on the X, such that f and g are cpc (resp. csc), and $g < f$. Set $h = (f+g)/2$, thus $g < h < f$, and by Corollary 3.2, since g and f are contra-$\alpha-$continuous functions hence h is a contra-$\alpha-$continuous function.■

Corollary 3.4. If for each pair of disjoint subsets G_1, G_2 of X , such that G_1 is preopen and G_2 is *semi*−open, there exist $\alpha-$closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X have the weak $c\alpha-$insertion property for (cpc, csc) and (csc, cpc).
Proof. Let g and f be real-valued functions defined on X, such that g is cpc (resp. csc) and f is csc (resp. cpc), with $g \le f$.If a binary relation ρ is defined by $A \ \rho \ B$ in case $s(A^\Lambda) \subseteq p(B^V)$ (resp. $p(A^\Lambda) \subseteq s(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is a *semi*−open (resp. preopen) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. *semi*−closed) set, it follows that $s(A(f,t_1)^\Lambda) \subseteq p(A(g,t_2)^V)$ (resp. $p(A(f,t_1)^\Lambda) \subseteq s(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1) \ \rho \ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.
Lemma 3.1. The following conditions on the space X are equivalent:

(i) For each pair of disjoint subsets G_1, G_2 of X, such that G_1 is preopen and G_2 is *semi*−open, there exist $\alpha-$closed subsets F_1, F_2 of X such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$.

(ii) If G is a *semi*−open (resp. preopen) subset of X which is contained in a preclosed (resp. *semi*−closed) subset F of X, then there exists an $\alpha-$closed subset H of X such that $G \subseteq H \subseteq \alpha(H^\Lambda) \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where G and F are *semi*−open (resp. preopen) and preclosed (resp. *semi*−closed) subsets of X, respectively. Hence, F^c is a preopen (resp. *semi*−open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint $\alpha-$closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an $\alpha-$open subset containing F_1, we conclude that $\alpha(F_1^\Lambda) \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq \alpha(F_1^\Lambda) \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is preopen and G_2 is *semi*−open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is a preclosed subset of X. Hence by (ii) there exists an $\alpha-$closed set H such that $G_2 \subseteq H \subseteq \alpha(H^\Lambda) \subseteq G_1^c$.
But

$$H \subseteq \alpha(H^\Lambda) \Rightarrow H \cap \alpha((H^\Lambda)^c) = \varnothing$$

and

$$\alpha(H^\Lambda) \subseteq G_1^c \Rightarrow G_1 \subseteq \alpha((H^\Lambda)^c).$$

Furthermore, $\alpha((H^\Lambda)^c)$ is an $\alpha-$closed subset of X. Hence $G_2 \subseteq H, G_1 \subseteq \alpha((H^\Lambda)^c)$ and $H \cap \alpha((H^\Lambda)^c) = \varnothing$. This means that condition (i) holds.■

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is preopen and G_2 is *semi*−open, can be separated by $\alpha-$closed subsets of X then there exists a contra-$\alpha-$continuous function $h : X \to [0,1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.
Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is *semi*−open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is a preclosed subset of X containing the *semi*−open subset G_2 of X,by Lemma 3.1, there exists an $\alpha-$closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq \alpha(H_{1/2}^\Lambda) \subseteq G_1^c.$$

Note that $H_{1/2}$ is also a preclosed subset of X and contains G_2, and G_1^c is a preclosed subset of X and contains the *semi*−open subset $\alpha(H_{1/2}^\Lambda)$ of X. Hence, by Lemma 3.1, there exists $\alpha-$closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq \alpha(H_{1/4}^\Lambda) \subseteq H_{1/2} \subseteq \alpha(H_{1/2}^\Lambda) \subseteq H_{3/4} \subseteq \alpha(H_{3/4}^\Lambda) \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $\alpha-$closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X, 0 \le h(x) \le 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-$\alpha-$continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \le 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are

$\alpha-$closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{\alpha((H_t^{\Lambda})^c) : t > \alpha\}$ hence, every of them is an $\alpha-$closed subset. Consequently h is a contra-$\alpha-$continuous function. ■

Lemma 3.3. Suppose that X is a topological space such that every two disjoint *semi*$-$open and preopen subsets of X can be separated by $\alpha-$closed subsets of X. The following conditions are equivalent:

(i) Every countable convering of *semi*$-$closed (resp. preclosed) subsets of X has a refinement consisting of preclosed (resp. *semi*$-$closed) subsets of X such that for every $x \in X$, there exists an $\alpha-$closed subset of X containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{G_n\}$ of *semi*$-$open (resp. preopen) subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of preclosed (resp. *semi*$-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of *semi*$-$open (resp. preopen) subsets of X with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of *semi*$-$closed (resp. preclosed) subsets of X . By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is an $\alpha-$closed subset of X and $\alpha(V_n^{\Lambda}) \subseteq G_n^c$. By setting $F_n = \alpha((V_n^{\Lambda})^c)$, we obtain a decreasing sequence of $\alpha-$closed subsets of X with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of *semi*$-$closed (resp. preclosed) subsets of X, we set for $n \in \mathbb{N}, G_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{G_n\}$ is a decreasing sequence of *semi*$-$open (resp. preopen) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of preclosed (resp. *semi*$-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.Now we define the subsets W_n of X in the following manner:

W_1 is an $\alpha-$closed subset of X such that $F_1^c \subseteq W_1$ and $\alpha(W_1^{\Lambda}) \cap G_1 = \varnothing$.

W_2 is an $\alpha-$closed subset of X such that $\alpha(W_1^{\Lambda}) \cup F_2^c \subseteq W_2$ and $\alpha(W_2^{\Lambda}) \cap G_2 = \varnothing$, and so on. (By Lemma 3.1, W_n exists).

Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of $\alpha-$closed sets. Moreover, we have

(i) $\alpha(W_n^{\Lambda}) \subseteq W_{n+1}$
(ii) $F_n^c \subseteq W_n$
(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus \alpha(W_{n-1}^{\Lambda})$.

Then since $\alpha(W_{n-1}^{\Lambda}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $\alpha-$closed sets and covers X. Furthermore, $S_i \cap S_j \neq \varnothing$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$$\begin{array}{llllll}
S_1 \cap H_1, & S_1 \cap H_2 & & & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & \\
\vdots & & & & & \\
S_i \cap H_1, & S_i \cap H_2, & S_i \cap H_3, & S_i \cap H_4, & \cdots, & S_i \cap H_{i+1} \\
\vdots & & & & &
\end{array}$$

These sets are $\alpha-$closed sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is an $\alpha-$closed set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $\alpha-$closed sets, and for every point in X we can find an $\alpha-$closed set containing the point that intersects only finitely many elements of that refinement.■

Corollary 3.5. If every two disjoint *semi*$-$open and preopen subsets of X can be separated by $\alpha-$closed subsets of X, and in addition, every countable covering of *semi*$-$closed (resp. preclosed) subsets of X has a refinement that consists of preclosed (resp. *semi*$-$closed) subsets of X such that for every point of X we can find an $\alpha-$closed subset containing that point such that it intersects only a finite number of refining members then X has the weakly $c\alpha-$insertion property for (cpc, csc) (resp. (csc, cpc)).

Proof. Since every two disjoint *semi*$-$open and preopen sets can be separated by $\alpha-$closed subsets of X, therefore by Corollary 3.4, X has the weak $c\alpha-$insertion property for (cpc, csc) and (csc, cpc). Now suppose that f and g are real-valued functions on X with $g < f$, such that g is cpc (resp. csc), f is csc (resp. cpc) and $f - g$ is csc (resp. cpc). For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}.$$

Since $f - g$ is csc (resp. cpc), hence $A(f-g, 3^{-n+1})$ is a *semi*$-$open (resp. preopen) subset of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of *semi*$-$open (resp. preopen) subsets of X and furthermore since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \varnothing$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of preclosed (resp. *semi*$-$closed) subsets of X such that $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma 3.2, the pair $A(f-g, 3^{-n+1})$ and $X \setminus D_n$ of *semi*$-$open (resp. preopen) and preopen (resp. *semi*$-$open) subsets of X can be completely separated by contra-$\alpha-$continuous functions. Hence by Theorem 2.2, there exists a contra-$\alpha-$continuous function h defined on X such that $g < h < f$, i.e., X has the weakly $c\alpha-$insertion property for (cpc, csc) (resp. (csc, cpc)).■

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