Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/050203

Numerical Method for Space- and Time-Fractional Telegraph Equation with Generalized Lagrange Multipliers

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Received: 21 May 2018, Revised: 8 Jul. 2018, Accepted: 4 Aug. 2018
Published online: 1 Apr. 2019

Abstract: In the present article, we applied a novel methodology for solving time-fractional and space-fractional telegraph equation with the help of new Lagrange multiplier. We compare the analytic solution and numerical solution obtained by iterative method using graphs, find analytic solution and plotted absolute error for different value of fractional order that proves obtained numerical solution by employed technique converges very fast to the analytic solution.

Keywords: Fractional variational iteration method, Mittag-Leffler function, generalized Lagrange multiplier, fractional telegraph equation, Caputo fractional derivative.

1 Introduction

In the previous some years, there has been found many uses of fractional calculus in numerous parts of engineering and physical sciences for example in the field of chemical engineering, electromagnetics, viscoelasticity, bioinformatics, control systems, robotics engineering, electrochemistry, acoustics, material science and almost in every part of day-today life by various techniques [1,2,3,4,5,6,7,8,9,10,11,12,13,14]. We have to face many linear and nonlinear problems in modeling of various projects and many times, it is very difficult to find the analytic solutions, so it is necessary to obtain approximate solutions of linear and nonlinear differential equations. We can make models of suspension flows by using parabolic partial differential equations usually and these flows also can be modelled better using hyperbolic partial differential equations using the telegraph equations having parabolic asymptotic.

Communication systems play a vital role in many of the real world problems. A typical engineering problem involves the transmission of signals between two points. A transmission media is part of the circuit and it describes a physical system that directly propagates signal among more than one points. Certainly, all the transmission media have loss in signal. To optimize the transmission media it is needed to sort out signal losses. To evaluate these losses, it is very essential to frame some sort of equations that can figure out these losses efficiently. Telegraph equations get up in the examine of propagation of electrical signals in a cable of transmission line and wave phenomena. Oliver Heaviside found the telegraph equations and developed the transmission line model. This model exhibits that the electromagnetic type waves can be amplified on the wire and wave patterns appear along the transmission line. The telegraph equations are in phrases of voltage and current for a segment of a transmission media and which can be applicable in numerous fields including wave propagation, random walk theory, signal evaluation and so on. In most of the practical situation, these telegraph equations arise in fractional order but not always in integer type order. There have been many analytic and numerical methods are available in the literature to solve fractional order telegraph equations. There are certain cases in which applied information studied in [15,16] seems to be better formed using the telegraph equations in comparison of heat equation. Main application of the telegraph equations lie in the analysis of transmission of signals and in the process of circulation of electrical signals and also in the models like diffusion of reaction [17,18]. Fractional telegraph equation was studied by Momani [19] using Adomain decomposition method, Azab and Gamel [20] using Rothe-Wavelet-Galerkin method, Biazar and Eslami [21] employed differential transform method, Huang [22] with the help of Cauchy problem,

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Chen et al. [23] using separable variables method, Yildirim [24] using homotopy perturbation method. Recently, Das et al. [25] used homotopy analysis method and Kumar [26] applied Homotopy analysis transform method to solution of time- and space-fractional telegraph equation respectively.

Motivated by the literature studied for telegraph equation and its applications in the world of communication system, in this paper, fractional variational iteration method is used to find solution of space-fractional and time-fractional telegraph equation with the help of new generalized Lagrange multipliers. In (1997) J. H. He [27,28,29,30] established a novel techniques, namely, varitational iteration method to find approximate solution of ordinary and partial differential equations. In (2009), Odibat et.al [31] and Molliq et al. [32] used VIM to find numerically solution of fractional Zakharov-Kuznetsov equations. In (2011) J. Lu and in (2012) Sakar et al. used VIM to find the numerical solution of Fornberg–Whitham equation [33,34], Prakash et al. studied coupled Burger's equation [35] and biological population model [36] in (2015) and (2016), respectively.

Consider, fractional telegraph equation in space [26] and fractional telegraph equation in time [25] as:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}=\frac{\partial^2 u(x,t)}{\partial t^2}+a\frac{\partial u(x,t)}{\partial t}+bu^m(x,t)+g(x,t),\ t\geq 0,\ 0<\alpha\leq 1, \tag{1}$$

which can be obtained by putting $R[u]$ and $N[u]$ terms in the following expression

$$D_x^{2\alpha}u(x,t)+R[u(x,t)]+N[u(x,t)]=g(x,t), \tag{2}$$

and

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}=\frac{\partial^2 u(x,t)}{\partial x^2}+a\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+bu^m(x,t)+g(x,t),\ t\geq 0,\ 0<\alpha\leq 1, \tag{3}$$

which can be obtained by putting $R[u]$ and $N[u]$ terms in the following expression

$$D_t^{2\alpha}u(x,t)+R[u(x,t)]+N[u(x,t)]=g(x,t). \tag{4}$$

where $R[u]$ denotes linear term and $N[u]$ denotes nonlinear term, a & b are constants, m is natural number and $g(x,t)$ is any source function.

Sevimlican [37] used the variational iteration method to given telegraph equation (1) and proposed correctional functional and Lagrange multiplier λ as

$$\begin{cases} u_{n+1}(x,t)=u_n(x,t)+\int_0^x\lambda(\frac{\partial^{2\alpha}u_n(\xi,t)}{\partial\xi^{2\alpha}}-\frac{\partial^2u_n(\xi,t)}{\partial t^2}-a\frac{\partial u_n(\xi,t)}{\partial t}-bu_n^m(\xi,t)-g(\xi,t))d\xi,\\ \lambda=\xi-x. \end{cases} \tag{5}$$

The present article represents a novel mode to find Lagrange multiplier λ, called generalized Lagrange multiplier for arbitrary value of fractional order $\alpha, 0<\alpha\leq 1$, causes an improvement in the correctional functional for equation (1), it can be derived as

$$\begin{cases} u_{n+1}(x,t)=u_n(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\lambda(\frac{\partial^{2\alpha}u_n(\xi,t)}{\partial\xi^{2\alpha}}-\frac{\partial^2u_n(\xi,t)}{\partial t^2}-a\frac{\partial u_n(\xi,t)}{\partial t}-bu_n^m(\xi,t)-g(\xi,t))(d\xi)^\alpha,\\ \lambda=\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}. \end{cases} \tag{6}$$

and for equation (3), it can be derived as

$$\begin{cases} u_{n+1}(x,t)=u_n(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^t\lambda(\frac{\partial^{2\alpha}u_n(x,\xi)}{\partial\xi^{2\alpha}}-\frac{\partial^2u_n(x,\xi)}{\partial x^2}-a\frac{\partial^\alpha u_n(x,\xi)}{\partial \xi^\alpha}-bu_n^m(x,\xi)-g(x,\xi))(d\xi)^\alpha,\\ \lambda=\frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}. \end{cases} \tag{7}$$

2 Preliminaries

Definition 2.. Riemann-Liouville fractional integral of order $\mu>0$, [37,38,39,40] of a function $f\in C_\beta,\ \beta\geq -1$ is defined as:

$I^\mu f(t)=\frac{1}{\Gamma(\mu)}\int_0^t\frac{f(\tau)}{(t-\tau)^{1-\mu}}d\tau=\frac{1}{\Gamma(1+\mu)}\int_0^t f(\tau)\,(d\tau)^\mu,$

$I^0 f(t)=f(t).$

Definition 2.2. Caputo fractional derivative of $f, f\in C_{-1}^m,\ m\in IN\cup\{0\}$ [37,38,39,40],

$$f(x)=\begin{cases}[I^{m-\mu}f^{(m)(t)}],\ m-1<\mu<m,\ m\in IN,\\ \frac{d^m f(t)}{dt^m},\ \mu=m.\end{cases} \tag{8}$$

$a.I_t^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s)ds, \alpha, t > 0.$
$b.D_t^{\alpha} u(x,t) = I_t^{m-\alpha} \frac{\partial^{\mu}(x,t)}{\partial t^m},\ m-1 < \alpha < m.$
$c.I^{\mu} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\mu)} t^{\gamma+\mu}.$
Definition 2.3. The Mittag-Leffler function $E_{\alpha} z$ for $\alpha > 0$ in the form of a series, domain in the complex plane [37,38,39,40] is $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)},\ \alpha > 0,\ z \in C.$

3 Generalized Lagrange multipliers for space and time-fractional telegraph equation

Theorem 3.1. If the correction functional for equation (2) is determined with the help of Riemann-Liouville integration

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \lambda(x,\xi)(D_{\xi}^{2\alpha} u_n + R(u_n) + N[u_n] - f(\xi,t))(d\xi)^{\alpha}, \tag{9}$$

where both $R[u_n]$ and $N[u_n]$ are restricted variations, generalized Lagrange multiplier can be found out as $\lambda = \frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}$.
Integrating and applying variation on both sides of equation (2), we get
$\delta u_{n+1} = \delta u_n + \delta(\lambda u^{\alpha})|_{\xi=x} - (\delta(\lambda^{(\alpha)} u_n))|_{\xi=x} + \frac{1}{\Gamma(1+\alpha)} \delta \int_0^x \lambda^{2\alpha} u_n (d\xi)^{\alpha}.$
From this we can obtain the equations
$1 - \lambda^{(\alpha)}|_{\xi=x} = 0, \lambda^{2\alpha} = 0$ and $\lambda|_{\xi=x} = 0.$
Finally, from these equations, we can obtain generalized Lagrange multiplier $\lambda = \frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}$, which complete the proof of this theorem.
Theorem 3.2. If the correction functional for equation (4) is determined with the help of Riemann-Liouville integration

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(t,\xi)(D_{\xi}^{2\alpha} u_n + R(u_n) + N[u_n] - f(x,\xi))(d\xi)^{\alpha}, \tag{10}$$

where both $R[u_n]$ and $N[u_n]$ are restricted variations, generalized Lagrange multiplier can be determined as $\lambda = \frac{(\xi-t)^{\alpha}}{\Gamma(1+\alpha)}$.
Integrating and applying variation on both sides of equation (4), we get

$$\delta u_{n+1} = \delta u_n + \delta(\lambda u^{\alpha})|_{\xi=t} - (\delta(\lambda^{(\alpha)} u_n))|_{\xi=t} + \frac{1}{\Gamma(1+\alpha)} \delta \int_0^t \lambda^{2\alpha} u_n (d\xi)^{\alpha}. \tag{11}$$

From this we can obtain the equations
$1 - \lambda^{(\alpha)}|_{\xi=t} = 0, \lambda^{2\alpha} = 0$ and $\lambda|_{\xi=t} = 0.$
Finally, from these equations, we can obtain generalized Lagrange multiplier $\lambda = \frac{(\xi-t)^{\alpha}}{\Gamma(1+\alpha)}$, which complete the proof of this theorem.

4 The Proposed FVIM for fractional Telegraph equation

In this segment, we represent the solution process for fractional Telegraph equation by fractional variational iteration method with generalized Lagrange multiplier.

4.1 Basic idea for space-fractional Telegraph equation

To find the process of solution for space-fractional telegraph equation (1) by using the proposed fractional variational iteration method, we take ensuing fractional differential equation as:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = \frac{\partial^2 u(x,t)}{\partial t^2} + a \frac{\partial u(x,t)}{\partial t} + b u^m(x,t) + g(x,t), t \geq 0,\ 0 < \alpha \leq 1, \tag{12}$$

where $a,\ b$ and m are given constants, $g(x,t)$ is integrable given function.
According to the FVIM, a correction functional [27] can be built for above equation as:

$$u_{n+1}(x,t) = u_n + \frac{1}{\Gamma(1+\alpha)} \int_0^x \lambda(\frac{\partial^{2\alpha} u_n}{\partial \xi^{2\alpha}} - \frac{\partial^2 \tilde{u}_n}{\partial t^2} - a\frac{\partial \tilde{u}_n}{\partial t} - b\tilde{u}_n^m + \tilde{g}(x,\xi))(d\xi)^{\alpha}. \tag{13}$$

Now using the theory of variation, Lagrangr multiplier λ satisfy equations as
$1-\lambda^{(\alpha)}|_{\xi=x}=0, \lambda^{2\alpha}=0$. Solving these equations, we get $\lambda=\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}$ and a new correction functional

$$u_{n+1}(x,t)=u_n(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}(\frac{\partial^{2\alpha}u_n(\xi,t)}{\partial\xi^{2\alpha}}-\frac{\partial^2 u_n(\xi,t)}{\partial t^2}-a\frac{\partial u_n(\xi,t)}{\partial t}-bu_n^m(\xi,t)-g(\xi,t))(d\xi)^\alpha. \quad (14)$$

In this way we obtain sequence of iteration $u_n, n\geq 0$ with the help of λ, a generalized Lagrange's multiplier, which can be found with the help of theory of variation. The functions u_n, is restricted variation so $\delta\tilde{u}_n=0$. Subsequently, first we find the generalized Lagrange multiplier λ, which is found easily with the help of integration. In same process we get, sequences $u_{n+1}(x,t), n\geq 0$, using this we get solution as $u(x,t)=\lim_{n\to\infty}u_n(x,t)$.

4.2 Basic idea for time-fractional Telegraph equation

To find the process of solution for time-fractional telegraph equation (3) with the help of fractional variational iteration method, we take ensuing fractional differential equation as:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}=\frac{\partial^2 u(x,t)}{\partial x^2}+a\frac{\partial^\alpha u}{\partial t^\alpha}+bu^m(x,t)+g(x,t),\ t\geq 0,\ 0<\alpha\leq 1, \quad (15)$$

where a,b and m are any arbitrary constants, $g(x,t)$ is any source function.
Using FVIM, a new correct functional [20] can be constructed for above equation as

$$u_{n+1}(x,t)=u_n+\frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda(\xi,t)(\frac{\partial^{2\alpha}u_n}{\partial\xi^{2\alpha}}-\frac{\partial^2\tilde{u}_n}{\partial x^2}-a\frac{\partial^\alpha u_n}{\partial t^\alpha}-b\tilde{u}_n^m+\tilde{g}(x,\xi))(d\xi)^\alpha. \quad (16)$$

Now using the theory of variational λ must satisfy
$1-\lambda^{(\alpha)}|_{\xi=t}=0, \lambda^{2\alpha}=0$.
From these equations, we obtain $\lambda=\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}$ and we build correction functional as:

$$u_{n+1}(x,t)=u_n(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}(\frac{\partial^{2\alpha}u_n(x,\xi)}{\partial\xi^{2\alpha}}-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}-a\frac{\partial^\alpha u_n}{\partial t^\alpha}-bu_n^m(x,\xi)-g(x,\xi))(d\xi)^\alpha. \quad (17)$$

We build the sequence of iterations $u_n,\ n\geq 0$ with the help of λ, which is generalized Lagrange's multiplier, which can be found out by the theory of variation. The functions u_n, is restricted variation so $\delta\tilde{u}_n=0$. So, first we take the generalized Lagrange multiplier λ, which is obtained easily using integration. In this process we obtain, iterations $u_{n+1}(x,t),\ n\geq 0$, using this we obtain solution as $u(x,t)=\lim_{n\to\infty}u_n(x,t)$.

5 Test Examples

In the present segment, we used proposed method to some test examples.
Example 5.1. Considering the ensuing telegraph equation [26] as

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}=\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+u,\ t\geq 0,\ 0<\alpha\leq 1, \quad (18)$$

with the given initial conditions $u(0,t)=e^{-t},\ u(x,0)=e^x,\ u_x(0,t)=e^{-t},\ 0<x<1$ and exact solution for standard case when $\alpha=1$ is $u(x,t)=e^{x-t}$.
Now using the given initial condition, we assume
$u_0(x,t)=(x+1)e^{-t}$.

Now by applying fractional variational iteration method

$$u_1(x,t)=u_0(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left(\frac{\partial^{2\alpha}u_0}{\partial\xi^{2\alpha}}-\frac{\partial^2u_0}{\partial t^2}-\frac{\partial u_0}{\partial t}-u_0\right)(d\xi)^\alpha,$$

$$=u_0(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}(-\xi-1)(d\xi)^\alpha,$$

$$=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}\right).$$

$$u_2(x,t)=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right).$$

$$u_3(x,t)=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}+\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}+\frac{x^{6\alpha+1}}{\Gamma(6\alpha+2)}\right).$$

$$u_n(x,t)=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}+\ldots+\frac{x^{2\alpha n}}{\Gamma(2\alpha n+1)}+\frac{x^{2\alpha n+1}}{\Gamma(2\alpha n+2)}\right).$$

when $\alpha=1$, we get
$u_n(x,t)=e^{-t}(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots+\frac{x^{2n}}{2n!}+\frac{x^{2n+1}}{(2n+1)!})$,
and in this process as $n\to\infty$, $u(x,t)=\lim_{n\to\infty}u_n(x,t)=e^{x-t}$, which is the exact solution of the basic telegraph equation (18).

Example 5.2. Considering the ensuing telegraph equation [26] as

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}=\frac{\partial^2u}{\partial t^2}+4\frac{\partial u}{\partial t}+4u,\ t\ge 0,\ 0<\alpha\le 1, \tag{19}$$

with the given initial conditions
$u(0,t)=e^{-2t}+1, u(x,0)=e^{2x}+1, u_x(0,t)=2, u_t(x,0)=-2,\ 0<x<1,$
and exact solution for standard case when $\alpha=1$ is $u(x,t)=e^{-2t}+e^{2x}$.
Now using the given initial condition, we can take initially $u_0(x,t)=e^{-2t}+2x+1$.
Now by using fractional variational iteration method

$$u_1(x,t)=u_0(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left(\frac{\partial^{2\alpha}u_0}{\partial\xi^{2\alpha}}-\frac{\partial^2u_0}{\partial t^2}-4\frac{\partial u_0}{\partial t}-4u_0\right)(d\xi)^\alpha,$$

$$=u_0(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}(-\xi-1)(d\xi)^\alpha,$$

$$=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}\right).$$

$$u_2(x,t)=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right).$$

$$u_3(x,t)=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}+\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}+\frac{x^{6\alpha+1}}{\Gamma(6\alpha+2)}\right).$$

$$u_n(x,t)=e^{-t}\left(1+x+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}+\ldots+\frac{x^{2\alpha n}}{\Gamma(2\alpha n+1)}+\frac{x^{2\alpha n+1}}{\Gamma(2\alpha n+2)}\right).$$

when $\alpha=1$, we get
$u_n(x,t)=e^{-2t}2x^2+1+2x+x^2+\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{1}{15}(4x^5)+\frac{4x^6}{45}+\ldots+\frac{(2x)^n}{n!}$,
and in this process as $n\to\infty, u(x,t)=\lim_{n\to\infty}u_n(x,t)=e^{-2t}+e^{2x}$,which is the exact solution of the standard telegraph equation (19).

Example 5.3. Considering the ensuing telegraph equation [26] as

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}=\frac{\partial^2u}{\partial t^2}+\frac{\partial u}{\partial t}+u-x^2-t+1, t\ge 0,\ 0<\alpha\le 1, \tag{20}$$

with the given conditions $u(0,t)=t,\ u(x,0)=x^2,\ u_x(0,t)=0,\ 0<x<1$
and $u(x,t)=t+x^2$ is an analytic solution of standard telegraph equation i.e. for $\alpha=1$.

Now using the given initial condition, we assume $u_0(x,t)=t$.
Now by using fractional variational iteration method

$$u_1(x,t)=u_0(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}(\frac{\partial^{2\alpha}u_0}{\partial\xi^{2\alpha}}-\frac{\partial^2u_0}{\partial t^2}-\frac{\partial u_0}{\partial t}-u_0+\xi^2+t-1)(d\xi)^\alpha,$$
$$=u_0(x,t)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}(-2+\xi^2)(d\xi)^\alpha,$$
$$=t+\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}.$$
$$u_2(x,t)=t+\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{2x^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{2x^{4\alpha+2}}{\Gamma(4\alpha+3)}.$$
$$u_3(x,t)=t+\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}+\frac{2x^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{2x^{4\alpha+2}}{\Gamma(4\alpha+3)}+\frac{2x^{6\alpha}}{\Gamma(6\alpha+1)}-\frac{2x^{6\alpha+2}}{\Gamma(6\alpha+3)}.$$
$$u_n(x,t)=t+\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}+...+\frac{2x^{2\alpha n}}{\Gamma(2\alpha n+1)}-\frac{2x^{2\alpha n+2}}{\Gamma(2\alpha n+3)}.$$

when $\alpha=1$, we get $u_n(x,t)=t+x^2$,
and in this process as $n\to\infty, u(x,t)=\underset{n\to\infty}{Lim}u_n(x,t)=t+x^2$, which is the exact solution of the standard telegraph equation (20).

Example 5.4. Consider the ensuing time fractional-order linear Telegraph equation as

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\frac{\partial^\alpha u}{\partial t^\alpha}+u=\frac{\partial^2u}{\partial x^2}, \tag{21}$$

with given initial conditions $u(x,0)=e^x, u_t(x,0)=-2e^x$.
and $u(x,t)=e^{x-2t}$ is an analytic solution of standard telegraph equation i.e. for $\alpha=1$. Now using the given initial condition, we can take initially $u_0(x,t)=e^x(1-\frac{2t^\alpha}{\Gamma(\alpha+1)})$.
Now by using fractional variational iteration method

$$u_1(x,t)=u_0(x,t)+\frac{1}{\Gamma(\alpha+1)}\int_0^t(\frac{\partial^{2\alpha}u}{\partial\xi^{2\alpha}}+2\frac{\partial^\alpha u}{\partial\xi^\alpha}+u-\frac{\partial^2u}{\partial x^2})(d\xi)^\alpha,$$
$$=e^x(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}\frac{+(-2)^2t^{2\alpha}}{\Gamma(2\alpha+1)}).$$
$$u_2(x,t)=e^x(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{8t^{3\alpha}}{\Gamma(3\alpha+1)}).$$
$$u_3(x,t)=e^x(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}\frac{+(-2)^2t^{2\alpha}}{\Gamma(2\alpha+1)})+\frac{(-2)^3t^{3\alpha}}{\Gamma(3\alpha+1)}.$$
$$u_n(x,t)=e^x(1+\frac{(-2)t^\alpha}{\Gamma(\alpha+1)}\frac{+(-2)^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(-2)^3t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{(-2)^4t^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{(-2)^5t^{5\alpha}}{\Gamma(5\alpha+1)}+...+\frac{(-2)^{n+1}t^{3\alpha}}{\Gamma(\alpha(n+1)+1)}).$$

and in this process as $n\to\infty$, $u(x,t)=\underset{n\to\infty}{\mathrm{Lim}}u_n(x,t)=e^xE_\alpha(-2t^\alpha)$, where $E_\alpha(-2t^\alpha)$ is Mittag Leffler function. which is the analytic solution of the telegraph equation (21).

6 Numerical results and discussion

Table 1-4, illustrates the comparative study of absolute error between approximate and analytic solution for various values of α with different order of approximations. From these table, we can conclude that absolute error is decreasing as the order of approximation is increased and at the tenth order approximation absolute error is negligible. Also, we can conclude that as the order of fractional derivative α increases, numerical solution is of high accuracy. Table 5-6, shows the comparison among the proposed method, exact solution, HATM and VIM for different values of x and t at $\alpha=1$. From these two tables, we can conclude easily that the applied technique is better than VIM and agree with HATM.

Table 1: Absolute error comparison between analytic and numerical solution for various values of α for Eq. (18) for Ex.5.1.

α	$\rightarrow$	0.50	0.75	1.00	0.50	0.75	1.00	0.50	0.75	1.00
x	t		$\lvert u-u_5\rvert$			$\lvert u-u_8\rvert$			$\lvert u-u_{10}\rvert$	
0.1	0.1	0	0	0	0	0	0	0	0	0
0.2	0.2	0	0	0	0	0	0	0	0	0
0.3	0.3	2.2×10^{-16}	0	0	0	0	0	2.2×10^{-16}		0
0.4	0.4	0	0	0	0	0	0	0	0	0
0.5	0.5	0	0	0	0	0	0	0	0	0
0.6	0.6	0	0	2.2×10^{-16}	0	0	0	0	0	2.2×10^{-16}
0.7	0.7	2.2×10^{-16}	0	0	2.2×10^{-16}	0	0	2.2×10^{-16}	0	0
0.8	0.8	0	2.2×10^{-16}	0	0	0	0	0	2.2×10^{-16}	0
0.9	0.9	4.4×10^{-16}	2.2×10^{-16}	0	0	0	0	4.4×10^{-16}	2.2×10^{-16}	
1.0	1.0	4.4×10^{-16}	0	0	0	0	0	4.4×10^{-16}	0	0

Table 2: Absolute error comparison between analytic and numerical solution for various values of α for Eq. (19) for Ex.5.2.

α	$\rightarrow$	0.50	0.75	1.00	0.50	0.75	1.00	0.50	0.75	1.00
x	t		$\lvert u-u_5\rvert$			$\lvert u-u_8\rvert$			$\lvert u-u_{10}\rvert$	
0.1	0.1	0	0	0	0	0	0	0	0	0
0.2	0.2	0	0	0	0	0	0	0	0	0
0.3	0.3	0	0	0	0	0	0	0	0	0
0.4	0.4	0	0	0	0	0	0	0	0	0
0.5	0.5	0	0	2.2×10^{-16}	0	0	2.2×10^{-16}	0	0	2.2×10^{-16}
0.6	0.6	6.6×10^{-16}	0	0	0	0	0	0	0	0
0.7	0.7	3.3×10^{-15}	0	0	0	2.2×10^{-16}	0	0	0	0
0.8	0.8	1.6×10^{-14}	0	0	2.2×10^{-16}	0	0	0	0	0
0.9	0.9	6.6×10^{-14}	0	2.2×10^{-16}	0	0	0	2.2×10^{-16}	0	2.2×10^{-16}
1.0	1.0	2.3×10^{-13}	0	0	0	0	0	0	0	0

Table 3: Absolute error comparison between analytic and numerical solution for various values of α for Eq. (20)for Ex.5.3.

α	$\rightarrow$	0.50	0.75	1.00	0.50	0.75	1.00	0.50	0.75	1.00
x	t		$\lvert u-u_5\rvert$			$\lvert u-u_8\rvert$			$\lvert u-u_{10}\rvert$	
0.1	0.1	0	0	0	0	0	0	0	0	0
0.2	0.2	9.9×10^{-16}	5.5×10^{-17}	0	1.1×10^{-16}	5.5×10^{-17}	0	0	5.5×10^{-17}	0
0.3	0.3	9.1×10^{-14}	1.1×10^{-16}	0	0	0	0	0	1.1×10^{-16}	0
0.4	0.4	2.1×10^{-12}	0	0	0	0	0	0	0	0
0.5	0.5	2.5×10^{-11}	0	0	2.2×10^{-16}	0	0	2.2×10^{-16}	0	1.1×10^{-16}
0.6	0.6	1.9×10^{-10}	0	0	0	0	1.1×10^{-16}	4.4×10^{-16}	0	1.1×10^{-16}
0.7	0.7	1.0×10^{-9}	0	0	4.4×10^{-16}	0	0	4.4×10^{-16}	0	0
0.8	0.8	4.5×10^{-9}	8.8×10^{-16}	1.1×10^{-16}	0	0	0	0	0	0
0.9	0.9	1.6×10^{-8}	4.0×10^{-15}	0	1.3×10^{-15}	0	2.2×10^{-16}	0	0	0
1.0	1.0	5.4×10^{-8}	2.4×10^{-14}	0	6.2×10^{-15}	0	0	8.8×10^{-16}	0	0

Table 4: Absolute error comparison between analytic and numerical solution for various values of α for Eq. (21) for Ex.5.4.

α	$\rightarrow$	0.50	0.75	1.00	0.50	0.75	1.00	0.50	0.75	1.00
x	t		$\lvert u-u_5\rvert$			$\lvert u-u_8\rvert$			$\lvert u-u_{10}\rvert$	
0.1	0.1	6.59×10^{-11}	0	0	0	0	0	0	1.73×10^{-18}	0
0.2	0.2	2.89×10^{-9}	2.22×10^{-16}	2.22×10^{-16}	0	0	0	0	0	0
0.3	0.3	2.63×10^{-8}	8.22×10^{-15}	0	0	0	0	0	0	0
0.4	0.4	1.26×10^{-7}	8.42×10^{-14}	2.22×10^{-16}	0	0	0	0	0	0
0.5	0.5	4.21×10^{-7}	5.26×10^{-13}	0	2.78×10^{-17}	0	0	0	0	0
0.6	0.6	1.13×10^{-6}	2.35×10^{-12}	2.22×10^{-16}	0	0	0	0	0	0
0.7	0.7	2.61×10^{-6}	8.35×10^{-12}	0	0	0	0	2.78×10^{-17}	0	0
0.8	0.8	5.37×10^{-6}	2.5×10^{-11}	2.22×10^{-16}	5.55×10^{-17}	0	0	5.55×10^{-17}	0	0
0.9	0.9	1.01×10^{-5}	6.57×10^{-11}	0	0	0	0	0	2.78×10^{-17}	0
1.0	1.0	1.79×10^{-5}	1.56×10^{-10}	4.44×10^{-16}	0	0	0	0	0	0

Table 5: Absolute error comparison between analytic and numerical solution at $\alpha = 1$ for Eq. (18) for Ex.5.1.

x	t	$Exact\ sol.$	$Proposed\ Method$	$HATM$[26]	VIM[37]
0.5	0.5	1.0	0.9999858350626777	0.9999858350626777	1.074222660600422
0.5	1.0	0.6065306597126334	0.6065220682438546	0.6065220682438546	0.6515489790122343
1.0	0.5	1.6487212707001282	1.6477416255526542	1.6477416255526542	1.925734844587611
1.0	1.0	1.0	0.9994058151824183	0.9994058151824183	1.1680172257193293
1.5	0.5	2.718281828459045	2.7061692168897262	2.7061692168897262	3.260576148033305
1.5	1.0	1.6487212707001282	1.6413746004141463	1.6413746004141463	1.9776394021099177

Table 6: Absolute error comparison between analytic and numerical solution at $\alpha = 1$ for Eq. (20) for Ex.5.3.

x	t	$Exact\ sol.$	$Proposed\ Method$	$HATM$[26]	VIM[37]
0.5	0.5	0.75	0.7499565972222222	0.7499565972222222	0.73953125
0.5	1.0	1.25	1.2499565972222222	1.2499565972222222	1.2395312500000002
1.0	0.5	1.5	1.4972222222222222	1.4972222222222222	1.3299999999999998
1.0	1.0	2.0	1.9972222222222222	1.9972222222222222	1.83
1.5	0.5	2.75	2.718359375	2.718359375	1.86828125
1.5	1.0	3.25	3.218359375	3.218359375	2.36828125

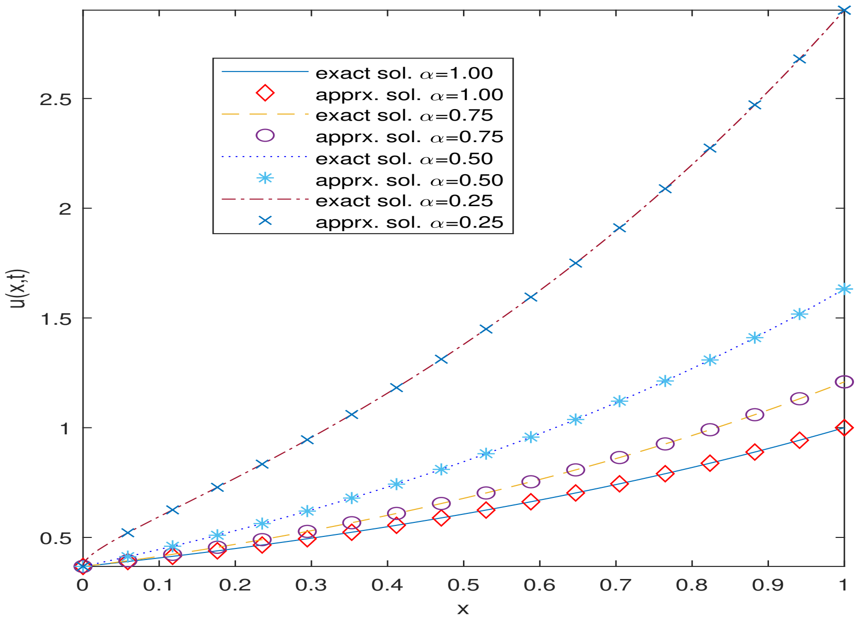

Fig. 1: Plot of tenth approximate solution for different values of α at $t = 1$ for telegraph equation (18) for Ex.5.1.

The numerical results obtained in table 5-6, demonstrate that FVIM is very well technique for this problem, even if we use second order iteration to find approximate solution. Fig. 1 illustrates the comparison between analytic and numerical solution for different value of fractional order α, here $u(x,t)$ increase as we decrease the value of α. Fig. 2 illustrates the comparison between analytic and numerical solution for different values of α, here $u(x,t)$ strictly increases as α decreases. Fig. 3 illustrates the comparison between analytic and numerical solution for different values of α, here $u(x,t)$ increase as we decrease the value of α and finally fig. 4 displays the comparative study of exact and approximate solution for various values of α, here $u(x,t)$ decrease as we decrease the value of α. It can be observed here that only tenth order

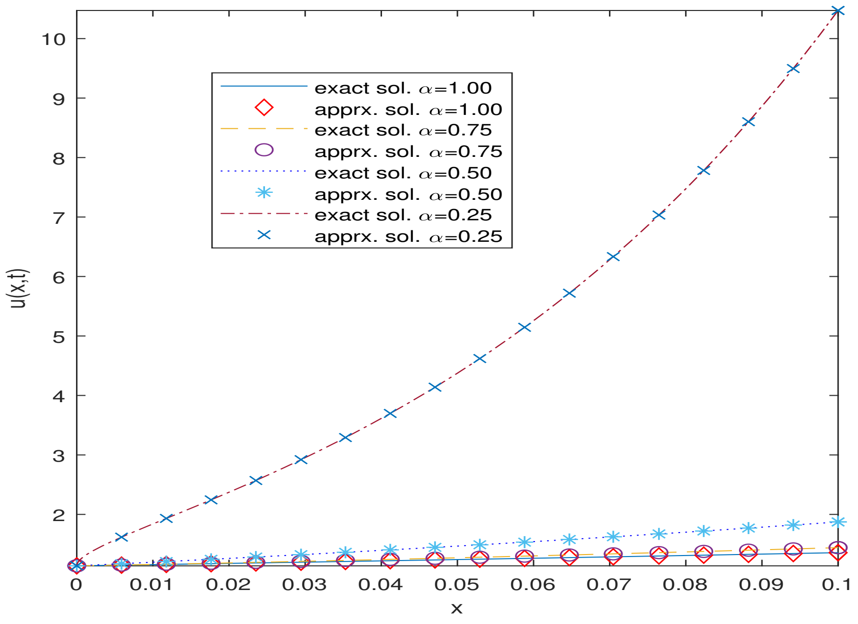

Fig. 2: Plot of tenth approximate solution for different values of α at $t = 1$ for telegraph equation (19) for Ex.5.2.

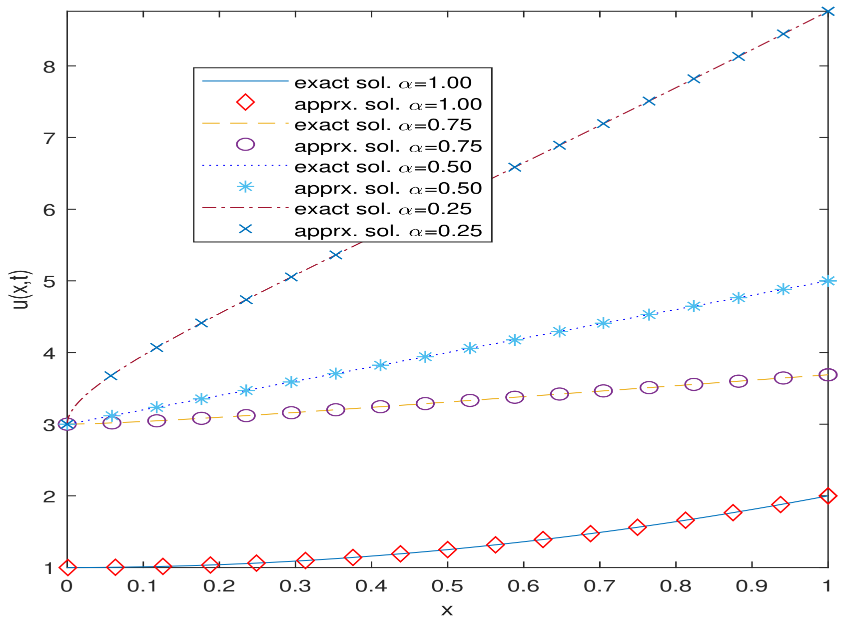

Fig. 3: Plot of tenth approximate solution for different values of α at $t = 1$ for telegraph equation (20) for Ex.5.3.

approximate solution is used in comparison of exact and approximate solution and its accuracy can be further improved by using higher-order approximation. Figs. 5-8, display the plot of absolute error, for Ex. 5.1-5.4, which is negligible, shows the accuracy of proposed technique.

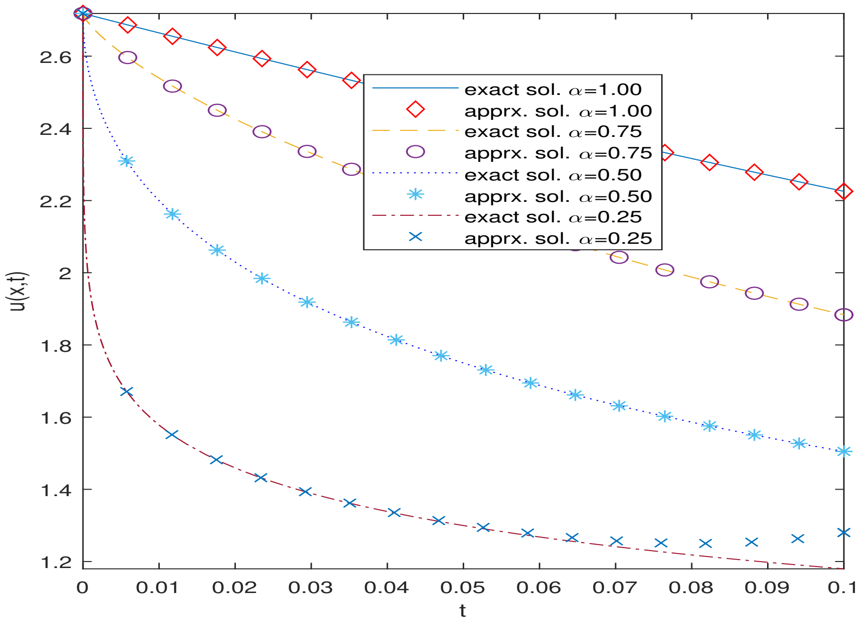

Fig. 4: Plot of tenth approximate solution for different values of α at $t = 1$ for telegraph equation (21) for Ex.5.4

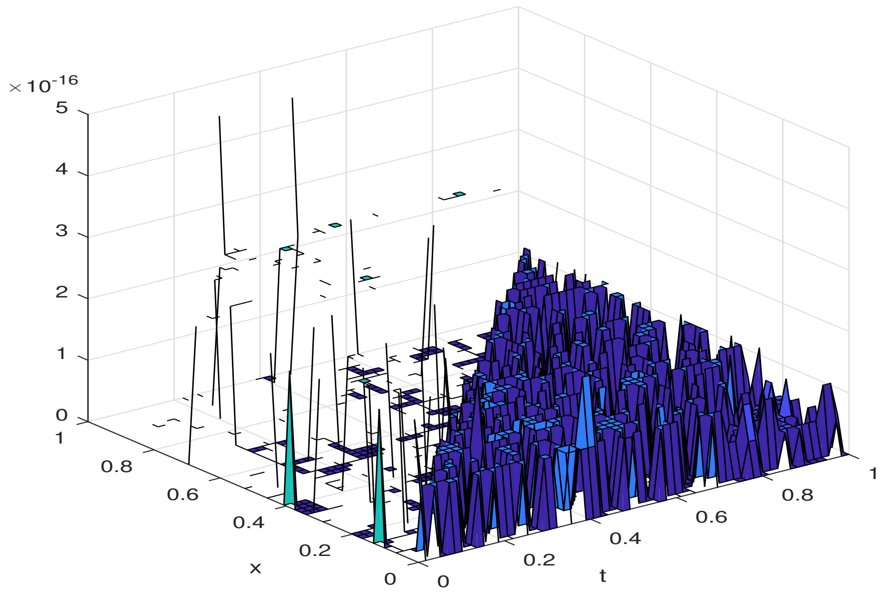

Fig. 5: Absolute error plot for the standard case $\alpha = 1$ for telegraph equation (18) for Ex.5.1.

7 Conclusion

In present article, Fractional variational iteration method (FVIM) with generalized Lagrange multipliers has been applied effectively for numerically solution of space- and time-fractional telegraph equations. By the use of generalized Lagrange

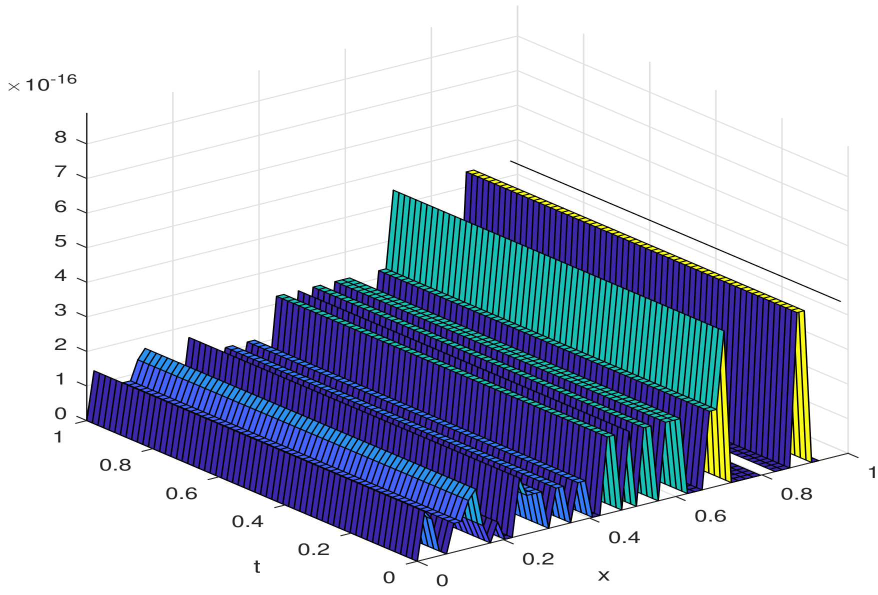

Fig. 6: Absolute error plot for the standard case $\alpha = 1$ for telegraph equation (19) for Ex.5.2.

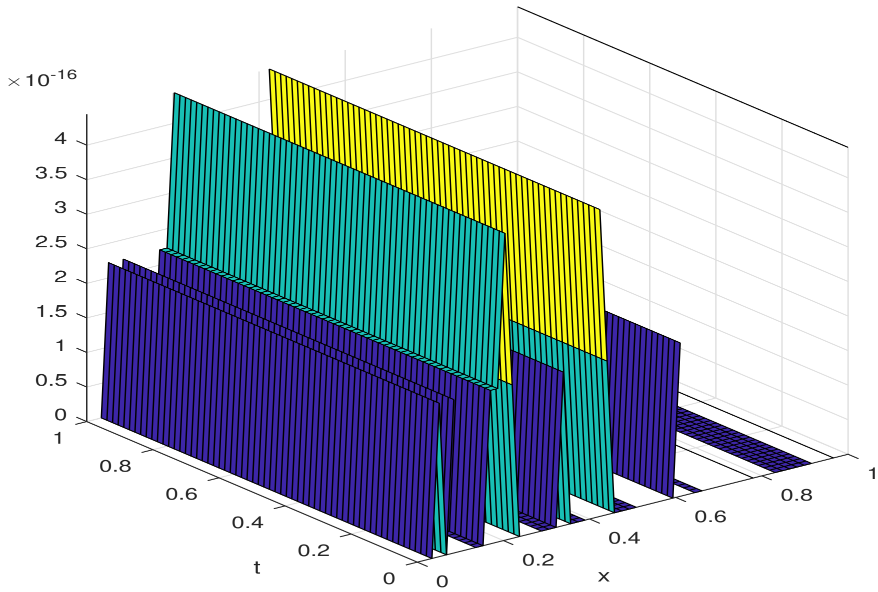

Fig. 7: Absolute error plot for the standard case $\alpha = 1$ for telegraph equation (20) for Ex.5.3.

multiplier, it is apparently seen that the numerical solution obtained in the form of rapidly convergent series with easily computable components and converges to the exact solution. So, FVIM is a very simple, efficient and powerful numerical

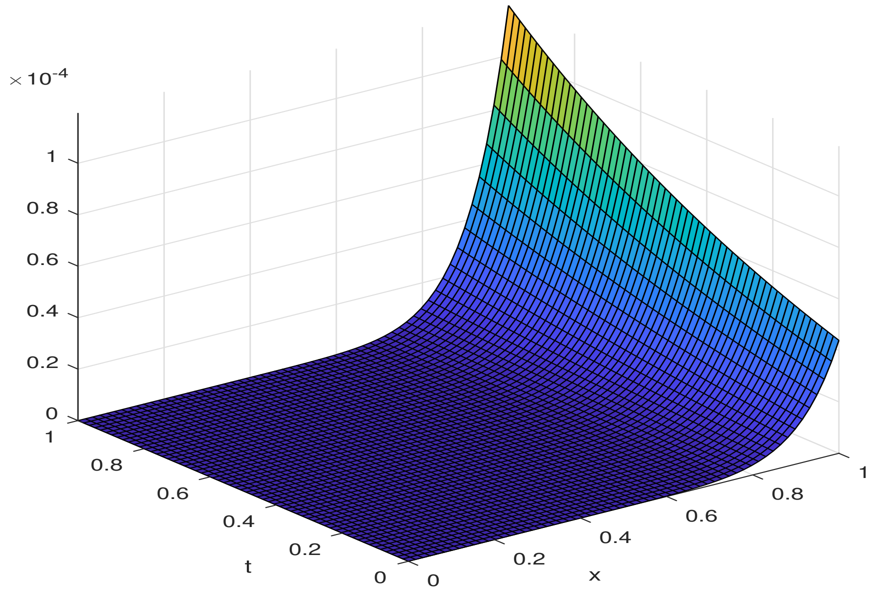

Fig. 8: Absolute error plot for the standard case $\alpha = 1$ for telegraph equation (21) for Ex.5.4.

method for solving any type of space- and time-fractional partial differential equations arising in various fields of science and engineering.

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