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Mixed Conformable and Iterated Fractional Approximation by Choquet Integrals

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Abstract: We analyse the approximation of the unit operator by positive sublinear operators of quantitative mixed conformable and iterated fractional type, including a precise Choquet integral interpretation of these operators. First of all, we examine the mixed conformable and iterated fractional rate of convergence of both the Bernstein-Kantorovich-Choquet operator and the Bernstein-Durrweyer-Choquet polynomial Choquet-integral operator. Next we use a representation theorem due to Schmeidler (1986) [1] in order to study some very general comonotonic positive sublinear operators. Finally, we give an approximation using some very general direct Choquet-integral form positive sublinear operators. The approximations of mixed conformable and iterated fractional type are given as inequalities which involve the modulus of continuity of the approximated function and its mixed conformable and iterated fractional derivatives.

Keywords: Jackson type inequality, Choquet integral, modulus of continuity, Bernstein-Kantorovich-Choquet operators, Bernstein-Durrmeyer-Choquet operators, mixed conformable and iterated fractional derivative.

1 Introduction

Capacities and what is now called the Choquet integral were introduced by Choquet (1953) [2]. They have been applied to potential theory and statistical mechanics, and have inspired studies of non-additive measure theory. Economists have also become interested in these ideas, partly due to the study of Shapley (1953) [3] in cooperative game theory., and so have decision theorists after the work of Schmeidler (1989) [4] using them in a model of choice with non-additive beliefs. Using Choquet capacities gives stronger results than using probability measures.

The Choquet integral has now discovered many applications in the world of economics and finance: for example in insurance, portfolio problems, and decision making under uncertainty and risk. Motivation for this study may also come from the foundations of subjective probability and Bayesian decision theory.

In the current work, inspired by the Choquet integral and its immense significance, we study related approximations by positive sublinear operators, specifically in the mixed conformable and iterated fractional sense.

The manuscript organization is given below. In Section 2 some Choquet integral properties are introduced. Section 3 deals with the conformable calculus. Some properties of the fractional calculus in the Caputo sense are given in Section 4. In Section 5 we discuss about representations of positive sublinear operators by Choquet integrals. Section 6 presents the main results of the manuscript.

2 Definitions and Properties - I

We make [5]

Definition 1. *Let $\Omega \neq \varnothing$ and let $\mathscr{C}$ be a σ-algebra of subsets in Ω.*

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(1) [6] $\mu_1 : \mathscr{C} \to [0,+\infty]$ *denotes a monotone set function if* $\mu_1(\varnothing) = 0$ *and* $\mu_1(A_1) \leq \mu_1(B_1)$ *for all* $A_1, B_1 \in \mathscr{C}$*, with* $A_1 \subset B_1$*. Also,* μ_1 *is called submodular if*

$$\mu_1(A_1 \cup B_1) + \mu_1(A_1 \cap B_1) \leq \mu_1(A_1) + \mu_1(B_1), \text{ for all } A_1, B_1 \in \mathscr{C}.$$

μ_1 *is called bounded if* $\mu_1(\Omega) < +\infty$ *and normalized if* $\mu_1(\Omega) = 1$.

(2) (see, e.g., [6], p. 233, or [2]) If μ_1 *is a monotone set function on* $\mathscr{C}$ *and if* $f_1 : \Omega \to \mathbb{R}$ *is* $\mathscr{C}$*-measurable (that is, for any Borel subset* $B_1 \subset \mathbb{R}$ *it follows* $f_1^{-1}(B_1) \in \mathscr{C}$*), then for any* $A_1 \in \mathscr{C}$*, the expression of Choquet integral becomes*

$$(C)\int_{A_1} f_1 d\mu_1 = \int_0^{+\infty} \mu_1(F_\beta(f_1) \cap A_1)\, d\beta + \int_{-\infty}^0 [\mu_1(F_\beta(f_1) \cap A_1) - \mu_1(A_1)]\, d\beta,$$

where we utilized the notation $F_\beta(f_1) = \{\omega \in \Omega : f_1(\omega) \geq \beta\}$*. Notice that if* $f_1 \geq 0$ *on* A_1*, then in the above formula we get* $\int_{-\infty}^0 = 0$.

The function f_1 *will be called Choquet integrable on* A_1 *if* $(C)\int_{A_1} f_1 d\mu_1 \in \mathbb{R}$.

Some Choquet integral properties are given below.

*Remark.*If $\mu_1 : \mathscr{C} \to [0,+\infty]$ is a monotone set function, then we have:

(1) For all $a \geq 0$ we have $(C)\int_{A_1} af_1 d\mu_1 = a \cdot (C)\int_{A_1} f d\mu_1$ (if $f_1 \geq 0$ then see, e.g., [6], Theorem 11.2, (5), p. 228 and if f_1 is arbitrary sign, then see, e.g., [7], p. 64, Proposition 5.1, (ii)).

(2) For all $c \in \mathbb{R}$ and f_1 of arbitrary sign, we have (see, e.g., [6], pp. 232-233, or [7], p. 65) $(C)\int_{A_1}(f_1 + c)\, d\mu_1 = (C)\int_{A_1} f_1 d\mu_1 + c \cdot \mu_1(A_1)$.

If μ_1 is submodular too, then for all f_1, g_1 of arbitrary sign and lower bounded, we have (see, e.g., [7], p. 75, Theorem 6.3)

$$(C)\int_{A_1}(f_1 + g_1)\, d\mu_1 \leq (C)\int_{A_1} f_1 d\mu_1 + (C)\int_{A_1} g_1 d\mu_1.$$

(3) If $f_1 \leq g_1$ on A_1 then $(C)\int_{A_1} f_1 d\mu_1 \leq (C)\int_{A_1} g_1 d\mu_1$ (see, e.g., [6], p. 228, Theorem 11.2, (3) if $f_1, g_1 \geq 0$ and p. 232 if f_1, g are of arbitrary sign).

(4) Let us consider $f_1 \geq 0$. If $A_1 \subset B_1$ then $(C)\int_{A_1} f_1 d\mu_1 \leq (C)\int_{B_1} f_1 d\mu_1$. Besides, if μ_1 is finitely subadditive, then

$$(C)\int_{A_1 \cup B_1} f_1 d\mu_1 \leq (C)\int_{A_1} f_1 d\mu_1 + (C)\int_{B_1} f_1 d\mu_1.$$

(5) We report that $(C)\int_{A_1} 1 \cdot d\mu_1(t) = \mu_1(A_1)$.

(6) The expression of $\mu_1(A_1) = \gamma(M(A_1))$, where $\gamma : [0,1] \to [0,1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and M represents a probability measure (or only finitely additive) on a σ-algebra on Ω (that is, $M(\varnothing) = 0$, $M(\Omega) = 1$ and M is countably additive), gives simple examples of normalized, monotone and submodular set functions (see [7], p. 16, Example 2.1). μ_1 is called distorsions of countably normalized, additive measures (or distorted measures). For a simple example, we can take $\gamma(t) = \frac{2t}{1+t}$, $\gamma(t) = \sqrt{t}$.

(7) If μ_1 is a countably additive bounded measure, then $(C)\int_{A_1} f_1 d\mu_1$ becomes to the usual Lebesgue type integral (see, e.g., [7], p. 62, or [6], p. 226).

(8) If $f_1 \geq 0$, then $(C)\int_{A_1} f_1 d\mu_1 \geq 0$.

(9) Let $\mu_1 = \sqrt{M}$, where M is the Lebesgue measure on $[0,+\infty)$, then μ_1 is a monotone and submodular set function, furthermore μ_1 is strictly positive, see [8].

We have

Definition 2.*([9]) For the* $\Omega \neq \varnothing$*, the power set* $\mathscr{P}(\Omega)$ *means the family of all subsets of* Ω.

(1) A function $\lambda : \Omega \to [0,1]$ *with the property* $\sup\{\lambda(s) : s \in \Omega\} = 1$*, denotes the possibility distribution on* Ω.

(2) $P : \mathscr{P}(\Omega) \to [0,1]$ *is called possibility measure, if it fulfills* $P(\varnothing) = 0$*,* $P(\Omega) = 1$*, and* $P(\cup_{i \in I} A_i) = \sup\{P(A_i) : i \in I\}$ *for all* $A_i \subset \Omega$*, and any* I*, an at most countable family of indices. Note that if* $A_1, B_1 \subset \Omega$*,* $A_1 \subset B_1$*, then the last property implies* $P(A_1) \leq P(B_1)$ *and that* $P(A_1 \cup B_1) \leq P(A_1) + P(B_1)$.

Any possibility distribution λ *on* Ω*, induces the possibility measure* $P_\lambda : \mathscr{P}(\Omega) \to [0,1]$, $P_\lambda(A_1) = \sup\{\lambda(s) : s \in A_1\}$*,* $A_1 \subset \Omega$*. If* $f_1 : \Omega \to \mathbb{R}_+$*, then the possibilistic integral of* f *on* $A_1 \subset \Omega$ *with respect to* P_λ *becomes* $(Pos)\int_{A_1} f_1 dP_\lambda = \sup\{f_1(t)\lambda(t) : t \in A_1\}$ *([9]).*

3 Definitions and Properties - II

We present [10] some basic definitions of conformable calculus [11].

Definition 3.[10] *Consider $a,b \in \mathbb{R}$. The expression of the left conformable derivative starting from a of $f_1 : [a,\infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is written as*

$$\left(T_\alpha^a f_1\right)(t) = \lim_{\varepsilon \to 0} \frac{f_1\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f_1(t)}{\varepsilon}. \tag{1}$$

If $\left(T_\alpha^a f_1\right)(t)$ exists on (a,b), then

$$\left(T_\alpha^a f_1\right)(a) = \lim_{t \to a+} \left(T_\alpha^a f_1\right)(t). \tag{2}$$

The form of the right conformable derivative of order $0 < \alpha \leq 1$ terminating at b of $f_1 : (-\infty, b] \to \mathbb{R}$ becomes

$$\left({}_\alpha^b T f_1\right)(t) = -\lim_{\varepsilon \to 0} \frac{f_1\left(t + \varepsilon (b-t)^{1-\alpha}\right) - f_1(t)}{\varepsilon}. \tag{3}$$

If $\left({}_\alpha^b T f_1\right)(t)$ exists on (a,b), then

$$\left({}_\alpha^b T f_1\right)(b) = \lim_{t \to b-} \left({}_\alpha^b T f_1\right)(t). \tag{4}$$

If f_1 is differentiable then we conclude

$$\left(T_\alpha^a f_1\right)(t) = (t-a)^{1-\alpha} f_1'(t), \tag{5}$$

and

$$\left({}_\alpha^b T f_1\right)(t) = -(b-t)^{1-\alpha} f_1'(t). \tag{6}$$

Denote as

$$\left(I_\alpha^a f_1\right)(t) = \int_a^t (x-a)^{\alpha-1} f_1(x)\, dx, \tag{7}$$

and

$$\left({}^b I_\alpha f_1\right)(t) = \int_t^b (b-x)^{\alpha-1} f_1(x)\, dx, \tag{8}$$

these are the forms of the left and right conformable integrals of order $0 < \alpha \leq 1$.

For the higher order case we conclude:

Definition 4.[10] *We consider $\alpha \in (n, n+1]$, and set $\beta = \alpha - n$. Then, the left conformable derivative starting from a of a function $f_1 : [a,\infty) \to \mathbb{R}$ of order α, where $f_1^{(n)}(t)$ exists and it reads as*

$$\left(\mathbf{T}_\alpha^a f_1\right)(t) = \left(T_\beta^a f_1^{(n)}\right)(t), \tag{9}$$

The expression of the right conformable derivative of order α terminating at b of $f_1 : (-\infty, b] \to \mathbb{R}$, where $f_1^{(n)}(t)$ exists is given by

$$\left({}_\alpha^b \mathbf{T} f_1\right)(t) = (-1)^{n+1} \left({}_\beta^b T f_1^{(n)}\right)(t). \tag{10}$$

For $\alpha = n+1$ we have $\beta = 1$ and $\mathbf{T}_{n+1}^a f_1 = f_1^{(n+1)}$.

If n is odd, then ${}_{n+1}^b \mathbf{T} f_1 = -f_1^{(n+1)}$, and if n is even, then ${}_{n+1}^b \mathbf{T} f_1 = f_1^{(n+1)}$.
When $n = 0$ (or $\alpha \in (0,1]$), then $\beta = \alpha$, and (9), (10) collapse to $\{(1)$- $(4)\}$, respectively.

Lemma 1.[10] *Consider $f_1 : (a,b) \to \mathbb{R}$ be continuously differentiable and $0 < \alpha \leq 1$. Thus, for all $t > a$ we conclude*

$$I_\alpha^a T_\alpha^a (f_1)(t) = f_1(t) - f_1(a). \tag{11}$$

We recall that

Definition 5.[10] *For $\alpha \in (n, n+1]$, then form of the left fractional integral of order α starting at a becomes*

$$\left(\mathbf{I}_{\alpha}^{a} f_1\right)(t) = \frac{1}{n!} \int_{a}^{t} (t-x)^{n} (x-a)^{\beta-1} f_1(x)\, dx. \tag{12}$$

Similarly [12] the expression of the right fractional integral of order α terminating at b reads as

$$\left({}^{b}\mathbf{I}_{\alpha} f_1\right)(t) = \frac{1}{n!} \int_{t}^{b} (x-t)^{n} (b-x)^{\beta-1} f_1(x)\, dx. \tag{13}$$

We need

Proposition 1.[10] *Consider $\alpha \in (n, n+1]$ and $f_1 : [a, \infty) \to \mathbb{R}$ be $(n+1)$ times continuously differentiable for $t > a$. Thus, for all $t > a$ we conclude*

$$\mathbf{I}_{\alpha}^{a} \mathbf{T}_{\alpha}^{a} (f_1)(t) = f_1(t) - \sum_{k=0}^{n} \frac{f_1^{(k)}(a)(t-a)^{k}}{k!}. \tag{14}$$

In addition we have

Proposition 2.[12] *Consider $\alpha \in (n, n+1]$ and $f_1 : (-\infty, b] \to \mathbb{R}$ be $(n+1)$ times continuously differentiable for $t < b$. Thus, for all $t < b$ we report*

$$-{}^{b}\mathbf{I}_{\alpha}\, {}_{a}^{b}\mathbf{T}(f_1)(t) = f_1(t) - \sum_{k=0}^{n} \frac{f_1^{(k)}(b)(t-b)^{k}}{k!}. \tag{15}$$

If $n = 0$ or $0 < \alpha \leq 1$, then (see also [10])

$${}^{b}I_{\alpha}\, {}_{\alpha}^{b}T(f_1)(t) = f_1(t) - f_1(b). \tag{16}$$

As a result we derive

Theorem 1.[12] *Let $\alpha \in (n, n+1]$ and $f_1 \in C^{n+1}([a,b])$, $n \in \mathbb{N}$. Then*

1)

$$f_1(t) - \sum_{k=0}^{n} \frac{f_1^{(k)}(a)(t-a)^{k}}{k!} = \frac{1}{n!} \int_{a}^{t} (t-x)^{n} (x-a)^{\beta-1} \left(\mathbf{T}_{\alpha}^{a}(f_1)\right)(x)\, dx, \tag{17}$$

and

2)

$$f_1(t) - \sum_{k=0}^{n} \frac{f_1^{(k)}(b)(t-b)^{k}}{k!} = -\frac{1}{n!} \int_{t}^{b} (b-x)^{\beta-1} (x-t)^{n} \left({}_{\alpha}^{b}\mathbf{T}(f_1)\right)(x)\, dx, \tag{18}$$

$\forall\, t \in [a,b]$.

We have

Remark.[12] Consider $\alpha \in (n, n+1]$ and $f_1 \in C^{n+1}([a,b])$, $n \in \mathbb{N}$. Thus ($\beta := \alpha - n$, $0 < \beta \leq 1$)

$$\left(\mathbf{T}_{\alpha}^{a}(f_1)\right)(x) = \left(T_{\beta}^{\alpha} f_1^{(n)}\right)(x) = (x-a)^{1-\beta} f_1^{(n+1)}(x), \tag{19}$$

and

$$\left({}_{\alpha}^{b}\mathbf{T}(f_1)\right)(x) = (-1)^{n+1} \left({}_{\beta}^{b}T f_1^{(n)}\right)(x) =$$

$$(-1)^{n+1}(-1)(b-x)^{1-\beta} f_1^{(n+1)}(x) = (-1)^{n}(b-x)^{1-\beta} f_1^{(n+1)}(x). \tag{20}$$

We conclude

$$\left(\mathbf{T}_{\alpha}^{a}(f_1)\right)(x),\ \left({}_{\alpha}^{b}\mathbf{T}(f_1)\right)(x) \in C([a,b]).$$

Also, we recall

$$\left(\mathbf{T}_{\alpha}^{a}(f_1)\right)(a) = \left({}_{\alpha}^{b}\mathbf{T}(f_1)\right)(b) = 0, \tag{21}$$

when $0 < \beta < 1$, i.e. when $\alpha \in (n, n+1)$.

If $f_1^{(k_1)}(a) = 0$, $k_1 = 1, \ldots, n$, then

$$f_1(t) - f_1(a) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} (\mathbf{T}_\alpha^a (f_1))(x)\, dx, \tag{22}$$

$\forall\, t \in [a,b]$.

If $f_1^{(k_1)}(b) = 0$, $k_1 = 1, \ldots, n$, then

$$f_1(t) - f_1(b) = -\frac{1}{n!} \int_t^b (b-x)^{\beta-1} (x-t)^n \left({}_\alpha^b\mathbf{T}(f_1) \right)(x)\, dx, \tag{23}$$

$\forall\, t \in [a,b]$.

For $f_1 \in C([a,b])$, $\delta > 0$, denote by

$$\omega_1(f_1, \delta) = \sup_{\substack{x,y \in [a,b]: \\ |x-y| \le \delta}} |f_1(x) - f_1(y)|$$

the (first) modulus of continuity of f_1.

We have

Theorem 2.[12] *Consider $\alpha \in (n, n+1)$, $n \in \mathbb{N}$, and $f_1 \in C^{n+1}([a,b])$, $x \in [a,b]$ and $f_1^{(k_1)}(x) = 0$, $k_1 = 1, \ldots, n$. Denote*

$$\omega_1({}^x\mathbf{T}_\alpha f_1, \delta) := \max \left\{ \omega_1(\mathbf{T}_\alpha^x f_1, \delta)_{[x,b]}, \omega_1({}_\alpha^x\mathbf{T} f_1, \delta)_{[a,x]} \right\}. \tag{24}$$

Then, over $[a,b]$, we conclude

$$|f_1(\cdot) - f_1(x)| \le \frac{\omega_1({}^x\mathbf{T}_\alpha f_1, \delta)}{\prod_{j=0}^{n-1} (\alpha - j)} \left[\frac{|\cdot - x|^\alpha}{(\alpha - n)} + \frac{|\cdot - x|^{\alpha+1}}{(\alpha+1)\delta} \right], \quad \delta > 0. \tag{25}$$

We have

Definition 6.*Here $C_+([a,b]) := \{f_1 : [a,b] \to \mathbb{R}_+$, continuous functions$\}$. Let $L_N : C_+([a,b]) \to C_+([a,b])$, operators, $\forall$ $N \in \mathbb{N}$, such that*

(1)

$$L_N(\alpha f_1) = \alpha L_N(f_1), \ \forall\, \alpha \ge 0, \ \forall\, f \in C_+([a,b]), \tag{26}$$

(2) if $f_1, g_1 \in C_+([a,b]) : f_1 \le g_1$, then

$$L_N(f_1) \le L_N(g_1), \ \ \forall\, N \in \mathbb{N}, \tag{27}$$

(3)

$$L_N(f_1 + g_1) \le L_N(f_1) + L_N(g_1), \ \ \forall\, f_1, g_1 \in C_+([a,b]). \tag{28}$$

Here $\{L_N\}_{N \in \mathbb{N}}$ denotes a positive sublinear operators.

We have

Theorem 3.[12] *Consider $\alpha \in (n, n+1)$, $n \in \mathbb{N}$, and $f_1 \in C^{n+1}([a,b], \mathbb{R}_+)$, $x \in [a,b]$ and $f_1^{(k)}(x) = 0$, $k = 1, \ldots, n$. Let $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall\, N \in \mathbb{N}$, be positive sublinear operators, such that $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$. Thus,*

$$|L_N(f_1)(x) - f_1(x)| \le \frac{\omega_1({}^x\mathbf{T}_\alpha f_1, \delta)}{\prod_{j=0}^{n-1} (\alpha - j)} \left[\frac{L_N(|\cdot - x|^\alpha)(x)}{(\alpha - n)} + \frac{L_N\left(|\cdot - x|^{\alpha+1}\right)(x)}{(\alpha+1)\delta} \right], \tag{29}$$

$\delta > 0$.

Besides we conclude

Theorem 4.[12] *Consider* $f_1 \in C^1([a,b])$, $\alpha \in (0,1)$, $x \in [a,b]$. *Let us denote*

$$\omega_1({}^{x}T_{\alpha} f_1, \delta) := \max\left\{\omega_1(T_{\alpha}^{x} f_1, \delta)_{[x,b]}, \omega_1({}_{\alpha}^{x}T f_1, \delta)_{[a,x]}\right\}, \quad \delta > 0. \tag{30}$$

Then over $[a,b]$ *we conclude*

$$|f_1(\cdot) - f_1(x)| \le \omega_1({}^{x}T_{\alpha} f_1, \delta)\left[\frac{|\cdot - x|^{\alpha}}{\alpha} + \frac{|\cdot - x|^{\alpha+1}}{(\alpha+1)\delta}\right], \quad \delta > 0. \tag{31}$$

We have

Theorem 5.[12] *Consider* $f_1 \in C^1([a,b], \mathbb{R}_+)$, $\alpha \in (0,1)$, *and let* $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall\, N \in \mathbb{N}$, *be positive sublinear operators, such that* $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$. *Then*

$$|L_N(f_1)(x) - f_1(x)| \le \omega_1({}^{x}T_{\alpha} f_1, \delta)\left[\frac{L_N(|\cdot - x|^{\alpha})(x)}{\alpha} + \frac{L_N\left(|\cdot - x|^{\alpha+1}\right)(x)}{(\alpha+1)\delta}\right], \tag{32}$$

$\forall\, N \in \mathbb{N}$, $\forall\, x \in [a,b]$, $\delta > 0$.

Also we will use

Theorem 6.[12] *Let us consider* $f_1 \in C^1([a,b], \mathbb{R}_+)$, $\alpha \in (0,1)$, $x \in [a,b]$. *Let* $L_N : C_+([a,b]) \to C_+([a,b])$, $\forall\, N \in \mathbb{N}$ *be positive sublinear operators, such that* $L_N(1) = 1$, *and* $L_N\left(|\cdot - x|^{\alpha+1}\right)(x) > 0$, $\forall\, N \in \mathbb{N}$. *Thus,*

$$|L_N(f_1)(x) - f_1(x)| \le$$

$$\frac{(2\alpha+1)}{\alpha(\alpha+1)}\omega_1\left({}^{x}T_{\alpha} f_1, \left(L_N\left(|\cdot - x|^{\alpha+1}\right)(x)\right)^{\frac{1}{\alpha+1}}\right)\left(L_N\left(|\cdot - x|^{\alpha+1}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}, \tag{33}$$

$\forall\, N \in \mathbb{N}$.

4 Definitions and Properties - III

Below some basic properties of Caputo integral and derivatives are given.

*Remark.*Let $f_1 : [a,b] \to \mathbb{R}$ such that $f_1' \in L_\infty([a,b])$, $x_0 \in [a,b]$, $0 < \alpha < 1$, the left Caputo fractional derivative of order α is defined as follows

$$\left(D_{*x_0}^{\alpha} f_1\right)(x_0) = \frac{1}{\Gamma(1-\alpha)}\int_{x_0}^{x}(x-t)^{-\alpha} f_1'(t)\,dt, \tag{34}$$

where Γ is the gamma function for all $x_0 \le x \le b$.

We observe that

$$\left|\left(D_{*x_0}^{\alpha} f_1\right)(x)\right| = \frac{1}{\Gamma(1-\alpha)}\int_{x_0}^{x}(x-t)^{-\alpha}\left|f_1'(t)\right| dt$$

$$\le \frac{\|f_1'\|_\infty}{\Gamma(1-\alpha)}\int_{x_0}^{x}(x-t)^{-\alpha}\,dt = \frac{\|f_1'\|_\infty}{\Gamma(1-\alpha)}\frac{(x-x_0)^{1-\alpha}}{(1-\alpha)} = \frac{\|f_1'\|_\infty (x-x_0)^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{35}$$

I.e.

$$\left|\left(D_{*x_0}^{\alpha} f_1\right)(x)\right| \le \frac{\|f_1'\|_\infty (x-x_0)^{1-\alpha}}{\Gamma(2-\alpha)} \le \frac{\|f_1'\|_\infty (b-x_0)^{1-\alpha}}{\Gamma(2-\alpha)} < +\infty, \tag{36}$$

$\forall\, x \in [x_0, b]$.

We conclude

$$\left(D_{*x_0}^{\alpha} f_1\right)(x_0) = 0. \tag{37}$$

We define $\left(D_{*x_0}^{\alpha} f_1\right)(x) = 0$, for $a \le x < x_0$.

Let $n \in \mathbb{N}$, we denote the iterated fractional derivative $D_{*x_0}^{n\alpha} = D_{*x_0}^{\alpha} D_{*x_0}^{\alpha} ... D_{*x_0}^{\alpha}$ (n-times).

Let us suppose that

$$D_{*x_0}^{k\alpha} f_1 \in C([x_0, b]), \; k = 0, 1, ..., n+1; \; n \in \mathbb{N}, 0 < \alpha < 1.$$

By [13], [14], pp. 156-158, we have the following generalized fractional Caputo type Taylor's formula:

$$f_1(x) = \sum_{i=0}^{n} \frac{(x-x_0)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{*x_0}^{i\alpha} f_1\right)(x_0) + \tag{38}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (x-t)^{(n+1)\alpha-1} \left(D_{*x_0}^{(n+1)\alpha} f\right)(t)\, dt,$$

$\forall\, x \in [x_0, b]$.

Based on the above (37) and (38), we conclude

$$f_1(x) - f_1(x_0) = \sum_{i=2}^{n} \frac{(x-x_0)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{*x_0}^{i\alpha} f_1\right)(x_0) + \tag{39}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (x-t)^{(n+1)\alpha-1} \left(D_{*x_0}^{(n+1)\alpha} f_1\right)(t)\, dt,$$

$\forall\, x \in [x_0, b]$, $0 < \alpha < 1$.

In case of $\left(D_{*x_0}^{i\alpha} f_1\right)(x_0) = 0$, $i = 2, 3, ..., n+1$, we get

$$f_1(x) - f_1(x_0) =$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (x-t)^{(n+1)\alpha-1} \left(\left(D_{*x_0}^{(n+1)\alpha} f_1\right)(t) - \left(D_{*x_0}^{(n+1)\alpha} f_1\right)(x_0) \right) dt, \tag{40}$$

$\forall\, x \in [x_0, b]$, $0 < \alpha < 1$.

We have

*Remark.*Let $f_1 : [a, b] \to \mathbb{R}$ such that $f_1' \in L_\infty([a, b])$, $x_0 \in [a, b]$, $0 < \alpha < 1$, the right Caputo fractional derivative of order α becomes

$$\left(D_{x_0-}^{\alpha} f_1\right)(x_0) = \frac{-1}{\Gamma(1-\alpha)} \int_{x}^{x_0} (z-x)^{-\alpha} f_1'(z)\, dz, \tag{41}$$

$\forall\, x \in [a, x_0]$.

We observe that

$$\left|\left(D_{x_0-}^{\alpha} f_1\right)(x)\right| = \frac{1}{\Gamma(1-\alpha)} \int_{x}^{x_0} (z-x)^{-\alpha} \left|f_1'(z)\right| dz \le$$

$$\frac{\|f_1'\|_\infty}{\Gamma(1-\alpha)} \left(\int_{x}^{x_0} (z-x)^{-\alpha} dz \right) = \frac{\|f_1'\|_\infty}{\Gamma(1-\alpha)} \frac{(x_0-x)^{1-\alpha}}{(1-\alpha)} = \frac{\|f_1'\|_\infty}{\Gamma(2-\alpha)} (x_0-x)^{1-\alpha}. \tag{42}$$

That is

$$\left|\left(D_{x_0-}^{\alpha} f_1\right)(x)\right| \le \frac{\|f_1'\|_\infty}{\Gamma(2-\alpha)} (x_0-x)^{1-\alpha} \le \frac{\|f_1'\|_\infty}{\Gamma(2-\alpha)} (x_0-a)^{1-\alpha} < \infty, \tag{43}$$

$\forall\, x \in [a, x_0]$.

We conclude

$$\left(D_{x_0-}^{\alpha} f_1\right)(x_0) = 0. \tag{44}$$

We define $\left(D_{x_0-}^{\alpha} f_1\right)(x) = 0$, for $x_0 < x \le b$.

For $n \in \mathbb{N}$, denote the iterated fractional derivative $D_{x_0-}^{n\alpha} = D_{x_0-}^{\alpha} D_{x_0-}^{\alpha} ... D_{x_0-}^{\alpha}$ (n-times).

In [15], we proved the following right generalized fractional Taylor's formula: Assume that

$$D_{x_0-}^{k\alpha} f_1 \in C([a, x_0]), \text{ for } k = 0, 1, ..., n+1, 0 < \alpha < 1.$$

Then

$$f_1(x)=\sum_{i=0}^{n}\frac{(x_0-x)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{x_0-}^{i\alpha}f\right)(x_0)+$$
$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^{x_0}(z-x)^{(n+1)\alpha-1}\left(D_{x_0-}^{(n+1)\alpha}f_1\right)(z)\,dz, \tag{45}$$

$\forall\, x\in[a,x_0]$.

Based on (44) and (45), we conclude

$$f_1(x)-f_1(x_0)=\sum_{i=2}^{n}\frac{(x_0-x)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{x_0-}^{i\alpha}f_1\right)(x_0)+$$
$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^{x_0}(z-x)^{(n+1)\alpha-1}\left(D_{x_0-}^{(n+1)\alpha}f_1\right)(z)\,dz, \tag{46}$$

$\forall\, x\in[a,x_0]$, $0<\alpha<1$.

In case of $\left(D_{x_0-}^{i\alpha}f_1\right)(x_0)=0$, for $i=2,3,...,n+1$, we get

$$f_1(x)-f_1(x_0)=$$
$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^{x_0}(z-x)^{(n+1)\alpha-1}\left(\left(D_{x_0-}^{(n+1)\alpha}f_1\right)(z)-\left(D_{x_0-}^{(n+1)\alpha}f_1\right)(x_0)\right)dz, \tag{47}$$

$\forall\, x\in[a,x_0]$, $0<\alpha<1$.

We need

Definition 7. *Let $D_{x_0}^{(n+1)\alpha}f_1$ denote any of $D_{*x_0}^{(n+1)\alpha}f_1$, $D_{x_0-}^{(n+1)\alpha}f_1$, and $\delta>0$. We have*

$$\omega_1\left(D_{x_0}^{(n+1)\alpha}f_1,\delta\right)=\max\left\{\omega_1\left(D_{*x_0}^{(n+1)\alpha}f_1,\delta\right)_{[x_0,b]},\omega_1\left(D_{x_0-}^{(n+1)\alpha}f_1,\delta\right)_{[a,x_0]}\right\}, \tag{48}$$

where $x_0\in[a,b]$. Here the moduli of continuity are considered over $[x_0,b]$ and $[a,x_0]$, respectively.

We give

Theorem 7. *[16] Let $0<\alpha<1$, $f_1:[a,b]\to\mathbb{R}$, $f_1'\in L_\infty([a,b])$, $x_0\in[a,b]$. Suppose that $D_{*x_0}^{k\alpha}f_1\in C([x_0,b])$, $k=0,1,...,n+1$; $n\in\mathbb{N}$, and $\left(D_{*x_0}^{i\alpha}f_1\right)(x_0)=0$, $i=2,3,...,n+1$. Suppose that $D_{x_0-}^{k\alpha}f_1\in C([a,x_0])$, for $k=0,1,...,n+1$, and $\left(D_{x_0-}^{i\alpha}f_1\right)(x_0)=0$, for $i=2,3,...,n+1$. Then*

$$|f_1(x)-f_1(x_0)|\le\frac{\omega_1\left(D_{x_0}^{(n+1)\alpha}f_1,\delta\right)}{\Gamma((n+1)\alpha+1)}\left[|x-x_0|^{(n+1)\alpha}+\frac{|x-x_0|^{(n+1)\alpha+1}}{\delta((n+1)\alpha+1)}\right], \tag{49}$$

$\forall\, x\in[a,b]$, $\delta>0$.

We will use

Theorem 8. *[16] Let $\frac{1}{n+1}<\alpha<1$, $n\in\mathbb{N}$, $f_1:[a,b]\to\mathbb{R}_+$, $f'\in L_\infty([a,b])$, $x_0\in[a,b]$. Suppose that $D_{*x_0}^{k_1\alpha}f_1\in C([x_0,b])$, $k_1=0,1,...,n+1$, and $\left(D_{*x_0}^{i_1\alpha}f_1\right)(x_0)=0$, $i_1=2,3,...,n+1$. Suppose that $D_{x_0-}^{k_1\alpha}f_1\in C([a,x_0])$, for $k_1=0,1,...,n+1$, and $\left(D_{x_0-}^{i\alpha}f_1\right)(x_0)=0$, for $i_1=2,3,...,n+1$. Let $\lambda=(n+1)\alpha>1$ and consider $L_N:C_+([a,b])\to C_+([a,b])$, $\forall\, N\in\mathbb{N}$, be positive sublinear operators, fulfilling $L_N(1)=1$, $\forall\, N\in\mathbb{N}$. Thus,*

$$|L_N(f_1)(x_0)-f_1(x_0)|\le\frac{\omega_1\left(D_{x_0}^{(n+1)\alpha}f_1,\delta\right)}{\Gamma(\lambda+1)}\cdot$$
$$\left[L_N\left(|\cdot-x_0|^{\lambda}\right)(x_0)+\frac{L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)}{(\lambda+1)\delta}\right], \tag{50}$$

$\delta>0$, $\forall\, N\in\mathbb{N}$.

We need

Theorem 9.[16] *Denote* $0<\alpha\leq\frac{1}{n+1}$ *and consider* $n\in\mathbb{N}$, $f_1:[a,b]\rightarrow\mathbb{R}_+$, $f_1'\in L_\infty([a,b])$, $x_0\in[a,b]$. *Suppose that* $D_{*x_0}^{k_1\alpha}f_1\in C([x_0,b])$, $k_1=0,1,...,n+1$, *and* $\left(D_{*x_0}^{i_1\alpha}f_1\right)(x_0)=0$, $i_1=2,3,...,n+1$. *Suppose that* $D_{x_0-}^{k\alpha}f_1\in C([a,x_0])$, *for* $k_1=0,1,...,n+1$, *and* $\left(D_{x_0-}^{i_1\alpha}f_1\right)(x_0)=0$, *for* $i_1=2,3,...,n+1$. *Let* $\lambda:=(n+1)\alpha\leq1$ *and considering* $L_N:C_+([a,b])\rightarrow C_+([a,b])$, $\forall\,N\in\mathbb{N}$, *be positive sublinear operators, obeying* $L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)>0$ *and* $L_N(1)=1$, $\forall\,N\in\mathbb{N}$. *Thus*

$$|L_N(f_1)(x_0)-f_1(x_0)|\leq\frac{\omega_1\left(D_{x_0}^{(n+1)\alpha}f,\delta\right)}{\Gamma(\lambda+1)}\cdot \tag{51}$$

$$\left[\left(L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)\right)^{\frac{\lambda}{\lambda+1}}+\frac{L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)}{(\lambda+1)\delta}\right],$$

$\delta>0$, $\forall\,N\in\mathbb{N}$.

We need

Theorem 10.[16] *Consider* $0<\alpha\leq\frac{1}{n+1}$, $n\in\mathbb{N}$, $f_1:[a,b]\rightarrow\mathbb{R}_+$, $f_1'\in L_\infty([a,b])$, $x_0\in[a,b]$. *Suppose that* $D_{*x_0}^{k_1\alpha}f_1\in C([x_0,b])$, $k_1=0,1,...,n+1$, *and* $\left(D_{*x_0}^{i_1\alpha}f_1\right)(x_0)=0$, $i_1=2,3,...,n+1$. *Suppose that* $D_{x_0-}^{k_1\alpha}f_1\in C([a,x_0])$, *for* $k_1=0,1,...,n+1$, *and* $\left(D_{x_0-}^{i_1\alpha}f_1\right)(x_0)=0$, *for* $i_1=2,3,...,n+1$. *Let* $\lambda:=(n+1)\alpha\leq1$ *and consider* $L_N:C_+([a,b])\rightarrow C_+([a,b])$, $\forall\,N\in\mathbb{N}$, *be positive sublinear operators, obeying* $L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)>0$ *and* $L_N(1)=1$, $\forall\,N\in\mathbb{N}$. *As a result we have*

$$|L_N(f_1)(x_0)-f_1(x_0)|\leq\frac{(\lambda+2)\,\omega_1\left(D_{x_0}^{(n+1)\alpha}f_1,\left(L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)}\cdot \tag{52}$$

$$\left(L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)\right)^{\frac{\lambda}{\lambda+1}},\quad\forall\,N\in\mathbb{N}.$$

Note: [16] From (52) we conclude: if $L_N\left(|\cdot-x_0|^{\lambda+1}\right)(x_0)\rightarrow0$, as $N\rightarrow+\infty$, then $L_N(f_1)(x_0)\rightarrow f_1(x_0)$, as $N\rightarrow+\infty$.

5 Definitions and Properties - IV

We have

Definition 8.[8] *Let* $I=[0,1]$, $\mathscr{B}_I$ *the* σ*-algebra of all Borel measurable subsets of* I, $(\Gamma_{N,y})_{N\in\mathbb{N},\,y\in I}$ *will be the collection of the family* $\Gamma_{N,y}=\{\mu_{N,k,y}\}_{k=0}^N$, *of monotone, submodular and strictly positive set functions* $\mu_{N,k,y}$ *on* $\mathscr{B}_I$.

Let $f_1:[0,1]\rightarrow\mathbb{R}_+$ *be a* B_I*-measurable function which is bounded, and swfine* $p_{N,k}(y)=\binom{N}{k}y^k(1-y)^{N-k}$, *for any* $y\in[0,1]$.

The formula of the Bernstein-Kantorovich-Choquet operators reads as

$$K_{N,\Gamma_{N,y}}(f_1)(y)=\sum_{k=0}^N p_{N,k}(y)\frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}}f_1(t)\,d\mu_{N,k,y}(t)}{\mu_{N,k,y}\left(\left[\frac{k}{(N+1)},\frac{(k+1)}{(N+1)}\right]\right)},\quad\forall\,y\in[0,1]. \tag{53}$$

If $\mu_{N,k,y}=\mu$, *for all* N,y,k, *we will denote* $K_{N,\Gamma_{N,y}}(f_1):=K_{N,\mu}(f_1)$.

We have

Theorem 11.[8] *Suppose that* $\mu_{N,k,y} = \mu := \sqrt{M}$, *for all* N, k *and* y *in such a way that* M *denotes the Lebesgue measure on* $[0,1]$. *Thus,*

$$\left|K_{N,\mu}(f_1)(y) - f_1(y)\right| \leq 2\omega_1\left(f_1, \frac{\sqrt{y(1-y)}}{\sqrt{N}} + \frac{1}{N}\right), \tag{54}$$

$\forall\, N \in \mathbb{N}$, $y \in [0,1]$, $f_1 \in C_+([0,1])$, *above* ω_1 *is over* $[0,1]$.

We make

*Remark.*From [8] we recall that

$$K_{N,\mu}(|\cdot - y|)(y) \leq \frac{\sqrt{y(1-y)}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\, N \in \mathbb{N}. \tag{55}$$

Let $m > 1$, recall that $|\cdot - y|^{m-1} \leq 1$, so

$$|\cdot - y|^m = |\cdot - y|\,|\cdot - y|^{m-1} \leq |\cdot - y|,$$

thus

$$K_{N,\mu}(|\cdot - y|^m)(y) \leq K_{N,\mu}(|\cdot - y|)(y),$$

that is

$$K_{N,\mu}(|\cdot - y|^m)(y) \leq \frac{\sqrt{y(1-y)}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\, y \in [0,1],\ N \in \mathbb{N},\ m \geq 1. \tag{56}$$

Notice that $K_{N,\mu}(1) = 1$, $\forall\, N \in \mathbb{N}$.

$K_{N,\mu}$ operators are positive sublinear operators from $C_+([0,1])$ into itself.

We mention

Definition 9.[5] *Below we discuss measures of possibility. Let* $p_{N,k}(y) = \binom{N}{k} y^k (1-y)^{N-k}$ *and we define*

$$\lambda_{N,k}(t) := \frac{p_{N,k}(t)}{k^k N^{-N} (N-k)^{N-k} \binom{N}{k}} = \frac{t^k (1-t)^{N-k}}{k^k N^{-N} (N-k)^{N-k}}, \quad k = 0, \ldots, N. \tag{57}$$

Assume that $0^0 = 1$ *in such a way that cases* $k = 0$, *and* $k = N$ *make sense. We notice that* $\frac{k}{N}$ *of* $p'_{N,k}(y)$, *then we report that*

$$\max\{p_{N,k}(t) : t \in [0,1]\} = k^k N^{-N} (N-k)^{N-k} \binom{N}{k},$$

which implies that each $\lambda_{N,k}$ *represents a possibility distribution on* $[0,1]$.

$P_{\lambda_{N,k}}$ *represents the possibility measure induced by* $\lambda_{N,k}$ *and* $\Gamma_{n,y} := \Gamma_N := \{P_{\lambda_{N,k}}\}_{k=0}^N$ *(that is* Γ_N *is independent of* y*), we define the nonlinear Bernstein-Durrmeyer-Choquet polynomial operators with respect to the set functions in* Γ_N *as*

$$D_{N,\Gamma_N}(f_1)(y) := \sum_{k=0}^{N} p_{N,k}(y) \frac{(C)\int_0^1 f(t)\, t^k (1-t)^{N-k}\, dP_{\lambda_{N,k}}(t)}{(C)\int_0^1 t^k (1-t)^{N-k}\, dP_{\lambda_{N,k}}(t)}, \tag{58}$$

$\forall\, y \in [0,1]$, $N \in \mathbb{N}$, $f_1 \in C_+([0,1])$.

We have

Theorem 12.[5] *For every* $f_1 \in C_+([0,1])$, $y \in [0,1]$ *and* $N \in \mathbb{N} - \{1\}$, *we have*

$$|D_{N,\Gamma_N}(f_1)(y) - f_1(y)| \leq 2\omega_1\left(f_1, \frac{\left(1+\sqrt{2}\right)\sqrt{y(1-y)} + \sqrt{2}\sqrt{y}}{\sqrt{N}} + \frac{1}{N}\right), \tag{59}$$

where ω_1 *is on* $[0,1]$.

We make

*Remark.*Using [5] we conclude

$$D_{N,\Gamma_N}(|\cdot - y|)(y) \leq \frac{\left(1+\sqrt{2}\right)\sqrt{y(1-y)}+\sqrt{2}\sqrt{y}}{\sqrt{N}} + \frac{1}{N}, \quad \forall N \in \mathbb{N} - \{1\}. \tag{60}$$

For $m > 1$ we conclude that $|\cdot - y|^{m-1} \leq 1$, therefore

$$|\cdot - y|^m = |\cdot - y|\,|\cdot - y|^{m-1} \leq |\cdot - y|,$$

so

$$D_{N,\Gamma_N}(|\cdot - y|^m)(y) \leq D_{N,\Gamma_N}(|\cdot - y|)(y),$$

which means that

$$D_{N,\Gamma_N}(|\cdot - y|^m)(y) \leq \frac{\left(1+\sqrt{2}\right)\sqrt{y(1-y)}+\sqrt{2}\sqrt{y}}{\sqrt{N}} + \frac{1}{N}, \tag{61}$$

$\forall N \in \mathbb{N} - \{1\}, m \geq 1, \forall y \in [0,1]$.

We conclude

*Remark.*When $y \in [0,1]$, then the $\max(y(1-y)) = \frac{1}{4}$, at $y = \frac{1}{2}$. Thus, we have

$$\frac{\sqrt{y(1-y)}}{\sqrt{N}} + \frac{1}{N} \leq \frac{1}{2\sqrt{N}} + \frac{1}{N}, \tag{62}$$

$\forall y \in [0,1], \forall N \in \mathbb{N}$.

We get

$$\frac{\left(1+\sqrt{2}\right)\sqrt{y(1-y)}+\sqrt{2}\sqrt{y}}{\sqrt{N}} + \frac{1}{N} \leq \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}, \tag{63}$$

$\forall y \in [0,1], \forall N \in \mathbb{N} - \{1\}$.

Corollary 1.*(to Theorem 11) We conclude*

$$\left\|K_{N,\mu}(f_1) - f_1\right\|_\infty \leq 2\omega_1\left(f_1, \frac{1}{2\sqrt{N}} + \frac{1}{N}\right), \tag{64}$$

$\forall N \in \mathbb{N}, f_1 \in C_+([0,1])$.

Corollary 2.*(to Theorem 12) We have*

$$\|D_{N,\Gamma_N}(f_1) - f_1\|_\infty \leq 2\omega_1\left(f_1, \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right), \tag{65}$$

$\forall N \in \mathbb{N} - \{1\}, f_1 \in C_+([0,1])$.

The form of the Bernstein-Kantorovich-Choquet operators $K_{N,\mu}$, where $\mu := \sqrt{M}$, with M the Lebesgue measure on $[0,1]$ becomes

$$K_{N,\mu}(f_1)(y) = \sum_{k=0}^{N} p_{N,k}(y) \frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}} f_1(t)\,d\mu(t)}{\mu\left(\left[\frac{k}{(N+1)}, \frac{(k+1)}{(N+1)}\right]\right)}, \tag{66}$$

$\forall y \in [0,1], \forall N \in \mathbb{N}, f_1 \in C_+([0,1])$.

Below we discuss about representations of positive sublinear operators by Choquet integrals:

We have

Definition 10. *Let Ω be a set, and let $f_1, g_1 : \Omega \to \mathbb{R}$ be bounded functions. We claim that f_1 and g_1 are comonotonic, if for every $\omega, \omega' \in \Omega$,*

$$\left(f_1(\omega) - f_1(\omega')\right)\left(g_1(\omega) - g_1(\omega')\right) \geq 0. \tag{67}$$

Schmeidler's Representation Theorem (Schmeidler 1986) is given below:

Theorem 13. *[1] Consider $\mathscr{L}_\infty(\mathscr{A})$ be the vector space of $\mathscr{A}$-measurable bounded real valued functions on Ω, where $\mathscr{A} \subset 2^\Omega$ is a σ-algebra. Given a real functional $\Gamma : \mathscr{L}_\infty(\mathscr{A}) \to \mathbb{R}$, suppose that for $f_1, g_1 \in \mathscr{L}_\infty(\mathscr{A})$:*
(1) $\Gamma(cf_1) = c\Gamma(f_1)$, $\forall\, c > 0$,
(2) $f_1 \leq g$, implies $\Gamma(f_1) \leq \Gamma(g_1)$,
and
(3) $\Gamma(f_1 + g_1) = \Gamma(f_1) + \Gamma(g_1)$, for any comonotonic f_1, g_1.
As a result $\gamma(A) := \Gamma(1_A)$, $\forall\, A \in \mathscr{A}$, defines a finite monotone set function on $\mathscr{A}$, and Γ is the Choquet integral with respect to γ, i.e.

$$\Gamma(f_1) = (C)\int_\Omega f_1(t)\, d\gamma(t), \quad \forall\, f_1 \in \mathscr{L}_\infty(\mathscr{A}). \tag{68}$$

We recall that 1_A denotes the characteristic function on A.

We have

Remark. Suppose $L_N(1) = 1$, $\forall\, N \in \mathbb{N}$. Thus, $\mathscr{L}_\infty(\mathscr{B}) \supset C_+([a,b])$. We consider $L_N|_{C_+([a,b])}$, denoted by L_N, $\forall\, N \in \mathbb{N}$.
Thus, $L_N(\cdot)(y) : \mathscr{L}_\infty(\mathscr{B}) \to \mathbb{R}$ is a functional, $\forall\, N \in \mathbb{N}$. The properties are given below:
(1)

$$L_N(cf)(y) = cL_N(f_1)(y), \;\forall\, c > 0, \;\forall\, f_1 \in \mathscr{L}_\infty(\mathscr{B}), \tag{69}$$

(2)

$$f_1 \leq g, \text{ implies } L_N(f_1)(y) \leq L_N(g_1)(y), \quad \text{where } f_1, g_1 \in \mathscr{L}_\infty(\mathscr{B}), \tag{70}$$

and
(3)

$$L_N(f+g)(y) \leq L_N(f_1)(y) + L_N(g_1)(y), \quad \forall\, f_1, g_1 \in \mathscr{L}_\infty(\mathscr{B}). \tag{71}$$

For comonotonic $f_1, g_1 \in \mathscr{L}_\infty(\mathscr{B})$, we further suppose that

$$L_N(f_1 + g_1)(y) = L_N(f_1)(y) + L_N(g_1)(y). \tag{72}$$

In that case L_N is called comonotonic.

Using the Theorem 13 we conclude:

$$\gamma_{N,y}(A) := L_N(1_A)(y), \quad \forall\, A \in \mathscr{B}, \forall\, N \in \mathbb{N}, \tag{73}$$

defines a finite monotone set function on $\mathscr{B}$, and

$$L_N(f_1)(y) = (C)\int_a^b f_1(t)\, d\gamma_{N,y}(t), \tag{74}$$

$\forall\, f \in \mathscr{L}_\infty(\mathscr{B})$, $\forall\, N \in \mathbb{N}$.
In particular (74) is valid for any $f_1 \in C_+([a,b])$. $\gamma_{N,y}$ is normalized, which means that $\gamma_{N,y}([a,b]) = 1$, $\forall\, N \in \mathbb{N}$.
A different type of general operators appears:

Remark. We suppose that all $\mu_{N,y}$ are normalized, which means that $\mu_{N,y}([a,b]) = 1$, and submodular. We discuss the operators $T_N : C_+([a,b]) \to C_+([a,b])$ described as

$$T_N(f_1)(y) = (C)\int_{[a,b]} f_1(t)\, d\mu_{N,y}(t), \tag{75}$$

$\forall\, N \in \mathbb{N}$, $\forall\, y \in [a,b]$.
In fact here $\mu_{N,y}$ are chosen in such a way that $T_N(C_+([a,b])) \subseteq C_+([a,b])$.
We report that:
(1)

$$T_N(\alpha f)(y) = \alpha T_N(f_1)(y), \;\forall\, \alpha \geq 0, \tag{76}$$

(2)

$$f_1 \leq g_1, \text{ implies } T_N(f_1)(y) \leq T_N(g_1)(y), \tag{77}$$

and

(3)

$$T_N(f_1+g_1)(y) \leq T_N(f_1)(y) + T_N(g_1)(y), \tag{78}$$

$\forall\, N \in \mathbb{N}, \forall\, y \in [a,b], \forall\, f_1, g_1 \in C_+([a,b])$.

6 Main Results

In this section we present our main results.

We start with the following result

Theorem 14. *Consider $\alpha \in (n, n+1)$, $n \in \mathbb{N}$, and $f_1 \in C^{n+1}([0,1], \mathbb{R}_+)$, $x \in [0,1]$ and $f_1^{(k_1)}(x) = 0, k_1 = 1, ..., n$. Then*

$$\left|K_{N,\mu}(f_1)(x) - f_1(x)\right| \leq \frac{\omega_1\left({}^{x}T_\alpha f_1, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\prod_{j=0}^{n-1}(\alpha - j)} \cdot$$

$$\left[\frac{1}{(\alpha-n)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \quad \forall\, N \in \mathbb{N}. \tag{79}$$

We notice that $\lim_{N\to\infty} K_{N,\mu}(f_1)(x) = f_1(x)$.

Proof. Using (29) we conclude that $(\delta > 0)$:

$$\left|K_{N,\mu}(f_1)(x) - f_1(x)\right| \leq \frac{\omega_1\left({}^{x}T_\alpha f_1, \delta\right)}{\prod_{j=0}^{n-1}(\alpha - j)} \cdot$$

$$\left[\frac{K_{N,\mu}\left(|\cdot - x|^{\alpha}\right)(x)}{(\alpha-n)} + \frac{K_{N,\mu}\left(|\cdot - x|^{\alpha+1}\right)(x)}{(\alpha+1)\delta}\right] \overset{(56)}{\leq}$$

$$\frac{\omega_1\left({}^{x}T_\alpha f_1, \delta\right)}{\prod_{j=0}^{n-1}(\alpha - j)}\left[\frac{1}{(\alpha-n)}\left(\frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)\delta}\left(\frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}\right)\right] \overset{(62)}{\leq} \tag{80}$$

$$\frac{\omega_1\left({}^{x}T_\alpha f_1, \delta\right)}{\prod_{j=0}^{n-1}(\alpha - j)}\left[\frac{1}{(\alpha-n)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)\delta}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)\right]$$

(for $\delta := \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}$, then $\delta^{\alpha+1} = \frac{1}{2\sqrt{N}} + \frac{1}{N}$, and $\delta^{\alpha} = \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}$)

$$= \frac{\omega_1\left({}^{x}T_\alpha f_1, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\prod_{j=0}^{n-1}(\alpha - j)} \cdot$$

$$\left[\frac{1}{(\alpha-n)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \tag{81}$$

ending the proof.

We also give

Theorem 15. *If $\alpha \in (n, n+1)$, $n \in \mathbb{N}$, and $f_1 \in C^{n+1}([0,1], \mathbb{R}_+)$, $x \in [0,1]$ and $f_1^{(k_1)}(x) = 0, k_1 = 1, ..., n$, then we have*

$$|D_{N,\Gamma_N}(f_1)(x) - f_1(x)| \leq \frac{\omega_1\left({}^{x}T_\alpha f_1, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\prod_{j=0}^{n-1}(\alpha - j)} \cdot$$

$$\left[\frac{1}{(\alpha-n)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \tag{82}$$

$\forall N \in \mathbb{N} - \{1\}$.

We notice that $\lim_{N\to\infty} D_{N,\Gamma_N}(f_1)(x) = f_1(x)$.

Proof. Using (29) we conclude that $(\delta > 0)$:

$$|D_{N,\Gamma_N}(f_1)(x) - f_1(x)| \leq \frac{\omega_1({}^{x}T_\alpha f_1, \delta)}{\prod_{j=0}^{n-1}(\alpha - j)} \cdot$$

$$\left[\frac{D_{N,\Gamma_N}(|\cdot - x|^\alpha)(x)}{(\alpha - n)} + \frac{D_{N,\Gamma_N}\left(|\cdot - x|^{\alpha+1}\right)(x)}{(\alpha+1)\delta}\right] \overset{(61)}{\leq}$$

$$\frac{\omega_1({}^{x}T_\alpha f_1, \delta)}{\prod_{j=0}^{n-1}(\alpha - j)}\left[\frac{1}{(\alpha-n)}\left[\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2x}}{\sqrt{N}} + \frac{1}{N}\right] + \right.$$

$$\left.\frac{1}{(\alpha+1)\delta}\left[\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2x}}{\sqrt{N}} + \frac{1}{N}\right]\right] \overset{(63)}{\leq}$$

$$\frac{\omega_1({}^{x}T_\alpha f_1, \delta)}{\prod_{j=0}^{n-1}(\alpha - j)}\left[\frac{1}{(\alpha-n)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)\delta}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)\right] \tag{83}$$

(choosing $\delta := \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}$, then $\delta^{\alpha+1} = \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}$, and $\delta^\alpha = \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}$)

$$= \frac{\omega_1\left({}^{x}T_\alpha f_1, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\alpha+1}}\right)}{\prod_{j=0}^{n-1}(\alpha - j)} \cdot$$

$$\left[\frac{1}{(\alpha-n)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\alpha+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\alpha}{\alpha+1}}\right], \tag{84}$$

$\forall N \in \mathbb{N} - \{1\}$, proving the claim.

We have

Theorem 16. *Consider* $\frac{1}{n+1} < \alpha < 1$, $n \in \mathbb{N}$, $f_1 : [0,1] \to \mathbb{R}_+$, $f_1' \in L_\infty([0,1])$, $x_0 \in [0,1]$. *Suppose that* $D_{*x_0}^{k\alpha} f \in C([x_0,1])$, $k = 0,1,\dots,n+1$, *and* $\left(D_{*x_0}^{i\alpha} f\right)(x_0) = 0$, $i = 2,3,\dots,n+1$. *Also, suppose that* $D_{x_0-}^{k\alpha} f_1 \in C([0,x_0])$, *for* $k = 0,1,\dots,n+1$, *and* $\left(D_{x_0-}^{i\alpha} f_1\right)(x_0) = 0$, *for* $i = 2,3,\dots,n+1$. *Consider* $\lambda = (n+1)\alpha > 1$.

$$\left|K_{N,\mu}(f_1)(x_0) - f_1(x_0)\right| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\lambda+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\lambda}{\lambda+1}}\right], \quad \forall N \in \mathbb{N}. \tag{85}$$

We notice that $\lim_{N\to\infty} K_{N,\mu}(f_1)(x_0) = f_1(x_0)$.

Proof. By (50) we conclude

$$\left|K_{N,\mu}(f_1)(x_0) - f_1(x_0)\right| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \delta\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[K_{N,\mu}\left(|\cdot - x_0|^\lambda\right)(x_0) + \frac{1}{(\lambda+1)\delta} K_{N,\mu}\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)\right] \overset{\text{(by (56), (62))}}{\leq}$$

$$\frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \delta\right)}{\Gamma(\lambda+1)} \left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\lambda+1)\delta}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)\right]$$

choosing $\delta := \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\lambda+1}}$, then $\delta^{\lambda+1} = \frac{1}{2\sqrt{N}} + \frac{1}{N}$, and $\delta^\lambda = \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\lambda}{\lambda+1}}$)

$$= \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\lambda+1)}\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\lambda}{\lambda+1}}\right], \tag{86}$$

proving the claim.

Theorem 17. *Same assumptions as in Theorem 16. As a result we get*

$$\left|D_{N,\Gamma_N}(f_1)(x_0) - f_1(x_0)\right| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(\lambda+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{\lambda}{\lambda+1}}\right], \quad \forall N \in \mathbb{N} - \{1\}. \tag{87}$$

We report that $\lim_{N\to\infty} D_{N,\Gamma_N}(f_1)(x_0) = f_1(x_0)$.

Proof. By (50) we conclude

$$\left|D_{N,\Gamma_N}(f_1)(x_0) - f_1(x_0)\right| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \delta\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[D_{N,\Gamma_N}\left(|\cdot - x_0|^\lambda\right)(x_0) + \frac{1}{(\lambda+1)\delta} D_{N,\Gamma_N}\left(|\cdot - x_0|^{\lambda+1}\right)(x_0)\right] \overset{\text{(by (61), (63))}}{\leq} \tag{88}$$

$$\frac{\omega_1\left(D_{x_0}^{(n+1)\alpha}f_1,\delta\right)}{\Gamma(\lambda+1)}\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)+\frac{1}{(\lambda+1)\delta}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)\right]$$

(Let $\delta:=\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{\lambda+1}}$, then $\delta^{\lambda+1}=\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}$, and $\delta^{\lambda}=\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{\lambda}{\lambda+1}}$)

$$=\frac{\omega_1\left(D_{x_0}^{(n+1)\alpha}f_1,\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+1)}\cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)+\frac{1}{(\lambda+1)}\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{\lambda}{\lambda+1}}\right],$$

$\forall\, N\in\mathbb{N}-\{1\}$, ending the proof.

Below we are considering the results of Theorem (13).
We have

Theorem 18. *Consider $\alpha\in(n,n+1)$, $n\in\mathbb{N}$, and $f_1\in C^{n+1}([a,b],\mathbb{R}_+)$, $x\in[a,b]$ and $f_1^{(k_1)}(x)=0, k_1=1,\ldots,n$. Consider $L_N:\mathscr{L}_\infty(\mathscr{B}([a,b]))\to C_+([a,b])$, $\forall\, N\in\mathbb{N}$, be positive sublinear comonotonic operators, such that $L_N(1)=1$, $\forall\, N\in\mathbb{N}$. Then*

$$|L_N(f_1)(x)-f_1(x)|\le\frac{\omega_1\left({}^{x}T_\alpha f_1,\delta\right)}{\prod_{j=0}^{n-1}(\alpha-j)}\cdot \tag{89}$$

$$\left[\frac{1}{(\alpha-n)}\left((C)\int_a^b|t-x|^{\alpha}d\gamma_{N,x}(t)\right)+\frac{1}{(\alpha+1)\delta}\left((C)\int_a^b|t-x|^{\alpha+1}d\gamma_{N,x}(t)\right)\right],$$

$\delta>0$.

Proof. By using the Theorem 3.

Theorem 19. *Consider $f_1\in C^1([a,b],\mathbb{R}_+)$, $\alpha\in(0,1)$, and let $L_N:\mathscr{L}_\infty(\mathscr{B}([a,b]))\to C_+([a,b])$, $\forall\, N\in\mathbb{N}$, be positive sublinear comonotonic operators obeying $L_N(1)=1$, $\forall\, N\in\mathbb{N}$. Thus*

$$|L_N(f_1)(x)-f_1(x)|\le\omega_1\left({}^{x}T_\alpha f_1,\delta\right)\cdot$$

$$\left[\frac{1}{\alpha}\left((C)\int_a^b|t-x|^{\alpha}d\gamma_{N,x}(t)\right)+\frac{1}{(\alpha+1)\delta}\left((C)\int_a^b|t-x|^{\alpha+1}d\gamma_{N,x}(t)\right)\right]; \tag{90}$$

$\delta>0$, $\forall\, N\in\mathbb{N}$, $\forall\, x\in[a,b]$.

Proof. By using the Theorem 5.

Theorem 20. *Consider $f_1\in C^1([a,b],\mathbb{R}_+)$, $\alpha\in(0,1)$, $x\in[a,b]$. Let $L_N:\mathscr{L}_\infty(\mathscr{B}([a,b]))\to C_+([a,b])$, $\forall\, N\in\mathbb{N}$, be positive sublinear comonotonic operators, such that $L_N(1)=1$, $\forall\, N\in\mathbb{N}$, and $(C)\int_a^b|t-x|^{\alpha+1}d\gamma_{N,x}(t)>0$, $\forall\, N\in\mathbb{N}$. As a result we get*

$$|L_N(f_1)(x)-f_1(x)|\le\frac{(2\alpha+1)}{\alpha(\alpha+1)}\omega_1\left({}^{x}T_\alpha f_1,\left((C)\int_a^b|t-x|^{\alpha+1}d\gamma_{N,x}(t)\right)^{\frac{1}{\alpha+1}}\right)\cdot$$

$$\left((C)\int_a^b|t-x|^{\alpha+1}d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}},\ \forall\, N\in\mathbb{N}. \tag{91}$$

If $(C)\int_a^b|t-x|^{\alpha+1}d\gamma_{N,x}(t)\to0$, then $L_N(f_1)(x)\to f_1(x)$, as $N\to\infty$.

Proof. Due to the Theorem 6.

We have

Theorem 21. *Consider* $\frac{1}{n+1} < \alpha < 1$, $n \in \mathbb{N}$, $f_1 : [a,b] \to \mathbb{R}_+$, $f_1' \in L_\infty([a,b])$, $x_0 \in [a,b]$. *Suppose that* $D_{*x_0}^{k\alpha} f_1 \in C([x_0,b])$, $k = 0,1,...,n+1$, *and* $\left(D_{*x_0}^{i_1\alpha} f_1\right)(x_0) = 0$, $i_1 = 2,3,...,n+1$. *Assume that* $D_{x_0-}^{k_1\alpha} f_1 \in C([a,x_0])$, *for* $k_1 = 0,1,...,n+1$, *and* $\left(D_{x_0-}^{i_1\alpha} f\right)(x_0) = 0$, *for* $i_1 = 2,3,...,n+1$. *Consider* $\lambda = (n+1)\alpha > 1$ *and* $L_N : \mathscr{L}_\infty(\mathscr{B}([a,b])) \to C_+([a,b])$, $\forall N \in \mathbb{N}$, *be positive sublinear comonotonic operators, such that* $L_N(1) = 1$, $\forall N \in \mathbb{N}$. *Thus*

$$|L_N(f_1)(x_0) - f_1(x_0)| \le \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \delta\right)}{\Gamma(\lambda+1)} \cdot \tag{92}$$

$$\left[\left((C)\int_a^b |t-x|^\lambda d\gamma_{N,x_0}(t)\right) + \frac{1}{(\lambda+1)\delta}\left((C)\int_a^b |t-x|^{\lambda+1} d\gamma_{N,x_0}(t)\right)\right],$$

$\delta > 0$, $\forall N \in \mathbb{N}$.

Proof. By using the Theorem 8.

Note: When $0 < \alpha \le \frac{1}{n+1}$ the Theorem 21 is true.

Theorem 22. *We consider* $0 < \alpha \le \frac{1}{n+1}$, $n \in \mathbb{N}$, $f_1 : [a,b] \to \mathbb{R}_+$, $f_1' \in L_\infty([a,b])$, $x_0 \in [a,b]$. *Suppose that* $D_{*x_0}^{k_1\alpha} f_1 \in C([x_0,b])$, $k_1 = 0,1,...,n+1$, *and* $\left(D_{*x_0}^{i\alpha} f_1\right)(x_0) = 0$, $i_1 = 2,3,...,n+1$. *Suppose that* $D_{x_0-}^{k_1\alpha} f_1 \in C([a,x_0])$, *for* $k_1 = 0,1,...,n+1$, *and* $\left(D_{x_0-}^{i\alpha} f\right)(x_0) = 0$, *for* $i_1 = 2,3,...,n+1$. *Let* $\lambda = (n+1)\alpha \le 1$. *Let* $L_N : \mathscr{L}_\infty(\mathscr{B}([a,b])) \to C_+([a,b])$, $\forall N \in \mathbb{N}$, *be positive sublinear comonotonic operators, such that* $L_N(1) = 1$, $\forall N \in \mathbb{N}$, *and* $(C)\int_a^b |t-x_0|^{\lambda+1} d\gamma_{N,x_0}(t) > 0$, $\forall N \in \mathbb{N}$. *Then, we have*

$$|L_N(f_1)(x_0) - f_1(x_0)| \le \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \delta\right)}{\Gamma(\lambda+1)} \cdot \tag{93}$$

$$\left[\left((C)\int_a^b |t-x_0|^{\lambda+1} d\gamma_{N,x_0}(t)\right)^{\frac{\lambda}{\lambda+1}} + \frac{1}{(\lambda+1)\delta}\left((C)\int_a^b |t-x_0|^{\lambda+1} d\gamma_{N,x_0}(t)\right)\right],$$

$\delta > 0$, $\forall N \in \mathbb{N}$.

Proof. The results arise from Theorem 9.

Theorem 23. *All as in Theorem 22. Then*

$$|L_N(f_1)(x_0) - f_1(x_0)| \le \frac{(\lambda+2)\,\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \left((C)\int_a^b |t-x_0|^{\lambda+1} d\gamma_{N,x_0}(t)\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)}$$

$$\left((C)\int_a^b |t-x_0|^{\lambda+1} d\gamma_{N,x_0}(t)\right)^{\frac{\lambda}{\lambda+1}}, \quad \forall N \in \mathbb{N}. \tag{94}$$

If $(C)\int_a^b |t-x_0|^{\lambda+1} d\gamma_{N,x_0}(t) \to 0$, *then* $L_N(f_1)(x_0) \to f_1(x_0)$, *as* $N \to \infty$.

Proof. Using the help of the Theorem 10.

We have

Theorem 24. *Use the Theorem 3 with* $L_N = T_N$, $\forall N \in \mathbb{N}$. *As a result*

$$|T_N(f_1)(x) - f_1(x)| \le \frac{\omega_1\left({}^xT_\alpha f_1, \delta\right)}{\prod_{j=0}^{n-1}(\alpha - j)} \cdot$$

$$\left[\frac{1}{(\alpha-n)}\left((C)\int_a^b |t-x|^\alpha d\mu_{N,x}(t)\right) + \frac{1}{(\alpha+1)\delta}\left((C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)\right)\right], \tag{95}$$

$\delta > 0$, $\forall N \in \mathbb{N}$.

*Proof.*Using the Theorem 3 we will obtained the result.

Theorem 25.*All as in Theorem 5, such that $L_N = T_N$, $\forall\, N \in \mathbb{N}$. Thus*

$$|T_N(f_1)(x) - f_1(x)| \leq \omega_1\left({}^{x}T_\alpha f_1, \delta\right) \cdot$$

$$\left[\frac{1}{\alpha}\left((C)\int_a^b |t-x|^\alpha d\mu_{N,x}(t)\right) + \frac{1}{(\alpha+1)\delta}\left((C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)\right)\right], \tag{96}$$

$\forall\, N \in \mathbb{N}$, $\forall\, x \in [a,b]$, $\delta > 0$.

*Proof.*Theorem 5 is used in this proof.

Theorem 26.*All as in Theorem 6, with $L_N = T_N$, with $(C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t) > 0$, $\forall\, N \in \mathbb{N}$. Thus, we have*

$$|T_N(f_1)(x) - f_1(x)| \leq \frac{(2\alpha+1)}{\alpha(\alpha+1)}\omega_1\left({}^{x}T_\alpha f_1, \left((C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)\right)^{\frac{1}{\alpha+1}}\right)$$

$$\cdot\left((C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}}, \tag{97}$$

$\forall\, N \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t) \to 0$, then $T_N(f_1)(x) \to f_1(x)$, as $N \to \infty$.

*Proof.*By using the Theorem 6.

We have

Theorem 27.*All conditions are as in the Theorem 8, such $L_N = T_N$, $\forall\, N \in \mathbb{N}$. Thus,*

$$|T_N(f_1)(x_0) - f_1(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f, \delta\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[(C)\int_a^b |t-x_0|^\lambda d\mu_{N,x_0}(t) + \frac{1}{(\lambda+1)\delta}(C)\int_a^b |t-x_0|^{\lambda+1} d\mu_{N,x_0}(t)\right], \tag{98}$$

$\delta > 0$, $\forall\, N \in \mathbb{N}$.

*Proof.*By making use of the Theorem 8.

Note: When $0 < \alpha \leq \frac{1}{n+1}$ the Theorem 27 is also true.

Theorem 28.*All as in Theorem 9, with $L_N = T_N$, with $(C)\int_a^b |t-x_0|^{\lambda+1} d\mu_{N,x_0}(t) > 0$, $\forall\, N \in \mathbb{N}$. Then we get*

$$|T_N(f_1)(x_0) - f_1(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \delta\right)}{\Gamma(\lambda+1)} \cdot$$

$$\left[\left((C)\int_a^b |t-x_0|^{\lambda+1} d\mu_{N,x_0}(t)\right)^{\frac{\lambda}{\lambda+1}} + \frac{1}{(\lambda+1)\delta}\left((C)\int_a^b |t-x_0|^{\lambda+1} d\mu_{N,x_0}(t)\right)\right], \tag{99}$$

$\delta > 0$, $\forall\, N \in \mathbb{N}$.

*Proof.*Proofs looks similar as in the Theorem 9.

Theorem 29.*All as presented Theorem 28. Then we have*

$$|T_N(f_1)(x_0) - f_1(x_0)| \leq \frac{(\lambda+2)\,\omega_1\left(D_{x_0}^{(n+1)\alpha} f_1, \left((C)\int_a^b |t-x_0|^{\lambda+1} d\mu_{N,x_0}(t)\right)^{\frac{1}{\lambda+1}}\right)}{\Gamma(\lambda+2)}$$

$$\cdot\left((C)\int_a^b |t-x_0|^{\lambda+1} d\mu_{N,x_0}(t)\right)^{\frac{\lambda}{\lambda+1}}, \quad \forall\, N \in \mathbb{N}. \tag{100}$$

If $(C)\int_a^b |t-x_0|^{\lambda+1} d\mu_{N,x_0}(t) \to 0$, then $T_N(f_1)(x_0) \to f_1(x_0)$, as $N \to \infty$.

*Proof.*Utilize the Theorem 10.

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