Applied Mathematics & Information Sciences 1(2)(2007), 185-194
— An International Journal

Common Fixed Point Theorems of Single and Set-Valued Mappings on 2-Metric Spaces

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Recived 11 Dec. 2006, Accepted 3 March 2007

The purpose of this paper is to study a common fixed point theorems on 2-metric spaces. Generalizations some definitions on 2-metric spaces and Fisher theorems [2] on 2-metric spaces.

Keywords: 2-metric spaces, common fixed point, weakly compatible mappings, compact 2-metric spaces.

1 Introduction

The concept of a 2-metric space is a natural generalization of a metric space. It has been investigated initially by Gähler [4]. Iseki [5] studied the fixed point theorems in 2-metric spaces. Sessa [17] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. In [7] Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems. In [8,9] Jungck and Rhoades defined the concepts of δ-compatible and weakly compatible mappings, which extend the concept of compatible mappings in the single-valued setting on metric spaces. Several authors used these concepts to prove some common fixed point theorems (See, e.g., [13-16]). In this paper we generalized some definitions on 2- metric spaces and studied common fixed point theorems for four mappings on 2- metric spaces.

2 Preliminaries

The concept of a 2-metric space is a natural generalization of a metric space by Gähler as the following definition.

Definition 2.1. [4] *A 2-metric space is a set with a real-valued function on satisfying the following conditions:*

(1) For distinct points $x, y \in X$, there exists a point $c \in X$ such that $d(x, y, c) \neq 0$;

(2) $d(x, y, c) = 0$ if at least two of x, y and c are equal equal;

(3) $d(x, y, c) = d(x, c, y) = d(c, y, x)$;

(4) $d(x, y, c) \leq d(x, y, z) + d(x, z, c) + d(z, y, c)\ \forall\ x, y, c, z \in X$.

The function is called a 2-metric for the space X and the pair (X, d) (denotes a 2-metric space. It has shown by Gählher that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric space d which is continuous in all of its arguments is said to be continuous.

Geometrically a 2-metric $d(x, y, c)$ represents the area of triangle with vertices x, y and c.

Throughout this paper, let (X, d) be 2-metric space unless mentioned otherwise and $B(X)$ is the set of all nonempty bounded subset of X.

Definition 2.2. [12] *A sequence $\{x_n\}$ in (X, d) is said to be convergent to a point x in X, denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} d(x_n, x, c) = 0$ for all c in X. The point x is called the limit of the sequence $\{x_n\}$ in X.*

Definition 2.3. [12] *A sequence $\{x_n\}$ in (X, d) is said to be Cauchy sequence if $\lim_{n\to\infty}$ $d(x_m, x_n, c) = 0$, for all c in X.*

Definition 2.4. [12] *The space (X, d) is said to be complete if every Cauchy sequence in converges to an element in X.*

Remark 1. *We note that, in a metric space a convergent sequence is a Cauchy sequence and in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X* [11].

For all A, B and C in $B(X)$, let $\delta(A, B, C)$ be the functions defined by

$$\begin{aligned} \delta(A, B, C) &= \sup\{d(a, b, c) : a \in A, b \in B, c \in C\}, \\ d(a, b, C) &= \inf\{d(x, y, c) : c \in C\}. \end{aligned}$$

If A consists of a single point a we write $\delta(A, B, C) = \delta(a, B, C)$, if B and C consist of a single point b and c respectively, we write $\delta(A, B, C) = \delta(a, b, c)$. It follows immediately from the definition that

$\delta(A,B,C) = \delta(A,C,B) = \delta(C,B,A) = \delta(C,A,B) = \delta(B,C,A) = \delta(B,A,C) \geq 0.$

$\delta(A,B,C) \leq \delta(A,B,E) + \delta(A,E,C) + \delta(E,B,C)$. For all A,B,C and E in $B(X)$

$\delta(A,B,C) = 0$. if at least two of A,B and C consist of equal single points.

Definition 2.5. *A sequence A_n of subsets of X is said to be convergent to a subset A of X if*

(i) given $a \in A$, there is a sequence a_n in X such that $a_n \in A_n$ for $n = 1,2,3,....$ $\{a_n\}$ converges to a.

(ii) given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Definition 2.6. *The mappings $F : X \longrightarrow B(X)$ and $I : X \longrightarrow X$ are said to be weakly commuting on X if $IFx \in B(X)$ and*

$$\delta(FIx, IFx, C) \leq \max\{\delta(Ix, Fx, C), \delta(IFx, IFx, C)\} \tag{2.1}$$

Note that if F is a single-valued mapping, and then the set IFx consists of a single point. Therefore, $\delta(IFx, IFx, C) = d(IFx, IFx, C) = 0$ and condition (2.1) reduces to the condition given by Khan [6] that is $\delta(FIx, IFx, C) \leq d(Ix, Fx, C)$. Two commuting mapping F and I clearly weakly commuting but the converse is false.

Definition 2.7. [10] *Two single-valued mappings f and g of (X, d) into itself are compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n, C) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n$ = $\lim_{n\to\infty} gx_n = t$, for some t in X.*

It can be seen that two weakly commuting mappings are compatible but the converse is false.

Definition 2.8. *The mappings $I : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ are δ-compatible if $\lim_{n\to\infty} \delta(FIx_n, IFx_n, C) = 0$ whenever $\{x_n\}$ is a sequence in X such that, $IFx \in B(X)$, $Fx_n \longrightarrow \{t\}$ and $Ix_n \longrightarrow t$ for some t in X.*

Definition 2.9. *The mappings $I : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ are weakly compatible if they commute at coincidence points. i.e. for each point u in X such that, $Fu = \{Iu\}$, we have $FIu = IFu$.*

Not that the equation $Fu = \{Iu\}$ implies that Fu is singleton. It can be seen that any δ-compatible pair $\{F, I\}$ is weakly compatible but the converse is false.

Definition 2.10. *A set-valued mapping F of X into $B(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X. if it is continuous at every point in X.*

In [2], Fisher proved the following theorem:

Theorem 2.11. *Let F, G be mappings of X into $B(X)$ and I, J be mappings of X into itself satisfying:*

$$\delta(Fx, Gy) \leq c \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\}, \text{ for all } x, y \in X$$

where $0 \leq c < 1$. If F commutes with I and G commutes with J, $G(x) \subseteq I(x)$, $F(x) \subseteq J(x)$ and I or J is continuous, then F, G, I and J have a unique common fixed point u in X.

Also, Fisher [2] proved the following theorem on compact metric space:

Theorem 2.12. *Let F, G be continuous mapping of a compact metric space (X, d) into $B(X)$ and I, J continuous mapping of X into itself satisfying the inequality:*

$$\delta(Fx, Gy) < c \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\}, \tag{2.2}$$

for all $x, y \in X$ for which the right hand side of the inequality (2.2) is positive. If the mappings F and I commute, G and J commute and $G(X) \subset I(X)$, $F(X) \subset J(X)$, then there is a unique point u in X such that $Fu = Gu = \{u\} = \{Iu\} = \{Ju\}$.

The main aim of the present paper is to prove common fixed point theorems on 2-metric spaces.

3 Some Auxiliary Lemmas and the Main Theorems

Let I, J be mappings from 2-metric space (X, d) into itself and $F, G : X \longrightarrow B(X)$ set-valued mappings such that

$$G(x) \subseteq I(x) \quad \text{and} \quad F(x) \subseteq J(x) \tag{3.1}$$

Also, the mappings F, G, J and I satisfy the following inequality:

$$\delta(Fx, Gy, C) \leq k \max\{d(Ix, Jy, C), \delta(Ix, Gy, C), \delta(Jy, Gy, C)\}, \tag{3.2}$$

for all x, y in X and $0 \leq k < 1$.

Since $F(x) \subseteq J(x)$, for an arbitrary point x_0 in X there exists a point x_1 in X such that $Jx_1 \in Fx_0$. Since $G(x) \subseteq I(x)$, for this point x_1 there exist a point x_2 in X such that $Ix_2 \in Gx_1$ and so on. Consequently, we can define a sequence $\{y_n\}$ in X such that $Jx_{2n+1} \in Fx_{2n} = y_{2n}$ and $Ix_{2n+2} \in Gx_{2n+1} = y_{2n+1}$, for all $n = 1, 2, 3, \dots$.

In the following we introduce some auxiliary lemmas are useful in the sequel.

Lemma 3.1. *Suppose that I, J be mappings from (X, d) into itself and $F, G : X \longrightarrow B(X)$ set-valued mappings such that conditions (3.1) and (3.2) are satisfying. Then for every $n \in N$, we have $\delta(y_n, y_{n+1}, y_{n+2}) = 0$.*

Proof. Since $\delta(y_{2n+2}, y_{2n+1}, y_{2n}) = \delta(Fx_{2n+2}, Gx_{2n+1}, y_{2n})$ and by using (3.2), we have

$$\begin{aligned} &\delta(y_{2n+2}, y_{2n+1}, y_{2n}) \\ =\ &\delta(Fx_{2n+2}, Gx_{2n+1}, y_{2n}) \\ \leq\ &c\ \max\{d(Ix_{2n+2}, Ix_{2n+1}, y_{2n}), \delta(Ix_{2n+2}, Fx_{2n+2}, y_{2n}), \delta(Jx_{2n+1}, Gx_{2n+1}, y_{2n})\} \\ \leq\ &c\ \max\{\delta(y_{2n+1}, y_{2n}, y_{2n}), \delta(y_{2n+1}, y_{2n+2}, y_{2n}), \delta(y_{2n}, y_{2n+1}, y_{2n})\}, \end{aligned}$$

Thus, we have $(1 - c)\delta(y_{2n+2}, y_{2n+1}, y_{2n}) \leq 0$. Then $\delta(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$. Similarly, we have $\delta(y_{2n+1}, y_{2n+2}, y_{2n+3}) = 0$. Hence $\delta(y_n, y_{n+1}, y_{n+2}) = 0$. □

Lemma 3.2. *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively. Then the sequence $\{\delta(A_n, B_n, C)\}$ converges to $\delta(A, B, C)$.*

Proof. Since $A_n \longrightarrow A$, $B_n \longrightarrow B$ and

$$d(a_n, b_n, c) \leq d(a_n, a, c) + d(a_n, b_n, a) + d(a, b, c) + d(b, b_n, c) + d(a, b_n, b)$$

we have $d(a_n, b_n, c) \leq d(a, b, c)$, $|\sup d(a_n, b_n, c) - \sup d(a, b, c)| = 0$.

Since $\delta(A_n, B_n, C) = \sup\{d(a_n, b_n, c) : a_n \in A_n, b_n \in B_n, c \in C\}$, $\delta(A, B, C) = \sup\{d(a, b, c) : a \in A, b \in B, c \in C\}$. Then the sequence $\{\delta(A_n, B_n, C)\}$ converges to $\delta(A, B, C)$. □

Now we can introduce the main theorems:

Theorem 3.3. *Let F, G be mappings of X into $B(X)$ and I, J be mappings of X into itself satisfying:*

$$G(x) \subseteq I(x), F(x) \subseteq J(x) \tag{3.3}$$

$$\delta(Fx, Gy, C) \leq k \max\{d(Ix, Jy, C), \delta(Ix, Fx, C), \delta(Jy, Gy, C)\}, \tag{3.4}$$

for all $x, y \in X$ where $0 \leq k < 1$. If both pairs $\{F, I\}$ and $\{G, J\}$ are weakly compatible and one of $I(X)$ or $J(X)$ is complete. Then F, G, I and J have a unique common fixed point u in X.

Proof. Let x_0 be an arbitrary point in X. By (3.3), we chose a point x_1 in X such that $Jx_1 \in Fx_0$. For this point x_1 there exist a point x_2 in X such that $Ix_2 \in Gx_1$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$Jx_{2n+1} \in Fx_{2n} = y_{2n}, Ix_{2n+2} \in Gx_{2n+1} = y_{2n+1} \tag{3.5}$$

for all $n = 0, 1, 2, \dots$. For simplicity, we put $V_n = \delta(y_n, y_{n+1}, C)$, By using (3.4), we have

$$\begin{aligned} V_{2n} &= \delta(y_{2n}, y_{2n+1}, C) = \delta(Fx_{2n}, Gx_{2n+1}, C) \\ &\leq k \ \max\{d(Ix_{2n}, Jx_{2n+1}, C), \delta(Ix_{2n}, Fx_{2n}, C), \delta(Jx_{2n+1}, Gx_{2n+1}, C)\} \\ &\leq k \ \max\{\delta(Gx_{2n-1}, Fx_{2n}, C), \delta(Gx_{2n-1}, Fx_{2n}, C), \delta(Fx_{2n}, Gx_{2n+1}, C)\} \\ &\leq k \ \max\{\delta(y_{2n-1}, y_{2n}, C), \delta(y_{2n}, y_{2n+1}, C)\} \\ &\leq k \ \max\{V_{2n-1}, V_{2n}\}. \end{aligned}$$

If $V_{2n-1} \leq V_{2n}$, thus $(1-k)V_{2n} \leq 0$. Since $0 \leq k < 1$, thus $V_{2n} \leq 0$, this is a contradiction implies

$$V_{2n} \leq kV_{2n-1}, \tag{3.6}$$

$$\begin{aligned} V_{2n+1} &= \delta(y_{2n+1}, y_{2n+2}, C) = \delta(Fx_{2n+1}, Gx_{2n+2}, C) \\ &\leq k \ \max\{d(Ix_{2n+1}, Jx_{2n+2}, C), \delta(Ix_{2n+1}, Fx_{2n+2}, C), \delta(Jx_{2n+2}, Gx_{2n+2}, C)\} \\ &\leq k \ \max\{\delta(y_{2n}, y_{2n+1}, C), \delta(y_{2n+1}, y_{2n+2}, C)\} \\ &\leq k \ \max\{V_{2n}, V_{2n+1}\}. \end{aligned}$$

Similarly, we obtain that

$$V_{2n+1} \leq kV_{2n}, \tag{3.7}$$

for all $n = 0, 1, 2, \cdots$. By (3.6) and (3.7), we have $V_n \leq kV_{n-1} \leq k^2 V_{n-2} \leq \cdots \leq k^n V_0$. Then

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} \delta(y_n, y_{n+1}, C) = 0. \tag{3.8}$$

For all $n < m$, we have that

$$\begin{aligned} \delta(y_n, y_m, C) &\leq \delta(y_n, y_{n+1}, y_{n+2}) + \delta(y_{n+1}, y_{n+2}, y_{n+3}) + \cdots \\ &+ \delta(y_{m-2}, y_{m-1}, y_m) + \delta(y_{m-1}, y_m, C). \end{aligned}$$

By taking the limit as $n, m \longrightarrow \infty$ and using Lemma 3.1 and (3.8), we obtain that

$$\lim_{n\to\infty} \delta(y_n, y_m, C) = 0.$$

Then $\{y_n\}$ is Cauchy sequence in X. Suppose that $J(X)$ is complete. Let $\{x_n\}$ be the sequence defined by $Jx_{2n+1} \in Fx_{2n} = y_{2n}$, for all $n = 0, 1, 2, \cdots$.

Since $\lim_{n,m\to\infty} d(Jx_{2m+1}, Jx_{2n+1}, C) \le \lim_{n,m\to\infty} \delta(Jx_{2m}, Jx_{2n}, C) = 0$, the sequence $\{Jx_{2n+1}\}$ is Cauchy and hence $Jx_{2n+1} \longrightarrow p = Jv \in J(X)$, for some $v \in X$. But $Jx_{2n} \in Gx_{2n-1} = y_{2n-1}$, so we obtain

$$\lim_{n,m\to\infty} d(Ix_{2n}, Jx_{2n+1}, C) \le \lim_{n\to\infty} \delta(y_{2n-1}, y_{2n}, C) = 0.$$

Consequently, $Ix_{2n} \longrightarrow p$. Moreover, we obtain

$$\delta(Fx_{2n}, p, C) \le \delta(Fx_{2n}, Ix_{2n}, C) + \delta(Ix_{2n}, p, C) + \delta(Fx_{2n}, p, Ix_{2n}).$$

Therefore, we have that $\lim_{n\to\infty} \delta(Fx_{2n}, p, C) = 0$. Similarly, we have

$$\lim_{n\to\infty} \delta(Gx_{2n-1}, p, C) = 0$$

Since $\delta(Fx_{2n}, Gv, C) \le k \max\{d(Ix_{2n}, Jv, C), \delta(Ix_{2n}, Fx_{2n}, C), \delta(Jv, Gv, C)\}$, $\delta(Ix_{2n}, Gv, C) \longrightarrow \delta(p, Gv, C)$. When, $Ix_n \longrightarrow p$, we get as $n \longrightarrow \infty$, $(1-k)\delta(p, Gv, C) \le 0$. Hence $Gv = \{p\} = \{Jv\}$.

Since $G(x) \subseteq I(x)$, so $u \in X$ exists such that $\{Iu\} = Gv = \{Jv\}$.

Now if $Fu \ne Gv$, this implies that $\delta(Fu, Gv, C) \ne 0$, so that we have

$$\delta(Fu, Gv, C) \le k \max\{d(Iu, Jv, C), \delta(Iu, Fu, C), \delta(Jv, Gv, C)\}.$$

Then $Fu = \{p\} = Gv = \{Iu\} = \{Jv\}$. Since $Fu = \{Iu\}$ and the pair $\{F, I\}$ is weakly compatible, we obtain $Fp = FIu = IFu = \{Ip\}$. By using (3.2), we obtain

$$\begin{aligned} \delta(Fp, p, C) &\le \delta(Fp, Gv, C) \\ &\le k \max\{d(Ip, Jv, C), \delta(Ip, Fp, C), \delta(Jv, Gv, C)\} \end{aligned}$$

Then $Fp = \{p\} = \{Ip\}$. Similarly, $Gp = \{p\} = \{Jp\}$ if the pair $\{G, J\}$ is weakly compatible, we obtain $\{p\} = \{Ip\} = \{Jp\} = Fp = Gp$. Similarly, if $I(X)$ is complete.

Now, we prove the uniqueness. To see the point p is unique, suppose that w is another common fixed point of F, G, J and I with $w \ne p$. Then we have

$$\begin{aligned} d(p, w, C) &\le \delta(Fp, Gw, C) \\ &\le k \max\{d(Ip, Jw, C), \delta(Ip, Fp, C), \delta(Jw, Gw, C)\} \\ &\le kd(p, w, C). \end{aligned}$$

This implies that $w = p$. □

Theorem 3.4. *Let I, J be function of a compact 2-metric space (x, d) into itself and $F, G : X \longrightarrow B(X)$ two set-valued functions with $G(x) \subseteq I(x), F(x) \subseteq J(x)$. Suppose that the inequality:*

$$\delta(Fx, Gy, C) < \max\{d(Ix, Jy, C), \delta(Ix, Fx, C), \delta(Jy, Gy, C)\}, \tag{3.9}$$

for all $x, y \in X$ holds whenever the right hand side of the inequality (3.9) is positive. If the pairs $\{F, I\}$ and $\{G, J\}$ are weakly compatible, and if the function F and I are continuous, then there is a unique point u in X such that

$$Fu = Gu = \{u\} = \{Iu\} = \{Ju\}.$$

Proof. Let $\eta = \inf_{x \in X} \{\delta(\mathrm{Ix}, \mathrm{Fx}, \mathrm{C})\}$. Since X is a compact 2-metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that

$$\delta(Ix_n, Fx_n, C) \longrightarrow \eta \text{ as } n \longrightarrow \infty.$$

Since

$$\begin{aligned}
\delta(Ix_0, Fx_0, C) &\leq d(Ix_0, Ix_n, C) + \delta(Ix_n, Fx_0, C) + \delta(Ix_0, Fx_0, Ix_n) \\
&\leq d(Ix_0, Ix_n, C) + \delta(Ix_n, Fx_n, C) + \delta(Fx_n, Fx_0, C) \\
&\quad + \delta(Ix_0, Fx_0, Fx_n) + \delta(Ix_0, Fx_0, Ix_n)
\end{aligned}$$

by the continuity of F and I and $\lim_{n \to \infty} x_n = x_0$, we get $\delta(Ix_0, Fx_0, C) \leq \eta$ and thus

$$\delta(Ix_0, Fx_0, C) = \eta.$$

Since $F(X) \subset J(X)$, there exists a point y_0 in X with $Jy_0 \in Fx_0$ and $\delta(Ix_0, Jy_0, C) \leq \eta$. If $\eta > 0$. By (3.9), we obtain

$$\begin{aligned}
\delta(Jy_0, Gy_0, C) &\leq d(Fx_0, Gy_0, C) \\
&< \max\{d(Ix_0, Jy_0, C), \delta(Ix_0, Fx_0, C), \delta(Jx_0, Gy_0, C)\} \\
&< \max\{\eta, \delta(Jx_0, Gy_0, C)\},
\end{aligned}$$

which implies that $\delta(Jy_0, Gy_0, C) \leq \eta$.
Since $G(X) \subset I(X)$, there exists a point z_0 in X with $Iz_0 \in Gy_0$ and $\delta(Iz_0, Jy_0, C) \leq \eta$. Hence

$$\begin{aligned}
\eta \leq \delta(Iz_0, Fz_0, C) &\leq \delta(Fz_0, Gy_0, C) \\
&< \max\{d(Iz_0, Jy_0, C), \delta(Iz_0, Fz_0, C), \delta(Jy_0, Gy_0, C)\} \\
&< \delta(Iz_0, Fz_0, C).
\end{aligned}$$

This contradiction demands that $\eta = 0$. Therefore, we have $Gy_0 = \{Jy_0\} = Fx_0 = \{Ix_0\} = \{Iz_0\}$. Since F and I are weakly compatible and $Fx_0 = \{Ix_0\}$, we obtain $F^2x_0 = FIx_0 = IFx_0 = \{I^2x_0\}$. If $I^2x_0 \neq Ix_0$, then we have

$$\begin{aligned}
\delta(I^2x_0, Ix_0, C) &= \delta(F^2x_0, Gy_0, C) \\
&< \max\{d(IFx_0, Jy_0, C), \delta(IFx_0, F^2x_0, C), \delta(Jy_0, Gy_0, C)\} \\
&= \delta(I^2x_0, Ix_0, C).
\end{aligned}$$

So we have $I^2x_0 = Ix_0$, and hence $FIx_0 = \{Ix_0\} = \{I^2x_0\}$. Similarly, we have $GJy_0 = \{Jy_0\} = \{J^2y_0\}$. Let $u = Ix_0 = Jy_0$, thus $Fu = \{u\} = \{Iu\} = \{Ju\} = Gu$. Suppose that the point y in X is a common fixed point of F, G, J and I. If either $\delta(y, Fy, C) \neq 0$ or $\delta(y, Gy, C) \neq 0$, then we have that

$$\begin{aligned}\delta(y, Fy, C) &\leq \delta(Fy, Gy, C)\\ &< \max\{d(Iy, Jy, C), \delta(Iy, Fy, C), \delta(Jy, Gy, C)\}\\ &< \max\{d(y, Fy, C), \delta(y, Gy, C)\}.\end{aligned}$$

This implies that $\delta(y, Fy, C) < \delta(y, Gy, C)$. By symmetry, we have that $\delta(y, Gy, C) < \delta(y, Fy, C)$, which is impossible. So $\delta(y, Fy, C) = \delta(y, Gy, C)$, and $Fy = Gy = \{y\}$.

Now, we prove the uniqueness. Let y, u in X are two common fixed points of F, G, J and I with $y \neq u$. On using (3.9), we have that

$$\begin{aligned}d(y, u, C) &= \delta(Fy, Gu, C)\\ &< \max\{d(Iy, Ju, C), \delta(Iy, Fu, C), \delta(Ju, Gu, C)\}\\ &< d(y, u, C).\end{aligned}$$

This implies that $y = u$. Then F, G, J and I have a unique common fixed point in X. □

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