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On a Problem of Large-Amplitude Oscillation of a Non-linear Conservative System with Inertia and Static Non-linearity

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In this paper we will study the non-linear oscillation of a conservative system having inertia and static non-linearities. By combining the linearization of the governing equation with the method of harmonic balance, we investigate analytical approximate solutions for the non-linear oscillations of the system. Unlike the classical harmonic balance method, linearization is performed prior to proceeding with harmonic balancing, thus resulting in a set of linear algebraic equations instead of one of non-linear algebraic equations. Hence, we are able to establish analytical approximate formulas for the exact frequency and periodic solution. These analytical approximate formulas show excellent agreement with the exact solutions, and are valid for small as well as large amplitudes of oscillation.

Keywords: Non-linear oscillation, large amplitude, linearization.

1 Introduction:

Perturbation method [1-5] is one of the most widely applied analytic tools for non-linear problem. The perturbation methods are, in principle, for solving problems with small parameter. In that case, the solution is analytically expanded in a power series of the parameter. The coefficients of the series are found as solutions of a set of linear problems. However, in both science and engineering, there exist many non-lineare problems in which parameters are not small. Also an analytical approximation given by perturbation method has, in most cases, a small range of validity, but one is often interested in the large parameter reoime of the theory under study.

The applications of the perturbation methods have been extended to oscillators with strong non-linearity [6,7]. However, the algebraic manipulation of the perturbation procedures involves excessive labor. Recently, a power-series method has been developed [8] and extended to the conservative oscillator with inertia and static non-linearities [9], which simulates the uni-modal large-amplitude free vibration of a cantilever beam carrying an intermediate lumped mass with a rotary inertia [10]. The method may be used to achieve numerical solutions in the regime of relatively large amplitude of oscillation where the usual perturbation method fails. But the method cannot give analytical approximate expressions for the exact frequency and periodic solution. There exist some approaches, such as the method of harmonic balance [1,2, 5, 11-13], sometimes capable of producing analytical approximations to the frequency and periodic solution of non-linear oscillation. These approximate solution are valid even for rather large amplitude of oscillation. However, it is usually rather difficult to apply these methods to produce higher-order analytical approximations to the exact frequency and periodic solution. This is due to the fact that, for a given initial condition, a set of non-linear equations has to be nummerically solved [13].

In this paper we have got an alternative approach to solving the above-mentioned non-linear oscillator [9, 10]. The approach is a generalization of a recent work concerning with the oscillation equation $d^2u/dt^2 + f(u) = 0$, where $f(u)$ is an odd non-linear function [14-17]. By combining the linearization of the governing equation with the method of harmonic balance, we establish analytical approximate solution for the frequency and periodic solution of the system. The most interesting features of this new approach are its simplicity and its very good accuracy in a wide range of amplitude of oscillation.

2 Formulation and Solution Method

Consider the non-linear oscillator

$$\ddot{u} + u + \alpha u^4 \ddot{u} + 2\alpha \dot{u}^2 u^3 + \beta u^5 = 0, \tag{2.1}$$

subject to the initial conditions

$$u(0) = A, \ \dot{u}(0) = 0. \tag{2.2}$$

The over-dot denotes differentiation with respect to time t. This system describes the uni-modal large-amplitude free vibrsations of a slender inextensible cantilever beam carrying an intermediate mass with a rotary inertia. The third and fourth terms in equation (2.1) represent inertia type fifth non-linearity arising form the inextensibility assumption. The last term is a static-type fifth nonlinearity associated with the potential energy stored in bending. The modal constants α and β result from the discretization procedure and they have specific values for each mode as described in [10].

Introduce a new independent variable, $\tau = \omega t$, then equations (2.1) and (2.2) become

$$\omega^2[(1 + \alpha u^4)u'' + 2\alpha u'^2 u^3] + u + \beta u^5 = 0 \tag{2.3}$$

and

$$u(0) = A, \quad u'(0) = 0, \tag{2.4}$$

where the prime denotes differentiation with respect to τ. The new independent variable τ is chosen in such a way that solution of equation (2.3), which satisfies the assigned initial conditions in equation (2.4), is a periodic function of τ, of period 2π. The period of the corresponding non-linear oscillation is given by $T = 2\pi/\omega$. Here, both the periodic solution $u(\tau)$ and frequency ω (also with period T) depend on A. Based on equation (2.3), the periodic solution $u(\tau)$ has a Fourier series expansion:

$$u(\tau) = \sum_{n=0}^{\infty} g_{2n+1} \cos[(2n+1)]\tau, \tag{2.5}$$

which contains only odd multiples of τ.

Following the lowest order harmonic balance method [5,13], a reasonable and simple initial approximation satisfying initial conditions in equation (2.4) can be taken as

$$u_0(\tau) = A\cos\tau. \tag{2.6}$$

Here, $u_0(\tau)$ is a periodic function of τ, of period 2π. Using equation (2.6) leads to the following Fourier series expansions:

$$(1+\alpha u_0^4)u_0'' + 2\alpha u_0^3 u_0'^2 = -(A + \frac{3}{8}\alpha A^5)\cos\tau - \frac{7}{16}\alpha A^5 \cos 3\tau - \frac{3}{16}\alpha A^5 \cos 5\tau, \tag{2.7}$$

$$u_0 + \beta u_0^5 = (A + \frac{5}{8}\beta A^5)\cos\tau + \frac{5}{16}\beta A^5 \cos 3\tau + \frac{1}{16}\beta A^5 \cos 5\tau. \tag{2.8}$$

Substituting equation (2.6) into equation (2.3), making use of equations (2.7) and (2.8), and setting the coefficient of the resulting term $\cos\tau$ equal to zero give

$$(A + \frac{5}{8}\beta A^5) - (A + \frac{3}{8}\alpha A^5)\omega^2 = 0, \tag{2.9}$$

which can be solved for the first analytical approximate frequency ω_0 as a function of A :

$$\omega_0(A) = \sqrt{\frac{8+5\beta A^4}{8+3\alpha A^4}}. \tag{2.10}$$

Therefore, the first analytical approximate periodic solution is given by

$$u_0(\tau) = A\cos\tau, \quad \tau = \omega_0(A)t. \tag{2.11}$$

Next, we express the periodic solution to equation (2.3) with assigned conditions in equation (2.4) in the form of $u_0(\tau) + v(\tau)$ which is composed of the harmonic of the motion. Here, $u_0(\tau)$ is the main part satisfying initial conditions in equation (2.4), and

$v(\tau)$ is the correction part. Making linearization of the governing equations (2.3) and (2.4) with respect to the correction $v(\tau)$ at $u(\tau) = u_0(\tau)$ leads to

$$\begin{aligned}\omega^2[(1+\alpha u_0^4)u_0''+2\alpha u_0^3u_0'^2]+u_0+u_0^5+\omega^2\left[(1+\alpha u_0^4)v''\right.\\ \left.+4\alpha u_0^3u_0'v'+(4\alpha u_0^3u_0''+6\alpha u_0'^2u_0^2)v\right]+(1+5\beta u_0^4)v &= 0 \end{aligned} \tag{2.12}$$

and

$$v'(0) = 0, \tag{2.13}$$

where $v(\tau)$ is a periodic function of τ,of period 2π, to be determined later. Solving the resulting linear equations (2.12) and (2.13) in $v(\tau)$ by the method of harmonic balance may achieve the approximate frequency and periodic solution.

Making use of equation (2.6), we have the following Fourier series expansions:

$$1+\alpha u_0^4 = (1+\frac{3}{8}\alpha A^4)+\frac{1}{2}\alpha A^4\cos 2\tau+\frac{1}{8}\alpha A^4\cos 4\tau, \tag{2.14}$$

$$4\alpha u_0^3u_0' = -\alpha A^4\sin 2\tau-\frac{1}{2}\alpha A^4\sin 4\tau, \tag{2.15}$$

$$4\alpha u_0^3u_0''+6\alpha u'^2u_0^2 = -\frac{3}{4}\alpha A^4-2\alpha A^4\cos 2\tau-\frac{5}{4}\alpha A^4\cos 4\tau, \tag{2.16}$$

$$1+5\beta u_0^4 = 1+\frac{15}{8}\beta A^4+\frac{5}{2}\beta A^4\cos 2\tau+\frac{5}{8}\beta A^4\cos 4\tau. \tag{2.17}$$

To obtain the second approximation to the exact solution, $v(\tau)$ in equation (2.12), which must satisfy the initial conditions in equation (2.13), takes the form

$$v(\tau) = x_1(\cos\tau-\cos 3\tau). \tag{2.18}$$

Substituting equations (2.7), (2.8) and (2.14)-(2.18) into equation (2.12), expanding the expression in a trigonometeric series and setting the coefficients of the resulting items $\cos\tau$ and $\cos 3\tau$ equal to zeros, respectively, yield

$$(A+\frac{5}{8}\beta A^5)-(A+\frac{3}{8}\alpha A^5)\omega^2+[1+\frac{25}{16}\beta A^4+(-1+\frac{5}{16}\alpha A^4)\omega^2]x_1 = 0, \tag{2.19}$$

$$\frac{5}{16}\beta A^5-(\frac{7}{16}\alpha A^5)\omega^2+[(9+\frac{31}{16}\alpha A^4)\omega^2-1-\frac{5}{16}\beta A^4]x_1 = 0. \tag{2.20}$$

Eliminating x_1 from equations (2.19) and (2.20), we get

$$\tilde{A}\,\omega^4+\tilde{B}\,\omega^2+\tilde{C}=0, \tag{2.21}$$

where

$$\widetilde{A} = 144A + 92\alpha A^5 + \frac{151}{16}\alpha^2 A^9, \tag{2.22}$$

$$\widetilde{B} = -160A - 44\alpha A^5 - 100\beta A^5 - \frac{245}{8}\alpha\beta A^9, \tag{2.23}$$

$$\widetilde{C} = 16A + 20\beta A^5 + \frac{175}{16}\beta^2 A^9. \tag{2.24}$$

Equation (2.21) can be solved for the second analytical approximate frequency ω_1 as a function of A :

$$\omega_1(A) = \frac{1}{\sqrt{2}}\sqrt{\frac{2560 + 704\alpha A^4 + 1600\beta A^4 + 490\alpha\beta A^8 + \sqrt{\Delta}}{2304 + 1472\alpha A^4 + 151\alpha^2 A^8}}, \tag{2.25}$$

where

$$\begin{aligned} \Delta \;=\; & (2560 + 704\alpha A^4 + 1600\beta A^4 + 490\alpha\beta A^8)^2 \\ & -4(2304 + 1472\alpha A^4 + 151\alpha^2 A^8)(256 + 320\beta A^4 + 175\beta^2 A^8). \end{aligned} \tag{2.26}$$

Here the root of $\omega^2$with " $-$ " sign preceding $\sqrt{\Delta}$ has been dropped by inserting the limit $\lim_{A\to 0}\omega_1(A) = 1$. Furthermore, x_1 in equation (2.18) can be obtained by using either equation (2.19) or equation (2.20), i. e.

$$x_1(A) = -\frac{(16A + 10\beta A^5) - (16A + 6\alpha A^5)\omega^2}{(16 + 25\beta A^4) + (-16 + 5\alpha A^4)\omega^2} \tag{2.27}$$

and the corresponding second analytical approximate periodic solution is given by

$$\begin{aligned} u_1(\tau) &= u_0(\tau) + v(\tau) = [A + x_1(A)]\cos\tau - x_1(A)\cos 3\tau, \\ \tau &= \omega_1(A)t. \end{aligned} \tag{2.28}$$

To construct the next approximate solution, we replace $v(\tau)$ in equation (2.18) by

$$v(\tau) = y_1[\cos\tau - \cos 3\tau] + y_2[\cos 3\tau - \cos 5\tau], \tag{2.29}$$

which satisfies the initial conditions in equation (2.13) at the outset. Substituting equations (2.7), (2.8), (2.14)-(2.17) and (2.29) into equation (2.12), expanding the expression in trigonometric series and setting the coefficients of the resulting items $[\cos\tau, \cos 3\tau, \cos 5\tau]$ to zeros, respectiviely, give three relations for y_1, y_2, ω. These can be solved to obtain ω as a function of A. It should be clear how the procedure works for constructing further approximate solutions. Since we wish to calculate manually the analytical approximations to the exact solutions, the number of harmonics as those in equation (2.29) has to be small. The major reason is the complexity of algebra involved. However, this is not a major restriction because, as we will show in the next section, formulas (2.25)-(2.28) are capable of providing excellent analytical approximate representations to the exact frequency and periodic solution for small as well as large amplitudes of oscillation.

Other alternative techniques like the method of harmonic balance my give the first approximation as those in equations (2.10) and (2.11). However, when the method of harmonic balance is used to determine higher-order approximations, a set of algebraic equations with third-order non-linearity has to be solved for each given amplitude A of oscillation [5,11,13]. The corresponding numerical computation is rather involved and complicated. In contrast, formulas (2.25)-(2.28) are simple analytical approximate formulas and easy to be implemented, and they allow the explicit discussion of the influence of parameters and initial conditions on the frequency and the corresponding periodic solution.

3 Results and Discussion

In this section, we illustrate the applicability, accuracy and effectiveness of the proposed approach by comparing the analytical approximate frequency and periodic solution with the exact solutions.

The non-linear oscillator described in equation (2.1) and (2.2) is a conservative system. By integrating equation (2.1) and using the initial conditions in equation (2.2), we arrive at

$$\frac{1}{2}(1+\alpha u^4)(\frac{du}{dt})^2+\frac{1}{2}u^2+\frac{1}{6}\beta u^6=\frac{1}{2}A^2+\frac{1}{6}\beta A^6, \quad \text{for all } \ t\succeq 0. \tag{3.1}$$

From the representation above, we have

$$\frac{du}{dt}=\pm\left[\frac{3(A^2-u^2)+\beta(A^6-u^6)}{3(1+\alpha u^{4)}}\right]^{1/2}. \tag{3.2}$$

The time required for u to change from 0 to A is one-sixth of the exact period $T_e(A)$. Hence

$$T_e(A)=4\int_0^A\left[\frac{3(1+\alpha u^4)}{3(A^2-u^2)+\beta(A^6-u^6)}\right]^{1/2}du. \tag{3.3}$$

Letting $u=A\cos\theta$ in equation (3.3) leads to

$$T_e(A)=4\int_0^{\frac{\pi}{2}}\left[\frac{3(1+\alpha A^4\cos^4\theta)\sin^2\theta}{3\sin^2\theta+\beta A^4(1-\cos^6\theta)}\right]^{1/2}d\theta, \tag{3.4}$$

$$\begin{aligned}\omega_e(A) &= \frac{2\pi}{T_e(A)}\\ &= \frac{\pi}{2\displaystyle\int_0^{\frac{\pi}{2}}\left[\frac{3(1+\alpha A^4\cos^4\theta)\sin^2\theta}{(3\sin^2\theta+\beta A^4(1-\cos^6\theta))}\right]^{1/2}d\theta}.\end{aligned} \tag{3.5}$$

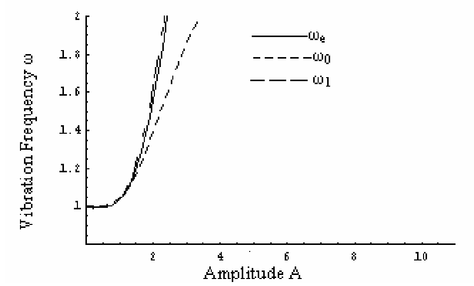

Figure 3.1: Dependence of the exact and the analytical approximate frequencies on the amplitude of oscillation for $\alpha = 0.1$, $\beta = 0.2$.

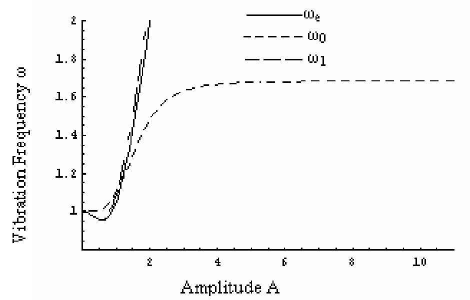

Figure 3.2: Dependence of the exact and the analytical approximate frequencies on the amplitude of oscillation for $\alpha = 1$, $\beta = 2$.

For $\alpha = 0.1$ and $\beta = 0.2$, the comparison of the exact frequency ω_e , obtained by integrating equation (3.5), with the first and second analytical approximate frequencies ω_0 and ω_1 computed, respectively, using equations (2.10) and (2.25), is illustrated in Figure 3.1. In Figure 3.1 indicates that the formula (2.25) is more accurate than the formula (2.10), and the former provides excellent approximation to the exact frequency for small as well as large values of amplitude of oscillation. The comparison of the corresponding analytical approximate frequencies with exact one for $\alpha = 1$ and $\beta = 1$ is shown in Figure 3.2. Again, similar agreement is observed. Furthermore, for any $\alpha > 0$, $\beta > 0$, we have

$$\lim_{A\to\infty} \omega_0(A) = \sqrt{\frac{5\beta}{3\alpha}}, \tag{3.6}$$

$$\lim_{A\to\infty} \omega_1(A) = \sqrt{\left(\frac{490 + 80\sqrt{21}}{151}\right)\left(\frac{\beta}{\alpha}\right)}, \tag{3.7}$$

$$\lim_{A\to\infty} \omega_e(A) = \frac{\pi\sqrt{3(\beta/\alpha)}}{6\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^2 t \sin t}{\sqrt{1-\cos^6 t}}\, dt}, \tag{3.8}$$

$$\lim_{A\to\infty}\frac{\omega_0(A)}{\omega_e(A)}=\frac{2\sqrt{5}}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos^2 t\sin t}{\sqrt{1-\cos^6 t}}dt\approx 0.74536, \tag{3.9}$$

$$\lim_{A\to\infty}\frac{\omega_1(A)}{\omega_e(A)}=\frac{2\sqrt{3}}{\pi}\sqrt{(\frac{490+80\sqrt{21}}{151})}\int_0^{\frac{\pi}{2}}\frac{\cos^2 t\sin t}{\sqrt{1-\cos^6 t}}dt\approx 0.89393 \tag{3.10}$$

Due to the increased significance of non-linear effects, the simple-mode harmonic approximation becomes insufficient. In contrast, the proposed formula (2.25) provides very accurate approximation. We can conclude that formula (2.25) is valid for the whole range of values of amplitude of oscillation and its maximum relative error$<10.607\%$, as obtained from equation (3.10).

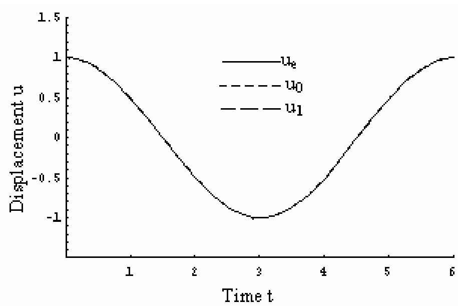

Figure 3.3: : Comparison of the analytical approximate periodic solution with exact solution for $\alpha=0.1$, $\beta=0.2$, and $A=1$.

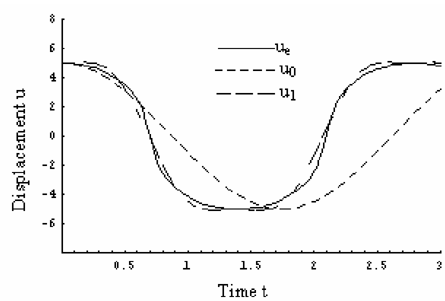

Figure 3.4: Comparison of the analytical approximate periodic solution with exact solution for $\alpha=0.1$, $\beta=0.2$, and $A=5$.

For $\alpha=0.1$, $\beta=0.2$, the exact periodic solution $u_e(t)$ achieved by integrating equations (2.1) and (2.2), the first and the second analytical approximate periodic solutions $u_0(t)$ and $u_1(t)$ computed, respectively, by equations (2.11) and (2.28), are plotted in Figures 3.3-3.5. The corresponding three solution are shown in Figures 3.6-3.8 for the case $\alpha=1$, $\beta=1$.

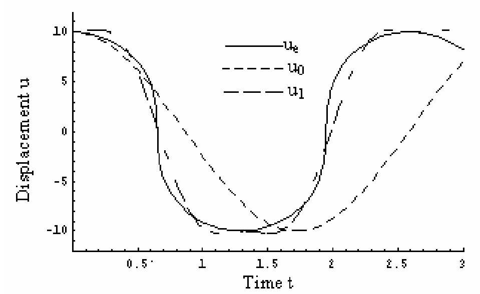

Figure 3.5: : Comparison of the analytical approximate periodic solution with exact solution for $\alpha = 0.1$, $\beta = 0.2$, and $A = 10$.

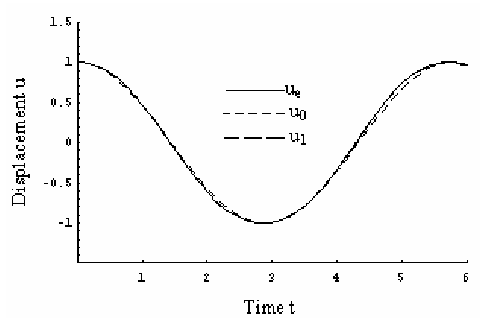

Figure 3.6: Comparison of the analytical approximate periodic solution with exact solution for $\alpha = 1$, $\beta = 2$, and $A = 1$.

These figures represent, respectively, three different amplitudes $A = 1$, 5 and 10. They show that the second analytical approximate periodic solutions provide the most excellent approximations to the exact periodic solutions for small as well as large amplitude of oscillation, but the first analytical approximate periodic solutions are generally acceptable only for small values of amplitude of oscillation. These figures also indicate that discrepancy of solutions widens as the modal constants α and β become larger. The above facts demonstrate that, unlike perturbation approximations, the present analytical approximate frequencies and periodic solutions apply well to small as well as large values of amplitude of oscillation.

4 Conclusions

The aims of the present work are to provide the combination of linearization of governing equation with the method of harmonic balance, a new approach has been proposed to solve the non-linear oscillation of a conservative system having inertia and static non-linearities. Unlike the classical harmonic balance method, linearization is performed perior

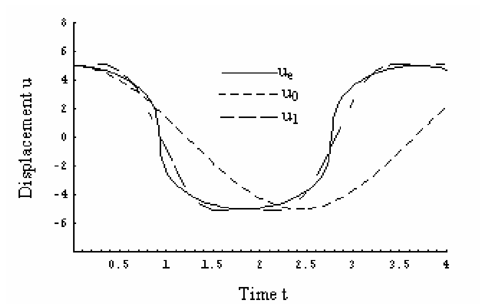

Figure 3.7: Comparison of the analytical approximate periodic solution with exact solution for $\alpha = 1$, $\beta = 2$, and $A = 5$.

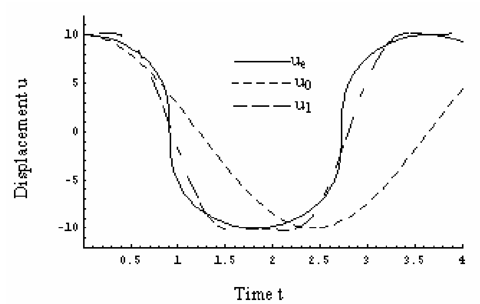

Figure 3.8: Comparison of the analytical approximate periodic solution with exact solution for $\alpha = 1$, $\beta = 2$, and $A = 10$.

to proceeding with harmonic balancing. As a result, we obtain a set of linear algebraic equations instead of one of non-linear algebraic equations, which enables us to establish analytical approximate formulas for the frequency and periodic solution. The method presented here is very simple in its principle, and is very easy to be applied. The analytical approximate formula shows excellent agreement with the exact solutions has been demonstrated and discussed. The most interesting features of this new approach are its simplicity and its very good accuracy in a wide range of amplitude of oscillation.

For any positive modal constants, the discrepancy of the second analytical approximate frequency with respect to the exact one will never exceed 10.607 %. The method proposed in this paper can also be used to find analytical approximate solution to other conservative oscillators.

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