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Property Q and a Common Fixed Point Theorem of (ψ, φ)-Weakly Contractive Maps in G-Metric Spaces

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Abstract: In this paper, we establish a coincidence and a common fixed point for a pair of weakly compatible mappings under (ψ, φ)-weakly contractive condition in G-metric spaces. Our results improve and generalize the results of Khandaqji et al. [20]. We also provide an example to support our results. Moreover, we prove that these mappings satisfy Property Q.

Keywords: Weak-compatible maps, coincidence point, common fixed point, G-metrics, property Q.

1. Introduction and preliminaries

Until 1968 Banach's contraction principle was the main tool used to establish the existence and uniqueness of fixed points. It has been used in many different fields of mathematics, but suffers from one drawback.

In order to use the contractive condition, a self-mapping f must be Lipschitz continuous, with Lipschitz constant less than 1. In particular, f must be continuous at all points of its domain. In 1968 Kannan [17] constructed a contractive condition, like that of Banach, possessed a unique fixed point, which could be obtained by starting at any point x_0 in the space, and using function iteration defined by $x_{n+1} = Tx_n$ (also called Picard iteration). However, unlike the Banach condition, there exist discontinuous functions satisfying the definition of Kannan, although such mappings are continuous at the fixed point. Following the appearance of [17] many authors created contractive conditions not requiring continuity of the mapping. Today fixed point literature of contractive mappings contains many such papers. One survey of a number of these conditions appears in [34].

Weak contraction principle is a generalization of Banach's contraction principle which was first given by Alber et al. in Hilbert spaces [5] and after that extended to metric spaces by Rhoades [35].

Khan et al. [19] initiated the use of a control function in metric fixed point theory, which they called an altering distance function. This function and its generalizations have been used in fixed point problems in metric and generalized metric spaces (see [6], [7], [8], [11], [20]).

The notion of G-metric space was introduced by Mustafa and Sims [26], [27] as a generalization of the notion of metric spaces. Afterwards Mustafa, Sims and others authors introduced and developed several fixed point theorems for mappings satisfying different contractive conditions in G-metric spaces, also extend known theorems in metric spaces to G-metric spaces see [1]-[4], [6], [7], [9], [18], [20]-[32], and many other papers.

The study of unique common fixed points of mappings satisfying strict contractive conditions has been at the center of rigorous research activity. Study of common fixed point theorems in G-metric spaces was initiated by Abbas and Rhoades [1]. Consistent with [27], the following definitions and results will be needed in the sequel. Now onwards, N will denote the set of natural numbers.

Definition 1 *Let X be a nonempty set and let $G : X^3 \to [0, \infty)$ be a function satisfying:*

(G_1)$G(x, y, z) = 0$ *if* $x = y = z$,
(G_2)$0 < G(x, x, y)$, *for all* $x, y \in X$, *with* $x \neq y$,
(G_3)$G(x, x, y) \leq G(x, y, z), \forall x, y, z \in X$, *with* $z \neq y$,
(G_4)$G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, *(symmetry in all three variables)*,

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(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z), \forall x,y,z,a \in X$, *(rectangle inequality).*

Then the function G is called a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 2 *Let (X,G) be a G-metric space, a sequence (x_n) is said to be*

(i)G-convergent if for every $\varepsilon > 0$, there exists an $x \in X$, and $k \in \mathbf{N}$ such that for all $m,n \geq k, G(x,x_n,x_m) < \varepsilon$.
(ii)G-Cauchy if for every $\varepsilon > 0$, there exists an $k \in \mathbf{N}$ such that for all $m,n,p \geq k, G(x_m,x_n,x_p) < \varepsilon$, that is $G(x_m,x_n,x_p) \to 0$ as $m,n,p \to \infty$.
(iii)A space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent.

Lemma 3 *Let (X,G) be a G-metric space. Then the following are equivalent:*

(i)(x_n) is convergent to x,
(ii)$G(x_n,x_n,x) \to 0$ as $n \to \infty$,
(iii)$G(x_n,x,x) \to 0$ as $n \to \infty$,
(iv)$G(x_n,x_m,x) \to 0$ as $n,m \to \infty$,

Lemma 4 *Let (X,G) be a G-metric space. Then the following are equivalent:*

(i)The sequence (x_n) is G-Cauchy,
(ii)for every $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$ for $m,n \geq k$.

Lemma 5 *Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 6 *A G metric space X is symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.*

Proposition 7 *every G-metric space (X,G) will define a metric space (X,d_G) by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \qquad \forall x,y \in X.$$

Proposition 8 *Let (X,G) be a G-metric space. Then for any x,y,z, and $a \in X$, it follows that*

(i)if $G(x,y,z) = 0$ then $x = y = z$,
(ii)$G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,
(iii)$G(x,y,y) \leq 2G(x,x,y)$,
(iv)$G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,
(v)$G(x,y,z) \leq \frac{2}{3}(G(x,y,a) + G(x,a,z) + G(a,y,z))$,
(vi)$G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$,

Jungck [14] proved a common fixed point theorem for commuting mappings as a generalization of the Banach's fixed point theorem. The concept of the commutativity has generalized in several ways. For this Sessa [37] introduced the concept of weakly commuting mappings, Jungck [15] extend this concept to compatible maps. In 1998, Jungck and Rhoades [16] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse need not to be true for example see [33].

Definition 9 *Let f and g be self maps of a nonempty set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f, g and w is called a point of coincidence of f and g.*

Definition 10 *[16] Two self mappings f and g are said to be weakly compatible if they commute at their coincidence points, that is, $fx = gx$ implies that $fgx = gfx$.*

We will denote the set all fixed points of a self mapping f from X into itself by $F(f)$, i.e., $F(f) = \{x \in X : fx = x\}$. It is obvious that if x is a fixed point of f then it is also a fixed point of f^n for each n, i. e., $F(f) \subseteq F(f^n)$ if $F(f) \neq \emptyset$. However converse is false. Indeed the mapping $f : R \to R$ defined by $fx = \frac{1}{2} - x$ has a unique fixed point $x = \frac{1}{4}$, but every $x \in R$ is a fixed point for f^n, for each even $n > 1$.

Jeong and Rhoades [12] showed that maps satisfying many contractive conditions have property P. They have [13] also shown that for a number of contractive conditions involving pairs of maps have property Q.

Several works has been done related to Property P and Q (see for instance [9], [10], [20] and [36]).

Definition 11 *(Property P [12]) Let f be a self-mapping of metric space with fixed point set $F(f) \neq \emptyset$. Then f is said to have property P if $F(f^n) = F(f)$, for each $n \in N$. Equivalently, a mapping has property P if every periodic point is a fixed point.*

Definition 12 *(Property Q [13]) Let f and g be self-mappings of metric space with $F(f)\cap F(g)\neq\emptyset$. f and g are said to have property Q if $F(f^n)\cap F(g^n)=F(f)\cap F(g)$, for each $n\in N$.*

Definition 13 *(Altering Distance Function [19]) A function $\psi:[0,\infty)\to[0,\infty)$ is called altering distance function if*

(i)ψ is increasing and continuous,
(ii)$\psi(t)=0$ if and only if $t=0$.

Recently Khandaqji et al. [20] proved the following Theorem

Theorem 14. *Let (X,G) be a complete G-metric space. Let f be a self mapping on X satisfying the following*

$$\psi(G(fx,fy,fz))\leq\psi(M(x,y,z))-\varphi(M(x,y,z)),$$

where

$$\begin{aligned}M(x,y,z)=\max\{&G(x,y,z),G(x,fx,fx),G(y,fy,fy),G(z,fz,fz),\\&\alpha G(fx,fx,y)+(1-\alpha)G(fy,fy,z),\\&\beta G(x,fx,fx)+(1-\beta)G(y,fy,fy)\},\end{aligned}$$

for all $x,y,z\in X$, where $0<\alpha,\beta<1$, ψ is an altering distance function, and $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function with $\varphi(t)=0$ if and only if $t=0$, then f has a unique fixed point.

The rest of this paper is organized as follows: in section 2 we establish a coincidence and a fixed point theorem for two weakly compatible mappings satisfying generalized (ψ,φ)-weakly contractive condition in which φ need not be continuous in G-metric spaces. The results in this section improve and extend Theorem (14). Also we give an example satisfying all requirements of our results. Finally in section 3 we prove that the mappings satisfying property Q.

In the sequel, we define
$\Psi=\{\psi:[0,\infty)\to[0,\infty)$ is a continuous non-decreasing function $\}$,
$\Phi=\{\varphi:[0,\infty)\to[0,\infty)$ is a non-decreasing function with $\varphi(t)=0$ if and only if $t=0\}$.

2. Main results

First of all we state the following Lemmas which are fundamental in the sequel.

Lemma 15 *[1] Let f and g be weakly compatible self mappings of nonempty set X. If f and g have a unique point of coincidence $w=fx=gx$, then w is the unique common fixed point of f and g.*

Lemma 16 *Let (X,G) be a G-metric space and $f,g:(X,G)\to(X,G)$ two mappings such that*

$$\psi(G(fx,fy,fz))\leq\psi(M(x,y,z))-\varphi(M(x,y,z)),\tag{1}$$

where

$$\begin{aligned}M(x,y,z)=\max\{&G(gx,gy,gz),G(gx,fx,fx),G(gy,fy,fy),G(gz,fz,fz),\\&\alpha G(fx,fx,gy)+(1-\alpha)G(fy,fy,gz),\\&\beta G(gx,fx,fx)+(1-\beta)G(gy,fy,fy)\},\end{aligned}$$

for all $x,y,z\in X$, where $0<\alpha,\beta<1$ and $\psi,\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)=0$ if and only if $t=0$. Then, f and g have at most a point of coincidence.

Proof. Suppose that $u=fp=gp$ and $v=fq=gq$. Then by (1) we have

$$\psi(G(fp,fp,fq))\leq\psi(M(p,p,q))-\varphi(M(p,p,q)),$$

where

$$\begin{aligned}M(p,p,q)=\max\{&G(gp,gp,gq),G(gp,fp,fp),G(gp,fp,fp),G(gq,fq,fq),\\&\alpha G(fp,fp,gp)+(1-\alpha)G(fp,fp,gq),\\&\beta G(gp,fp,fp)+(1-\beta)G(gp,fp,fp)\},\end{aligned}$$

$$M(p,p,q)=\max\{G(u,u,v),0,0,0,(1-\alpha)G(u,u,v),0\}=G(u,u,v).$$

Then

$$\psi(G(u,u,v))\leq\psi(G(u,u,v))-\varphi(G(u,u,v)).$$

Therefore $\varphi(G(u,u,v))=0$, hence $u=v$.

Theorem 17. *Let (X,G) be a G-metric space and $f,g:(X,G)\to(X,G)$ satisfying inequality (1) where $\psi\in\Psi$ and $\varphi\in\Phi$. If $f(X)\subseteq g(X)$ and $f(X)$ or $g(X)$ is a G-complete metric subspace of X, then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X since $f(X)\subseteq g(X)$ we can choose $x_1\in X$ such that $fx_0=gx_1$. Continuing this process, we get $y_n=fx_n=gx_{n+1}$. If $y_n=y_{n+1}$ for some n, then $y_{n+1}=fx_{n+1}=gx_{n+1}$ yields f and g have a coincidence point.

We may assume that $y_n\neq y_{n+1}$ for each n. Then from (1) we have

$$\begin{aligned}\psi(G(y_n,y_{n+1},y_{n+1}))&=\psi(G(fx_n,fx_{n+1},fx_{n+1}))\\&\le\psi(M(x_n,x_{n+1},x_{n+1}))-\varphi(M(x_n,x_{n+1},x_{n+1})),\end{aligned}\tag{2}$$

where

$$\begin{aligned}M(x_n,x_{n+1},x_{n+1})=\max\{&G(gx_n,gx_{n+1},gx_{n+1}),G(gx_n,fx_n,fx_n),\\&G(gx_{n+1},fx_{n+1},fx_{n+1}),G(gx_{n+1},fx_{n+1},fx_{n+1}),\\&\alpha G(fx_n,fx_n,gx_{n+1})+(1-\alpha)G(fx_{n+1},fx_{n+1},gx_{n+1}),\\&\beta G(gx_n,fx_n,fx_n)+(1-\beta)G(gx_{n+1},fx_{n+1},fx_{n+1})\},\end{aligned}$$

yields,

$$\begin{aligned}M(x_n,x_{n+1},x_{n+1})=\max\{&G(y_{n-1},y_n,y_n),G(y_{n-1},y_n,y_n),G(y_n,y_{n+1},y_{n+1}),\\&\alpha G(y_n,y_n,y_n)+(1-\alpha)G(y_{n+1},y_{n+1},y_n),\\&\beta G(y_{n-1},y_n,y_n)+(1-\beta)G(y_n,y_{n+1},y_{n+1})\}.\end{aligned}$$

Therefore

$$M(x_n,x_{n+1},x_{n+1})=\max\{G(y_{n-1},y_n,y_n),G(y_n,y_{n+1},y_{n+1})\}.$$

If for some $n\in N$, $M(x_n,x_{n+1},x_{n+1})=G(y_n,y_{n+1},y_{n+1})$, from (2) we obtain

$$\psi(G(y_n,y_{n+1},y_{n+1}))\le\psi(G(y_n,y_{n+1},y_{n+1}))-\varphi(G(y_n,y_{n+1},y_{n+1})).$$

Hence $\varphi(G(y_n,y_{n+1},y_{n+1}))=0$, implies that $y_n=y_{n+1}$, which is a contradiction with $y_n\neq y_{n+1}$. Thus $M(x_n,x_{n+1},x_{n+1})=G(y_{n-1},y_n,y_n)$ for each $n\in N$ and (2) becomes

$$\begin{aligned}\psi(G(y_n,y_{n+1},y_{n+1}))&\le\psi(G(y_{n-1},y_n,y_n))-\varphi(G(y_{n-1},y_n,y_n))\\&\le\psi(G(y_{n-1},y_n,y_n)).\end{aligned}\tag{3}$$

Since ψ is an increasing function, then from (3)we have

$$G(y_n,y_{n+1},y_{n+1})\le G(y_{n-1},y_n,y_n)\qquad\forall n\in N.$$

Therefore, $\{G(y_{n-1},y_n,y_n),\ n\in N\}$ is a non-increasing sequence of positive real numbers. Hence there exists $\delta\ge0$ such that

$$\lim_{n\to\infty}G(y_{n-1},y_n,y_n)=\delta\quad\text{and}\quad 0\le\delta\le G(y_{n-1},y_n,y_n)\qquad\forall n\in N.\tag{4}$$

From (3) we get

$$\varphi(G(y_{n-1},y_n,y_n))\le\psi(G(y_{n-1},y_n,y_n))-\psi(G(y_n,y_{n+1},y_{n+1})).\tag{5}$$

By (4), (5) and Since φ is non-decreasing function we obtain

$$\begin{aligned}0\le\varphi(\delta)&\le\varphi(G(y_{n-1},y_n,y_n))\\&\le\psi(G(y_{n-1},y_n,y_n))-\psi(G(y_n,y_{n+1},y_{n+1})).\end{aligned}\tag{6}$$

Letting $n\to\infty$, and by continuity of ψ it follows that

$$0\le\varphi(\delta)\le\lim_{n\to\infty}\varphi(y_{n-1},y_n,y_n)\le\psi(\delta)-\psi(\delta).$$

Hence $\varphi(\delta) = 0$, that means $\delta = 0$, so we have

$$\lim_{n\to\infty} G(y_{n-1}, y_n, y_n) = 0. \tag{7}$$

In other words, from Proposition (8) we obtain

$$0 \leq G(y_n, y_{n-1}, y_{n-1}) \leq 2G(y_{n-1}, y_n, y_n).$$

Letting $n \to \infty$, and using (7), we find that

$$\lim_{n\to\infty} G(y_n, y_{n-1}, y_{n-1}) = 0. \tag{8}$$

Now, we show that the sequence $\{y_n\}$ is a G-Cauchy sequence in X. Suppose that $\{y_n\}$ is not. Then there exist $\varepsilon > 0$, and subsequences $\{y_{m(k)}\}$, $\{y_{n(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) > k$ such that

$$G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \geq \varepsilon. \tag{9}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (9). Then

$$G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) < \varepsilon. \tag{10}$$

We want to prove that
(i) $\lim_{k\to\infty} G(y_{n(k)}, y_{m(k)}, y_{m(k)}) = \varepsilon$, (ii) $\lim_{k\to\infty} G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) = \varepsilon$,
(iii) $\lim_{k\to\infty} G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) = \varepsilon$.
By (9), (10) and G_5 we have

$$\begin{aligned} \varepsilon \leq G(y_{n(k)}, y_{m(k)}, y_{m(k)}) &\leq G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) \\ &< \varepsilon + G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}). \end{aligned} \tag{11}$$

Now, letting $k \to \infty$ in (11) and by (8) we conclude that

$$\lim_{k\to\infty} G(y_{n(k)}, y_{m(k)}, y_{m(k)}) = \varepsilon. \tag{12}$$

Moreover, we obtain

$$\begin{aligned} G(y_{n(k)}, y_{m(k)}, y_{m(k)}) &\leq G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \\ &+ G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}), \end{aligned}$$

$$\begin{aligned} G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) &\leq G(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\ &+ G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}). \end{aligned}$$

Letting $k \to \infty$, in the two above inequalities and using (7), (8) and (12) we have

$$\lim_{k\to\infty} G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) = \varepsilon. \tag{13}$$

similarly,

$$G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \leq G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}),$$

$$G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) \leq G(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)}).$$

Letting $k \to \infty$ in the two above inequalities and using (7), (8) and (12) we find that

$$\lim_{k\to\infty} G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) = \varepsilon. \tag{14}$$

Now, we have

$$\begin{aligned}M(x_{m(k)},x_{m(k)},x_{n(k)}) = \max\{&G(gx_{m(k)},gx_{m(k)},gx_{n(k)}),G(gx_{m(k)},fx_{m(k)},fx_{m(k)}),\\&G(gx_{m(k)},fx_{m(k)},fx_{m(k)}),G(gx_{n(k)},fx_{n(k)},fx_{n(k)}),\\&\alpha G(fx_{m(k)},fx_{m(k)},gx_{m(k)})+(1-\alpha)G(fx_{m(k)},fx_{m(k)},gx_{n(k)}),\\&\beta G(gx_{m(k)},fx_{m(k)},fx_{m(k)})+(1-\beta)G(gx_{m(k)},fx_{m(k)},fx_{m(k)})\}\\=\max\{&G(y_{m(k)-1},y_{m(k)-1},y_{n(k)-1}),G(y_{m(k)-1},y_{m(k)},y_{m(k)}),\\&G(y_{m(k)-1},y_{m(k)},y_{m(k)}),G(y_{n(k)-1},y_{n(k)},y_{n(k)}),\\&\alpha G(y_{m(k)},y_{m(k)},y_{m(k)-1})+(1-\alpha)G(y_{m(k)},y_{m(k)},y_{n(k)-1}),\\&\beta G(y_{m(k)-1},y_{m(k)},y_{m(k)})+(1-\beta)G(y_{m(k)-1},y_{m(k)},y_{m(k)})\}.\end{aligned}$$

Taking the limits as $k\to\infty$, and using (7), (13) and (14) we obtain

$$\lim_{k\to\infty}M(x_{m(k)},x_{m(k)},x_{n(k)})=\max\{\varepsilon,0,(1-\alpha)\varepsilon\}=\varepsilon. \tag{15}$$

Setting $x=y=x_{m(k)}$ and $z=x_{n(k)}$ in (1) we conclude that

$$\begin{aligned}\psi(G(y_{m(k)},y_{m(k)},y_{n(k)}))&=\psi(G(fx_{m(k)},fx_{m(k)},fx_{n(k)}))\\&\le\psi(M(x_{m(k)},x_{m(k)},x_{n(k)}))-\varphi(M(x_{m(k)},x_{m(k)},x_{n(k)})).\end{aligned}$$

This gives that

$$\varphi(M(x_{m(k)},x_{m(k)},x_{n(k)}))\le\psi(M(x_{m(k)},x_{m(k)},x_{n(k)}))-\psi(G(y_{m(k)},y_{m(k)},y_{n(k)})).$$

Since

$$\lim_{k\to\infty}M(x_{m(k)},x_{m(k)},x_{n(k)})=\varepsilon \quad\text{and}\quad \lim_{k\to\infty}G(y_{n(k)},y_{m(k)},y_{m(k)})=\varepsilon,$$

we get $\frac{1}{2}\varepsilon\le M(x_{m(k)},x_{m(k)},x_{n(k)})$ for sufficiently large k. Since φ is non-decreasing we obtain

$$0\le\varphi(\frac{1}{2}\varepsilon)\le\varphi(M(x_{m(k)},x_{m(k)},x_{n(k)})),$$

for sufficiently large k. Therefore we find that

$$\begin{aligned}0\le\varphi(\frac{1}{2}\varepsilon)&\le\varphi(M(x_{m(k)},x_{m(k)},x_{n(k)}))\\&\le\psi(M(x_{m(k)},x_{m(k)},x_{n(k)}))-\psi(G(y_{m(k)},y_{m(k)},y_{n(k)})).\end{aligned}$$

for sufficiently large k.

Letting $k\to\infty$ and by (12), (15) and the continuity of ψ in the last inequality we get

$$\begin{aligned}0\le\varphi(\frac{1}{2}\varepsilon)&\le\lim_{k\to\infty}\varphi(M(x_{m(k)},x_{m(k)},x_{n(k)}))\\&\le\lim_{k\to\infty}(\psi(M(x_{m(k)},x_{m(k)},x_{n(k)}))-\psi(G(y_{m(k)},y_{m(k)},y_{n(k)})))\\&=\psi(\varepsilon)-\psi(\varepsilon)=0.\end{aligned}$$

Hence $\varphi(\frac{1}{2}\varepsilon)=0$. Thus from the property of φ we have $\varepsilon=0$ which is a contradiction since $\varepsilon>0$. Then $\{y_n\}$ is a G-Cauchy sequence.

Suppose that $g(X)$ is G-complete subspace of X, so there exists a point $q\in g(X)$ such that $\lim_{n\to\infty}y_n=\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_{n+1}=q$. Also, we can find a point $p\in X$ such that $gp=q$.

Now, we prove that $fp=q$. By (1) we have

$$\psi(G(fx_n,fp,fp))\le\psi(M(x_n,p,p))-\varphi(M(x_n,p,p)), \tag{16}$$

where

$$\begin{aligned}M(x_n,p,p) &= \max\{G(gx_n,gp,gp),G(gx_n,fx_n,fx_n),G(gp,fp,fp),\\ &\quad G(gp,fp,fp),\alpha G(fx_n,fx_n,gp)+(1-\alpha)G(fp,fp,gp),\\ &\quad \beta G(gx_n,fx_n,fx_n)+(1-\beta)G(gp,fp,fp)\}\\ &= \max\{G(gx_n,q,q),G(gx_n,fx_n,fx_n),G(q,fp,fp),\\ &\quad \alpha G(fx_n,fx_n,q)+(1-\alpha)G(fp,fp,q),\\ &\quad \beta G(gx_n,fx_n,fx_n)+(1-\beta)G(q,fp,fp)\}.\end{aligned}$$

Letting $n\to\infty$ we have

$$\lim_{n\to\infty} M(x_n,p,p)=G(q,fp,fp). \tag{17}$$

From (16) one gets

$$\varphi(M(x_n,p,p))\le\psi(M(x_n,p,p))-\psi(G(fx_n,fp,fp)). \tag{18}$$

Since

$$\lim_{k\to\infty} M(x_n,p,p)=G(q,fp,fp),$$

we get $\frac{1}{2}G(q,fp,fp)\le M(x_n,p,p)$ for sufficiently large n. Since φ is nondecreasing we have

$$0\le\varphi(\frac{1}{2}G(q,fp,fp))\le\varphi(M(x_n,p,p)), \tag{19}$$

for sufficiently large n. Therefore, (18) and (19) imply that

$$\begin{aligned}0&\le\varphi(\frac{1}{2}G(q,fp,fp))\le\varphi(M(x_n,p,p))\\ &\le\psi(M(x_n,p,p))-\psi(G(fx_n,fp,fp)),\end{aligned}$$

for sufficiently large n. Further more, letting $n\to\infty$ and using (17) and the continuity of ψ in the last inequality we get

$$\begin{aligned}0&\le\varphi(\frac{1}{2}G(q,fp,fp))\le\lim_{n\to\infty}\varphi(M(x_n,p,p))\\ &\le\psi(G(q,fp,fp)))-\psi(G(fx_n,fp,fp))=0,\end{aligned}$$

hence $\varphi(\frac{1}{2}G(q,fp,fp))=0$, so that $G(q,fp,fp)=0$, and then $fp=q$.

Then q is a point of coincidence of f and g. So from lemma (16) q is the unique point of coincidence, and from lemma (15) q is the unique common fixed point of f and g. The proof is similar when we assume that $f(X)$ is complete since $f(X)\subseteq g(X)$.

Corollary 18 *Let (X,G) be a G-metric space and $f,g:(X,G)\to(X,G)$ satisfying the following inequality*

$$\begin{aligned}G(fx,fy,fz)\le\lambda\max\{&G(gx,gy,gz),G(gx,fx,fx),G(gy,fy,fy),G(gz,fz,fz),\\ &\alpha G(fx,fx,gy)+(1-\alpha)G(fy,fy,gz),\\ &\beta G(gx,fx,fx)+(1-\beta)G(gy,fy,fy)\},\end{aligned}$$

for all $x,y,z\in X$, where $0<\lambda,\alpha,\beta<1$, If $f(X)\subseteq g(X)$ and $f(X)$ or $g(X)$ is a G-complete metric subspace of (X,G), then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Define $\psi,\varphi:[0,\infty)\to[0,\infty)$ by $\psi(t)=t$, and $\varphi(t)=t-\lambda t$, then it is clearly that $\psi\in\Psi$ and $\varphi\in\Phi$. So the result follows by taking $\psi(t)=t$ and $\varphi(t)=t-\lambda t$ in Theorem (17).

If we put $g=I$, where I is the identity mapping, in Theorem (17), we have the following Corollary.

Corollary 19 *Let (X,G) be a complete G-metric space. Let f be a self mapping on X satisfying the following*

$$\psi(G(fx,fy,fz))\le\psi(M(x,y,z))-\varphi(M(x,y,z)),$$

where

$$\begin{aligned}M(x,y,z)=\max\{&G(x,y,z),G(x,fx,fx),G(y,fy,fy),G(z,fz,fz),\\ &\alpha G(fx,fx,y)+(1-\alpha)G(fy,fy,z),\\ &\beta G(x,fx,fx)+(1-\beta)G(y,fy,fy)\},\end{aligned}$$

for all $x,y,z\in X$, where $0<\alpha,\beta<1$, where $\psi\in\Psi$ and $\varphi\in\Phi$. Then f has a unique fixed point.

The following example was give by Abbas [4], which satisfying the hypotheses of Theorem (17).

Example 1. Let $X=\{0,1,2\}$ be a set with G-metric defined by

(x,y,z)	$G(x,y,z)$
$(0,0,0),(1,1,1),(2,2,2),$	0
$(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(0,0,2),(0,2,0),(2,0,0),$	1
$(0,2,2),(2,0,2),(2,2,0),(1,1,2),(1,2,1),(2,1,1),(1,2,2),(2,1,2),(2,2,1),$	2
$(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0),$	2

One note that G is a non-symmetric as $G(0,0,2)\neq G(0,2,2)$. For f and g are self mappings of X defined by

x	$f(x)$	$g(x)$
0	0	0
1	1	2
2	0	1

It is clearly that $f(X)\subseteq g(X)$ and the pair of mappings (f,g) is weakly compatible where 0 is the only coincidence point of f and g and $fg0=f0=0=g0=gf0$. Also, $M(x,y,z)\leq 2$. If we define $\psi,\varphi:[0,\infty)\to[0,\infty)$ by

$$\psi(t)=t^3+1 \quad and \quad \varphi(t)=\begin{cases}\frac{t}{4}, & \text{if } 0\leq t<1\\ \frac{1}{3}, & \text{if } t=1\\ \frac{t^3}{2}, & \text{if } t>1.\end{cases}$$

Then ψ and φ have the properties mentioned in Theorem (17). If

$$\begin{aligned}(x,y,z)\in&\{(0,0,0),(1,1,1),(2,2,2),(0,0,2),(0,2,0),\\&(2,0,0),(0,2,2),(2,0,2),(2,2,0)\},\end{aligned}$$

then $G(fx,fy,fz)=0$ and $M(x,y,z)\in\{0,1,2\}$ therefore

$$\psi(G(fx,fy,fz))=1\leq\psi(M(x,y,z))-\varphi(M(x,y,z),$$

then (1) holds.
On otherwise one find that $G(fx,fy,fz)=1$ and $M(x,y,z)=2$. Hence

$$\psi(G(fx,fy,fz))=2\leq\psi(M(x,y,z))-\varphi(M(x,y,z))=9-\frac{8}{2}=5.$$

Then condition (1) satisfied for all $x,y,z\in X$. Hence all hypotheses of Theorem (17) are satisfied and 0 is the unique common fixed point of f and g. We note that φ is not a lower semi-continuous function.

3. Mappings with Property Q

Theorem 20. *Under the condition of Theorem (17), If f and g are commuting then f and g has property Q.*

Proof. From Theorem (17), $F(f)\cap F(g)\neq\emptyset$. Therefore $F(f^n)\cap F(g^n)\neq\emptyset$ for each positive integer n. Let n be a fixed positive integer greater than 1 and suppose that $p\in F(f^n)\cap F(g^n)$. We claim that $p\in F(f)\cap F(g)$.

Let $p\in F(f^n)\cap F(g^n)$. Then, for any positive integers i,j,k,l,r,s satisfying $1\leq i,j,k,l,r,s\leq n$, we have

$$\begin{aligned}\psi(G(f^ig^jp,f^kg^lp,f^rg^sp))&=\psi(G(f(f^{i-1}g^jp),f(f^{k-1}g^lp),f(f^{r-1}g^sp))\\&\leq\psi(M(f^{i-1}g^jp,f^{k-1}g^lp,f^{r-1}g^sp))-\varphi(M(f^{i-1}g^jp,f^{k-1}g^lp,f^{r-1}g^sp)),\end{aligned}$$

where

$$\begin{aligned}M(f^{i-1}g^jp,&f^{k-1}g^lp,f^{r-1}g^sp)=\max\{G(f^{i-1}g^{j+1}p,f^{k-1}g^{l+1}p,f^{r-1}g^{s+1}p),\\&G(f^{i-1}g^{j+1}p,f^ig^jp,f^ig^jp),G(f^{k-1}g^{l+1}p,f^kg^lp,f^kg^lp),\\&G(f^{r-1}g^{s+1}p,f^rg^sp,f^rg^sp),\\&\alpha G(f^ig^jp,f^ig^jp,f^{k-1}g^{l+1}p)+(1-\alpha)G(f^kg^lp,f^kg^lp,f^{r-1}g^{s+1}p),\\&\beta G(f^{i-1}g^{j+1}p,f^ig^jp,f^ig^jp)+(1-\beta)G(f^{k-1}g^{l+1}p,f^kg^lp,f^kg^lp)\},\end{aligned}$$

Define

$$\delta = \max_{1\le i,j,k,l,r,s\le n}\{G(f^i g^j p, f^k g^l p, f^r g^s p)\}$$

Assume that $\delta > 0$, then from (1) it follows that

$$\psi(\delta) \le \psi(M_\delta) - \varphi(M_\delta) \tag{20}$$

where M_δ is $M(x, y, z)$ corresponding δ.

Since $M_\delta \le \delta$, then $\psi(M_\delta) \le \psi(\delta)$, so we get

$$\psi(\delta) \le \psi(M_\delta) - \varphi(M_\delta) \le \psi(\delta) - \varphi(M_\delta).$$

Hence $\varphi(M_\delta) = 0$, so $M_\delta = 0$. Substituting $M_\delta = 0$. in (20) $\psi(\delta) \le \psi(0) - \varphi(0)$, therefore $\psi(\delta) \le \psi(0)$, since ψ is non-decreasing then $\delta \le 0$, which is a contradiction for $\delta > 0$. Hence $\delta = 0$. In particular if

$$i = 1, \quad j = l = k = s = r = n \quad \text{and} \quad j = 1, \quad i = l = k = s = r = n,$$

we conclude that

$$G(fg^n p, f^n g^n p, f^n g^n p) = 0 \quad \text{and} \quad G(f^n gp, f^n g^n p, f^n g^n p) = 0,$$

this means

$$G(fp, p, p) = 0 \quad \text{and} \quad G(gp, p, p) = 0.$$

Hence $fp = gp = p$, implies that $p \in F(f) \cap F(g)$. Hence f and g have Property Q.

Example 2. Let $X = R$ with the G-metric space

$$G(x, y, z) = |x - y| + |y - z| + |z - x|,$$

and f and g are self mappings of X defined by $f(x) = 2$ and $g(x) = 2x - 2$. We take $\psi(t) = t$ and $\varphi(t) = \frac{1}{2}t$, for $t \in [0, \infty)$ and $\alpha, \beta \in (0, 1]$. So we have

$$\psi(M(x, y, z)) - \varphi(M(x, y, z)) = \frac{1}{2}M(x, y, z).$$

It is clearly that $f(X) \subseteq g(X)$ and (f, g) is commuting and hence weakly compatible. Also,

$$0 = \psi(G(fx, fy, fz)) \le \frac{1}{2}M(x, y, z), \quad \forall x, y, z \in X.$$

Therefore condition (1) holds for all $x, y, z \in X$, and hypotheses of Theorem (17) are satisfied, and 2 is the unique common fixed point of the mappings f and g.
Moreover, if $p \in F(f^n) \cap F(g^n)$, then $p = 2$ and so f and g have Property Q

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