Applied Mathematics & Information Sciences **2**(2) (2008), 211–223
– An International Journal
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On Dissipative Quadratic Stochastic Operators

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Received November 20, 2007; Revised January 4, 2008; Accepted January 31, 2008

In present paper we introduce a notion of dissipative stochastic operator. Furthermore, we prove necessary conditions for dissipativity of quadratic stochastic operators. We also study certain limit behavior of such operators. Finally, we prove ergodic theorem for any dissipative stochastic operator.

Keywords: Dissipative stochastic operators, quadratic stochastic operator, majorization, ergodic.

2000 Mathematics Subject Classification: 15A51, 47H60, 46T05, 92B99.

1 Introduction

It is known [7] that the theory of quadratic stochastic operators frequently arises in many models of physics, biology and so on. Let us briefly mention how such kind of operators appears in population genetics. Consider biological population, that is, a community of organisms closed with respect to reproduction [1]. Assume that every individual in this population belongs to one of the species $1, 2, \ldots, m$. Species of parents i and j uniquely determine the probability k of their direct descendant. We denote this probability (the heredity coefficient) via $p_{ij,k}$ and $\sum_{k=1}^{m} p_{ij,k} = 1$ for all i, j. Assume that the population is so large that frequency fluctuations can be neglected. Then the state of the population can be described by the tuple $x = (x_1, x_2, \ldots, x_m)$ of species probabilities, that is, x_i is the fraction of the species i in the population. In the case of panmixia (random interbreeding), the parent pairs i and j arise for a fixed state $x = (x_1, x_2, \ldots, x_m)$ with probability $x_i x_j$. Hence

$$x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j$$

is the total probability of the species in the first generation of direct descendants. Note that the concept of quadratic stochastic operator was first introduced by Bernstein in [1]. A lot of papers were devoted to study such operators (see for example, [3], [7], [6], [15], [11], [13]). One of the central problem in this theory is the study limit behavior and ergodic properties of trajectories of quadratic operators ([2], [9], [10], [13]). Note that the studying of a such properties of quadratic stochastic operators is very difficult. Even in the two-dimensional simplex the problem is still unsolved. This problem is well studied for Volterra quadratic stochastic operators ([3], [5]).

In [4] a class of quadratic operators, called bistochastic, is outlined. Such operators are characterized by a property $Vx \prec x$ for all $x \in S^{m-1}$ (see next section for notations). There it was showed that such operators form convex set and studied its extremal points.

The aim of this paper is to study some limiting behavior of nonlinear operators satisfying the condition $Vx \succ x$ for every $x \in S^{m-1}$, which will be called *dissipative*. Here we will restrict ourself to the case of quadratic. Note that intersection of the classes of bistochastic and dissipative operators consist of only permutation operators. It will be shown that quadratic dissipative operators are not Volterra.

This paper is organized as follows. In section 2 we give some preliminaries on quadratic stochastic operators. There we show that the set of dissipative operators does not form a convex set, while the class of bistochastic operators is convex. In section 3 we study certain limit properties of dissipative ones. Moreover, we describe all such operators in small dimensions. In section 4 we prove that every dissipative stochastic operator satisfies an ergodic theorem. Finally, in section 5 we give a conclusion of obtained results.

2 Preliminaries

Let $S^{m-1} = \{x \in R^m : x_i \geq 0, \ \sum\limits_{i=1}^{m} x_i = 1\}$ be a $(m-1)-$dimensional simplex. Then the vectors $e_k = (0, 0, \ldots, \underbrace{1}_{k}, \ldots, 0)$,$(k = \overline{1, m})$ are its vertices.

Let $x, y \in R^m$. Let's put $x_{\downarrow} = (x_{[1]}, x_{[2]}, \ldots x_{[m]})$, where $(x_{[1]}, x_{[2]}, \ldots, x_{[m]})$- decreasing rearrangement of $(x_1, x_2, \ldots, x_m)$, that is $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[m]}$.

Definition 2.1. We say that x *majorized* by y (or y *majorates* x), and write $x \prec y$ (or $y \succ x$) if the following conditions are fulfilled:

1) $\sum\limits_{i=1}^{k} x_{[i]} \leq \sum\limits_{i=1}^{k} y_{[i]}, \ \ k = \overline{1, m-1}$,

2) $\sum\limits_{i=1}^{m} x_{[i]} = \sum\limits_{i=1}^{m} y_{[i]}$.

Lemma 2.1. [8] *For any $x = (x_1, x_2, \ldots, x_m) \in S^{m-1}$ we have*

$$(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}) \prec x \prec (1, 0, \ldots, 0).$$

Remark 2.1. It should be noted that " $\prec$ " is not a partial ordering, because from $x \prec y$ and $y \prec x$ it only follows that $x_{\downarrow} = y_{\downarrow}$.

We call any operator V mapping S^{m-1} into itself *stochastic operator*.

Definition 2.2. A stochastic operator V is called *dissipative* if

$$Vx \succ x, \ \ \forall x \in S^{m-1}. \tag{2.1}$$

Observation. Let us consider the case when V is a linear dissipative operator, that is $Vx = Ax$, here $A = (a_{ij})_{i,j=\overline{1,m}}$. Now we show that only permutation linear operators are dissipative. Indeed, since $Vx \succ x$ then by putting $x = e_i$ we have $Ae_i \succ e_i$. From the lemma 2.1. we obtain that $(Ae_i)_{\downarrow} = (e_i)_{\downarrow}$. The last means that only one component of the vector Ae_i is 1 and the others are 0, which implies that the desired assumption. So, a linear case studying dissipative operators is very simple. Therefore, it is more interesting to study non-linear dissipative ones. In what follows, we consider quadratic operators.

Recall that a stochastic operator is called a *quadratic stochastic operator (q.s.o. in short)* if it has a form

$$(Vx)_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \tag{2.2}$$

where $x = (x_1, x_2, \ldots, x_m) \in S^{m-1}$. Here the coefficients $p_{ij,k}$ satisfy the following conditions

$$p_{ij,k} = p_{ji,k} \geq 0, \ \ \sum_{k=1}^{m} p_{ij,k} = 1. \tag{2.3}$$

It is easy to see that q.s.o. is well defined, i.e. it maps the simplex into itself.

It should be stressed that dissipative operators are not well studied, except for so called F- *quadratic stochastic operators* [12]. For the sake of completeness we recall it here. Let $E = \{0, 1, \ldots, m-1\}$. Fix a set $F \subset E$ and call this set of “females” and the rest set $M = E \setminus F$ set of “males”. The element 0 will play a role of empty body. The heredity coefficients of F-q.s.o. is defined by

$$p_{ij,k} = \begin{cases} 1, & if\ k = 0, i, j \in F \cup 0 \ or \ \ i, j \in M \cup 0 \\ 0, & if\ k \neq 0, i, j \in F \cup 0 \ or \ \ i, j \in M \cup 0 \\ \geq 0, & if\ i \in F, j \in M, \forall k. \end{cases}$$

Biological treatment of the above coefficients is very clear: a “child” k can be generated if its parents are taken from different classes F and M. In general, $p_{ij,0}$ can be strictly positive for $i \in F$ and $j \in M$, this corresponds, for instance, to the case when “female” i with “male” j can not generate a “child”, since one of them (or both) is (are) ill. In general F- q.s.o. for the case $F = \{1\}$ can be represented by

$$V : \begin{cases} (Vx)_0 = 1 - 2x_1 \sum\limits_{i=2}^{m} (1 - p_{1i,0}) x_i, \\ (Vx)_k = 2x_1 \sum\limits_{i=2}^{m} p_{1i,k} x_i, \ \ k = 1, 2, \ldots, m-1. \end{cases}$$

In [12], a limit behavior of such operators has been studied.

Now recall the term of well known Volterra q.s.o. A q.s.o. (2.2) is called Volterra q.s.o. if it satisfies an additional assumption $p_{ij,k} = 0,\ \forall k \notin \{i,j\}$. By changing $a_{ki} = 2p_{ik,k} - 1$ one can write down the following canonical form:

$$(Vx)_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) \tag{2.4}$$

In [3], it was proved that for any non-fixed initial point from the interior of the simplex, trajectory approaches a bound of the simplex. We will show that dissipative q.s.o. can not be Volterra q.s.o.

Now let us introduce the last notations which will be useful for the next sections.

The point $x^0 \in S^{m-1}$ is called *a fixed point* if $Vx^0 = x^0$. As a rule there are three types of fixed point. We call a fixed point x^0 *elliptic (hyperbolic; parabolic)* if the spectrum of the Jacobian $J(x^0)$ restricted to the invariant plane $\sum_{i=1}^m x_i = 0$ lies inside of the unit ball (respectively, outside the closure of the unit ball; inside the unit circle).

3 Dissipative Quadratic Stochastic Operators and Their Limit Behavior

In this section we first give some properties of dissipative q.s.o. and by means of them we will study the regularity of certain class of dissipative q.s.o. Moreover, we provide an example of dissipative q.s.o. which has infinitely many fixed points and study its limit behavior.

Note that the following example shows that the set of dissipative q.s.o. is non-empty.

$$\begin{aligned}
(Vx)_1 &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3,\\
(Vx)_2 &= x_1x_2 + x_1x_3,\\
(Vx)_3 &= x_2x_3.
\end{aligned}$$

Given q.s.o. V we denote $a_{ij} = (p_{ij,1}, p_{ij,2}, \dots, p_{ij,m})\ \ \forall i,j = \overline{1,m}$, where $p_{ij,k}$ are the coefficients of q.s.o. V (see (2.2)). One can see that $a_{ij} \in S^{m-1}$, for all $i,j \in \overline{1,m}$

Lemma 3.1. *Let V be a dissipative q.s.o. Then the following conditions hold*

$$(a_{ii})_\downarrow = e_1\ \ \forall i = \overline{1,m}.$$

Proof. Due to dissipativity of V one has $Vx \succ x\ \ \forall x \in S^{m-1}$. Now putting $x = e_i$ we get $e_i \prec Ve_i$. On the other hand, from Lemma 2.1 it follows that $e_i \succ x\ \ \forall x \in S^{m-1}$. That's why $(e_i)_\downarrow = (Ve_i)_\downarrow$. Then the equality $Ve_i = a_{ii}$ implies the assertion. □

Remark 3.1. Note that in [4] quadratic bistochastic operators were studied, that is, operators satisfying the condition $x \succ Vx \ \forall x \in S^{m-1}$. It was proved that such operators form a convex compact set and its extreme points were studied. The situation under consideration is different. Indeed, let us consider the following operators:

$$
\begin{aligned}
(V_0x)_1 &= x_1x_2 + x_1x_3,\\
(V_0x)_2 &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3,\\
(V_0x)_3 &= x_2x_3.
\end{aligned}
$$

$$
\begin{aligned}
(V_1x)_1 &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3,\\
(V_1x)_2 &= x_1x_2 + x_1x_3,\\
(V_1x)_3 &= x_2x_3
\end{aligned}
$$

One can see that these operators are dissipative. However, Lemma 3.1 implies that the operator $V_\lambda = \lambda V_1 + (1-\lambda)V_0$ is not dissipative for any $\lambda \in (0,1)$. Hence, all dissipative q.s.o. do not form a convex set.

Let V be a dissipative q.s.o. Then thanks to Lemma 3.1 it can represented by

$$
(Vx)_k = \sum_{i\in\alpha_k} x_i^2 + 2\sum_{i<j} p_{ij,k}x_ix_j \ \ k = \overline{1,m}, \tag{3.1}
$$

where $\alpha_k \subset I = \{1,2\ldots,m\}$, $\alpha_i \cap \alpha_j = \emptyset$, $i \neq j$, $\bigcup_{k=1}^m \alpha_k = I$.

Lemma 3.2. *Let (3.1) be a dissipative q.s.o.*

(i) *If $j \in \alpha_{k_0}$, then $p_{ij,k_0} = (a_{ij})_{[1]} \geq 1/2$, $\quad \forall i = \overline{1,m}$.*

(ii) *If $m \geq 3$, then $(a_{ij})_{[k]} = 0 \ \forall k \geq 3$, $\quad \forall i = \overline{1,m}$.*

Proof. (i). Let $j \in \alpha_{k_0}$ and $x = (1-\lambda)e_j + \lambda e_i$. Here, as before, e_i, e_j are the vertices of the simplex and λ is sufficiently small positive number. It is easy to see that $x_{[1]} = 1-\lambda$ and $(Vx)_{[1]} = (Vx)_{k_0}$. Since $Vx \succ x$ then $x_{[1]} \leq (Vx)_{[1]}$, so $1-\lambda \leq (Vx)_{k_0}$ or

$$
1-\lambda \leq (1-\lambda)^2 + 2p_{ij,k_0}\lambda(1-\lambda).
$$

The last inequality implies that $p_{ij,k_0} \geq 1/2$.

(ii). Denote $p_{ij,k^*} = \max_{t\neq k_0} p_{ij,t}$. One can see that $(Vx)_{k^*} = (a_{ij})_{[2]}$. Now from

$$
x_{[1]} + x_{[2]} \leq Vx_{[1]} + Vx_{[2]}.
$$

we obtain

$$
1 \leq (1-\lambda)^2 + 2(p_{ij,k_o} + p_{ij,k^*})\lambda(1-\lambda).
$$

Assuming $\lambda \to 0$ one gets $p_{ij,k_o} + p_{ij,k^*} \geq 1$. This yields that $p_{ij,k_o} + p_{ij,k^*} = 1$ and $(a_{ij})_{[k]} = 0 \ \forall k \geq 3, \forall i = \overline{1,m}$. □

Observation. Note that statements of Lemmas 3.1 and 3.2 are the necessary conditions for q.s.o. to be dissipative. It turns out that at $m = 2$ the statements are sufficient. Indeed, in this case only dissipative q.s.o. are the identity operator and the following one

$$(Vx)_1 = x_1^2 + x_2^2 + ax_1x_2,$$
$$(Vx)_2 = (2-a)x_1x_2,$$

up to permutation of the coordinates. Here $1 \le a \le 2$.

However, when $m \ge 3$ then the statements are not sufficient. Consider the following example of q.s.o.

$$(Vx)_1 = x_1 + x_2 - x_1x_2,$$
$$(Vx)_2 = 0.8x_1x_2,$$
$$(Vx)_3 = x_3 + 0.2x_1x_2.$$

One can see that it satisfies the mentioned statements. But for $x^0 = (0.5; 0.49; 0.01)$ we have $Vx^0 \not\succ x^0$, which means that it is not dissipative.

Studying of limit behavior of all dissipative q.s.o. is a difficult problem. Therefore, we are going to consider some particular cases.

First, recall a q.s.o. $V : S^{m-1} \to S^{m-1}$ is called *regular* if the trajectory of any $x \in S^{m-1}$ converges to a unique fixed point.

Note that regular operators a'priori must have a unique fixed point and its fixed point is attracting.

Now let consider the case $\alpha_1 = I$ and $\alpha_k = \emptyset$ for $k \neq 1$. Then the operator has the following form

$$\left.\begin{aligned} (Vx)_1 &= \sum_{i=1}^{m} x_i^2 + 2\sum_{i<j} p_{ij,1}x_ix_j \\ (Vx)_k &= 2\sum_{i<j} p_{ij,k}x_ix_j,\ 2 \le k \le m \end{aligned}\right\} \tag{3.2}$$

Theorem 3.1. *A dissipative q.s.o. given by (3.2) is regular. Its unique fixed point is* e_1.

Proof. Let us first prove that there is a unique fixed point. The existence of the fixed point follows from the Bohl-Brower theorem. Denote it by $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_m^{(0)})$. It is clear that it satisfies the following equality:

$$x_1^{(0)} = \sum_{i=1}^{m} (x_i^{(0)})^2 + 2\sum_{i<j} p_{ij,1}x_i^{(0)}x_j^{(0)},$$

which can be rewritten by

$$(x_1^{(0)})^2 + x_1^{(0)}x_2^{(0)} + \cdots + x_1^{(0)}x_m^{(0)} = \sum_{i=1}^{m} (x_i^{(0)})^2 + 2\sum_{i<j} p_{ij,1}x_i^{(0)}x_j^{(0)}$$

or

$$(x_2^{(0)})^2 + (x_3^{(0)})^2 + \cdots + (x_m^{(0)})^2 + \sum_{i=2}^{m}(2p_{ij,1} - 1)x_1^{(0)}x_i^{(0)} + \sum_{2\leq i<j} p_{ij,1}x_i^{(0)}x_j^{(0)} = 0.$$

From Lemma 3.2 we have $2p_{ij,1} - 1 \geq 0$. Therefore, the left hand side is positive, which means that the equality holds iff $x_2 = x_3 = \cdots = x_m = 0$. Hence, q.s.o. (3.2) has a unique fixed point e_1.

Now let us show that the operator is regular. Consider a function $\varphi : S^{m-1} \to R$, defined by

$$\varphi(x) = x_2 + x_3 + \cdots + x_m.$$

Then

$$\varphi(Vx) = \sum_{i<j}\sum_{k=2}^{m} 2p_{ij,k}x_ix_j.$$

One can see that $2\sum_{k=2}^{m} p_{ij,k} \leq 1$ since $2p_{ij,1} - 1 \geq 0$. Hence,

$$\varphi(Vx) \leq \sum_{i<j} x_ix_j \leq \sum_{i=2}^{m} x_i \sum_{i=1}^{m} x_i = \sum_{i=2}^{m} x_i = \varphi(x).$$

Consequently, $\{\varphi(V^nx)\}$ is a decreasing sequence. Therefore it converges.

Denote

$$\lim_{n\to\infty} \varphi(V^nx) = C.$$

The equality

$$(V^{n+1}x)_1 = \sum_{i=1}^{m}((V^nx)_i^2 + 2\sum_{i<j} p_{ij,1}(V^nx)_i^{(n)}(V^nx)_j$$

with $2p_{ij,1} - 1 \geq 0$ implies

$$\begin{aligned}(V^{n+1}x)_1 =& \sum_{i=1}^{m}((V^nx)_i^2 + 2\sum_{i<j} p_{ij,1}(V^nx)_i(V^nx)_j \\ \geq& \sum_{i=1}^{m}((V^nx)_i)^2 + \sum_{i<j}(V^nx)_i(V^nx)_j \\ =& (V^nx)_1 + \sum_{i=2}^{m}((V^nx)_i)^2 + \sum_{1<i<j}^{m}(V^nx)_i(V^nx)_j.\end{aligned}$$

Since $\lim_{n\to\infty}(V^{n+1}x)_1 = \lim_{n\to\infty}(V^nx)_1 = 1 - C$, we have

$$\lim_{n\to\infty}(\sum_{i=2}^{m}((V^nx)_i)^2 + \sum_{1<i<j}^{m}(V^nx)_i(V^nx)_2 \leq 0,$$

which means that $\lim_{n\to\infty}(V^nx)_i = 0$ for all $i \geq 2$, Hence $\lim_{n\to\infty}(V^nx) = e_1$. Therefore we deduce that the operator (3.2) is regular. $\square$

Remark 3.2. If the trajectory of q.s.o. belongs to the edge of the simplex, then it means that in process of time some species of the population are in the bound of disappearing. In our case we can conclude that almost all species will disappear.

Remark 3.3. As we know that dissipative q.s.o. has a form (3.1), but the converse is not true (see example below). Therefore, Theorem 3.1 holds for wider class of q.s.o. Indeed, consider the following example.

$$
\begin{aligned}
(V_1x)_1 &= x_1^2 + x_2^2 + x_3^2 + a_1x_1x_2 + b_1x_2x_3 + c_1x_1x_3,\\
(V_1x)_2 &= a_2x_1x_2 + b_2x_2x_3 + c_2x_1x_3,\\
(V_1x)_3 &= a_3x_1x_2 + b_3x_2x_3 + c_3x_1x_3,
\end{aligned}
$$

where $a_1, b_1, c_1 \geq 1$ and $\sum_{i=1}^3 a_i = \sum_{i=1}^3 b_i = \sum_{i=1}^3 c_i = 2$. If the coefficient are strictly positive, then q.s.o. is not dissipative. However, the proof of the theorem3.1 is valid for above operator, not only for dissipative ones.

On the other hand, we mentioned that in [12] a F-q.s.o. has been studied. Such operators can be represented in (3.2) form. For the case $F = \{1\}$ if we suppose that the coefficients of F-q.s.o. $p_{1i,0} \geq 1/2$ then this operator becomes dissipative. Only in this case our result extends a result obtained in [12]. In all other cases classes dissipative operators and F-q.s.o., respectively, do not intersect.

Corollary 3.1. *In the case of $\alpha_k = I$ and $\alpha_j = \emptyset$ for $j \neq k$ dissipative q.s.o. is regular and has a unique fixed point e_k.*

Let V be a q.s.o. Then the set $\omega(x^0) = \bigcap_{k\geq 0} \overline{\bigcup_{n\geq k}\{V^n x^0\}}$ is called *ω-limit set* of trajectory of initial point $x^0 \in S^{m-1}$. From the compactness of the simplex one can deduce that $\omega(x^0) \neq \emptyset$ for all $x^0 \in S^{m-1}$.

Now we turn to another case, namely let $\alpha_1 = I\backslash\{l\}$ $\alpha_2 = \{l\}$ (actually we can put $\alpha_{k_0} = \{l\}$ for some k_0) and $\alpha_k = \emptyset$ $\forall k \geq 3$. Then operator (3.1) has the following form

$$
\left.
\begin{aligned}
(Vx)_1 &= \sum_{i=1,\ i\neq l}^{m} x_i^2 + 2\sum_{i<j} p_{ij,1}x_ix_j\\
(Vx)_2 &= x_l^2 + 2\sum_{i<j} p_{ij,2}x_ix_j\\
(Vx)_k &= 2\sum_{i<j} p_{ij,k}x_ix_j,\ \ 3 \leq k \leq m
\end{aligned}
\right\}
\tag{3.3}
$$

Theorem 3.2. *If $l \neq 2$ then the dissipative q.s.o. (3.3) is regular and has a unique fixed point e_1. If $l = 2$ then the dissipative q.s.o. (3.3) has infinitely many fixed points and all of them are parabolic. Moreover, ω-limit set of trajectory of any initial point x^0 belongs to $co\{e_1, e_2\}$, here coA denotes the convex hull of a set A.*

Proof. The first part of the proof is similar to the proof of the Theorem 3.1. Therefore, consider when $l = 2$. From Lemma 3.2 it follows that $2p_{ij,1} \geq 1$ and $2p_{i2,2} \geq 1$, hence $p_{i2,1} + p_{i2,2} \geq 1$. But $\sum_{k=1}^{m} p_{i2,k} = 1$ implies that $2p_{i2,1} = 2p_{i2,2} = 1$ and $p_{i2,k} = 0$ for all $k \geq 3$. Now we can rewrite operator (3.1) as:

$$\left.\begin{aligned} (Vx)_1 &= x_1 + \sum_{i=3}^{m} x_i^2 + 2 \sum_{1<i<j} p_{ij,1} x_i x_j \\ (Vx)_2 &= x_2 + 2 \sum_{1<i<j \quad i\neq 2} p_{ij,2} x_i x_j \\ (Vx)_k &= 2 \sum_{1<i<j} p_{ij,k} x_i x_j, \ \ 3 \leq k \leq m \end{aligned}\right\} \tag{3.4}$$

Putting $x_\lambda = \lambda e_1 + (1-\lambda)e_2$, where $0 \leq \lambda \leq 1$, we get $Vx_\lambda = x_\lambda$. Therefore, V has infinitely many fixed points x_λ. Simple calculations show that Jacobian $J(x_\lambda)$ of the fixed point x_λ has a following form

$$\begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$$

So, one can see that the only eigenvalue of this matrix is 1, which belongs to the unit ball. Therefore, all of fixed points are parabolic. Now consider a function $\varphi : S^{m-1} \to R$, defined by $\varphi(x) = x_3 + x_4 + \cdots + x_m$. Then $\forall x \in S^{m-1}$ we have

$$\varphi(Vx) = \sum_{1<i<j} \sum_{k=3}^{m} 2p_{ij,k} x_i x_j.$$

The inequality $2p_{ij,1} - 1 \geq 0$ implies $2 \sum\limits_{k=2}^{m} p_{ij,k} \leq 1$, which yields

$$\varphi(Vx) \leq \sum_{1<i<j} x_i x_j \leq \sum_{i=3}^{m} x_i \sum_{i=1}^{m} x_i = \sum_{i=3}^{m} x_i = \varphi(x).$$

Consequently, $\{\varphi(V^n x)\}$ is a decreasing sequence. Therefore it converges.

Denote

$$\lim_{n\to\infty} \varphi(V^n x) = C.$$

From (3.4) one gets

$$(V^{n+1}x)_1 + (V^{n+1}x)_2 = (V^n x)_1 + (V^n x)_2 + 2 \sum_{1<i<j} p_{ij,1} (V^n x)_i (V^n x)_j$$
$$+ 2 \sum_{1<i<j} p_{ij,2} (V^n x)_i (V^n x)_j.$$

According to

$$\lim_{n\to\infty}((Vx)_1^{(n+1)} + (Vx)_2^{(n+1)}) = \lim_{n\to\infty}((Vx)_1^{(n)} + (Vx)_2^{(n)}) = 1 - C$$

we have

$$\lim_{n\to\infty}(\sum_{1<i<j} p_{ij,1}(Vx)_i^{(n)}(Vx)_j^{(n)} + \sum_{1<i<j} p_{ij,2}(Vx)_i^{(n)}(Vx)_j^{(n)}) = 0,$$

which means that $C = 0$, and therefore $\omega(x) \in co\{e_1, e_2\}$. □

Remark 3.4. We showed that all of the fixed points are parabolic, this means that trajectory in a neighborhood of these points is nonstable.

Observation. In last two theorems we have seen that the trajectory of an initial point tend to the bound of the simplex. Therefore it is natural to ask whether dissipative q.s.o. and Volterra q.s.o coincide. The answer is negative. Namely, a dissipative q.s.o. (3.1) can be Volterra if and only if $\alpha_k = \{k\}$. On the other hand, from the Lemma 3.2 it automatically follows that $p_{ik,k} = 1/2$ and $a_{ki} = 2p_{ik,k} - 1 = 0$. Therefore only identity operator can be contemporary dissipative and Volterra q.s.o.

4 Ergodicity of Dissipative Operators

Let us recall that a stochastic operator V is called *ergodic* if the following limit exists

$$\lim_{n\to\infty} \frac{x + Vx + \cdots + V^{n-1}x}{n}$$

for any $x \in S^{m-1}$.

In this section we are going to show that any dissipative stochastic operators is ergodic.

Note that Ulam [14] formulated a conjecture that *is any q.s.o. ergodic*. However, Zakharevich [16] showed that it is not so. He considered the following q.s.o.

$$(Vx)_1 = x_1^2 + 2x_1x_2,$$
$$(Vx)_2 = x_2^2 + 2x_2x_3,$$
$$(Vx)_3 = x_3^2 + 2x_1x_3$$

and proved that such an operator is not ergodic. From section 2 one can see that such an operator is Volterra.

Now we show that Ulam's conjecture is true for all dissipative stochastic operators, i.e. not only for quadratic ones.

Theorem 4.1. *Any dissipative operator is ergodic.*

Proof. Let $V : S^{m-1} \to S^{m-1}$ be a dissipative stochastic operator. Then we have

$$x \prec Vx \prec V^2x \prec V^3x \prec \cdots .$$

It means that

$$\begin{aligned}
&x_{[1]} \le (Vx)_{[1]} \le (V^2x)_{[1]} \le \cdots \\
&x_{[1]} + x_{[2]} \le (Vx)_{[1]} + (Vx)_{[2]} \le (V^2x)_{[1]} + (V^2x)_{[2]} \le \cdots \\
&\vdots \\
&\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} (Vx)_{[i]} \le \sum_{i=1}^{k} (V^2x)_{[i]} \le \cdots
\end{aligned}$$

The sequences $\{\sum_{i=1}^{k} (V^n(x))_{[i]},\ \ n = 1, 2, \ldots\}\ \forall k = \overline{1,m}$ are increasing and bounded, consequently convergent. The last means that the following sequences are also convergent

$$\{(V^n(x))_{[k]}\ \ n = 1, 2, \ldots\}\ \forall k = \overline{1,m}.$$

Let's denote $y_k = \lim_{n\to\infty}(V^n x)_{[k]}$ and $y = (y_1, y_2, \ldots, y_m)$.

If $z = (z_1, z_2, \ldots, z_m) \in \omega(x^0)$, then there exists $\{x^{(n_j)}\}$, such that $(V^{n_j}x) \to z$. Therefore we have $(V^{n_j}x)_\downarrow \to z_\downarrow$. On the other hand $(V^{n_j}x)_\downarrow \to y$, since y is a limit of the sequence $(V^n x),\ n = 1, 2, \ldots$ That's why $z_\downarrow = (y_1, y_2, \ldots, y_m) = y$. Therefore, we infer that any element of $\omega(x)$ is some kind of rearrangement of $(y_1, y_2, \ldots, y_m) = y$. This means that the cardinality of $\omega(x)$ cannot be greater than $m!$

Let $|\omega(x)| = p$, then the trajectory of the $\{V^n x\}$ tends to the cycle of order p, i.e. the trajectory is divided into p convergent subsequences. The operator V acts as a cyclic permutation of their limits. Therefore we conclude that V is ergodic. $\square$

Remark 4.1. Now we proved that limit set of the trajectory is finite. From biological point of view this means that there are periodical evolutions, since there are periodical points of the dissipative q.s.o. Indeed, all points of permutation operators are periodic. So we can infer that permutation operators have periodic trajectory. Now, let us consider nontrivial example.

$$\begin{aligned}
(Vx)_1 &= x_2 + x_3 + (c_1 - 2)x_2x_3, \\
(Vx)_2 &= x_1 + c_2x_2x_3, \\
(Vx)_3 &= c_3x_2x_3.
\end{aligned}$$

Here $c_1 + c_2 + c_3 = 2,\ \ c_1 \ge 1$. The last conditions tell us that above operator is dissipative(one can easily check it). This operator has a unique fixed point $(1/2, 1/2, 0)$ and two periodical points e_1 and e_2. That's why there is a periodical evolutions.

Corollary 4.1. *Let V be a dissipative stochastic operator. Then for any subsequence $\{n_k\} \subset N$ the following limit exists*

$$\lim_{k\to\infty} \frac{V^{n_1}x + V^{n_2}x + \cdots + V^{n_k}x}{k}$$

Proof. The proof immediately follows from that $\omega(x)$ is finite,for all $x \in S^{m-1}$. □

5 Conclusion.

The main achievement of the present paper is that it introduced and studied a new class of stochastic operators, called dissipative ones. Dissipative stochastic operators have various application in mathematical genetics and one can use given results [5], [7]. The main results of this work are Theorems 3.1, 3.2, and 4.1 . The methods, which were used for proving results are different from those well-known ones. One can use these methods and techniques for proving other results, not only in the theory of quadratic stochastic operators, but also in other disciplines of mathematics, namely nonlinear analysis, dynamical systems and ergodic theory. Nevertheless, in a class of dissipative q.s.o. there are some open problems.

Prove or disprove the following statements.

Problem 1. Any dissipative q.s.o. has either unique or infinitely many fixed points.

Problem 2. If V is a dissipative q.s.o., then $\omega-$ limit set of any non-fixed initial belongs to the bound of the simplex.

Acknowledgements

I wish to express my sincere gratitude to Professor R. N. Ganikhodzhaev and Professor F. M. Mukhamedov for their constant guidance, supervision and encouragement during the preparation of the present work. The work also partially supported by ICTP, OEA-AC-84.

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