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A GENERALIZATION OF MITTAG-LEFFLER FUNCTION AND INTEGRAL OPERATOR ASSOCIATED WITH FRACTIONAL CALCULUS

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Abstract. This paper is devoted for the study of a new generalized function of Mittag-Leffler type. Its various properties including differentiation, Laplace transform, Beta transform, Mellin transform, Whittaker transform, generalized hypergeometric series form, Mellin-Barnes integral representation and its relationship with Fox's H-function and Wright hypergeometric function are investigated and established. Further properties of generalized Mittag- Leffler function associated with fractional differential and integral operators are considered. Also an integral operator associated with fractional calculus operators is studied

1. Introduction

The Swedish mathematician Mittag-Leffler [5] introduced the function $E_\alpha(z)$ defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \tag{1}$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$.
The Mittag-Leffler function is a direct generalization of $\exp(z)$ in which $\alpha = 1$. Mittag - Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations.
A generalization of $E_\alpha(z)$ was studied by Wiman [14] where he defined the function $E_{\alpha,\beta}(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \tag{2}$$

$(\alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0,\ \Re(\beta) > 0)$ which is also known as Mittag-Leffler function or Wiman's function.
Prabhakar [6] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form(see also Kilbas et al. [4]

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$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{3}$$

$(\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0)$
Shukla and Prajapati [10] (see also Srivastava and Tomovski [13]) defined and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\gamma_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{4}$$

where $(\alpha, \beta, \gamma \in \mathbb{C};\ \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0\ , q \in (0, 1) \cup \mathbb{N}$ and $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol which in particular reduces to $q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n$ if $q \in \mathbb{N}$
A new generalization of Mittag-Leffler function was defined by Salim [8] as

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n} \tag{5}$$

where $(\alpha, \beta, \gamma, \delta \in \mathbb{C};\ \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0$
In this paper , we introduce a new generalization of Mittag-Leffler function defined as

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{\gamma_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_{pn}} \tag{6}$$

where

$$\alpha, \beta, \gamma, \delta \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0;\ p, q > 0\ \ and\ \ q \le \Re\alpha + p \tag{7}$$

Equation (6) is a generalization of equations (1) -(5).
- Setting $p = q = 1$, it reduces to Eq. (5) defined by Salim [8].
- Setting $\delta = p = 1$, it reduces to Eq. (4) defined by Shukla and Prajapati [10], in addition of that if $q = 1$, then we get Eq. (3) defined by Prabhakar [6] .
- On putting $\gamma = \delta = p = q = 1$ in (6) it reduces to Wiman's function, moreover if $\beta = 1$, Mittag-Leffler function $E_\alpha(z)$ will be the result.

Some recurrence relations, derivation formulas, Laplace transform, Beta transform, Mellin-Barnes integral of $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ will be established, also its relationship to Fox's H-function and Wright hypergeometric function will be established.
The integral operator defined by

$$\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a^+}(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(w(x-t)^\alpha)\varphi(t)dt \tag{8}$$

which contains the generalized Mittag-Leffler function (6) in its kernel is investigated and its boundedness is proved under certain conditions.
Theorems of composition of fractional calculus operators

$$\left(I^\lambda_a \varphi\right)(x) = \frac{1}{\Gamma(\lambda)} \int_a^x (x-t)^{\lambda-1}\varphi(t)dt \qquad (\lambda \in \mathbb{C}, \Re(\lambda) > 0) \tag{9}$$

and

$$\left(D_a^{\lambda}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n (I_a^{n-\lambda}\varphi)(x) \qquad n = [\Re(\lambda)] + 1 \tag{10}$$

with integral operators defined in (8) are given and proved. As a matter of fact if $w = 0,\ q = 1$ and $p = 1$, then the integral operator corresponds essentially to the Riemann-Liouville fractional integral operator defined in (9). The generalized fractional derivative operator $D_a^{u,v}\varphi$ known as Hilfer's fractional derivative (see Hilfer [2]) is written as

$$\left(D_a^{u,v}\varphi\right)(x) = \left(I_a^{v(1-u)}\frac{d}{dx}\left(I_a^{(1-v)(1-u)}\varphi\right)\right)(x) \tag{11}$$

$D_a^{u,v}$ yields the classical Riemann-Liouville fractional derivative D_a^u when $v = 0$; also if $v = 1$ it reduces to Caputo fractional derivative.

Throughout this paper , we need the following well-known facts and rules.

- Beta transform (Sneddon [11])

$$B\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz, \qquad \Re(a) > 0, \Re(b) > 0 \tag{12}$$

- Laplace transform (Sneddon [11])

$$\mathcal{L}\{f(z); s\} = \int_0^\infty e^{-sz} f(z)dz, \qquad \Re(s) > 0 \tag{13}$$

- Convolution theorem of Laplace transform (Finney et al. [1])

$$\mathcal{L}\left(f * g\right)(s) = \mathcal{L}\{\int_0^t f(t-\xi)f(\xi)d\xi\} = \mathcal{L}\left(f\right)(s)\mathcal{L}\left(g\right)(s);$$

$$\mathcal{L}\{\frac{t^{n-1}}{\Gamma(n)}; s\} = \frac{1}{s^n}, \qquad n > 0 \tag{14}$$

- Mellin transform (Sneddon [11])

$$\mathcal{M}\{f(x); s\} = f^*(s) = \int_0^\infty z^{s-1} f(z)dz \tag{15}$$

and the inverse Mellin transform is given by

$$f(z) = \mathcal{M}^{-1}\{f^*(s); z\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s} f^*(s)ds, \qquad c \in \mathbb{R} \tag{16}$$

- Confluent hypergeometric function (Rainville [7])

$$\Phi(a, b, z) = {}_1F_1(a, b, z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{z^n}{n!} \tag{17}$$

- Wright generalized hypergeometric function (Srivastava and Manocha [12]).

$$ {}_p\Psi_q\left(\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array};z\right)=\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{p}\Gamma(a_i+A_in)}{\prod\limits_{j=1}^{q}\Gamma(b_j+B_jn)}\frac{z^n}{n!} \tag{18}$$

- Fox's H-function (Kilbas and Saigo [3])

$$H_{p,q}^{m,n}\left[z\left|\begin{array}{c}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{array}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_js)\prod\limits_{j=1}^{n}\Gamma(1-a_j-\alpha_js)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_js)\prod\limits_{j=n+1}^{p}\Gamma(a_j+\alpha_js)}z^{-s}ds \tag{19}$$

- The generalized hypergeometric function (Rainville [7])

$$ {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)=\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{p}(\alpha_i)_n}{\prod\limits_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!} \tag{20}$$

- Whittaker transform (Whittaker and Watson [15])

$$\int_0^{\infty}e^{-t/2}t^{v-1}W_{\lambda,\mu}(t)dt=\frac{\Gamma(\frac{1}{2}+\mu+v)\Gamma(\frac{1}{2}-\mu+v)}{\Gamma(1-\lambda+v)} \tag{21}$$

where $\Re(\mu\pm v)>-1/2$ and $W_{\lambda,\mu}(t)$ is the Whittaker confluent hypergeometric function.

- Fubini's theorem (Dirichlet formula) (Samko et al. [9])

$$\int_a^b dx\int_a^x f(x,t)dt=\int_a^b dt\int_t^b f(x,t)dx; \tag{22}$$

$$\frac{d}{dx}\int_a^x h(x,t)dt=\int_a^x\frac{\partial}{\partial x}h(x,t)dt+h(x,x). \tag{23}$$

2. Basic properties

Theorem 2.1 The series in (6) is absolutely convergent for all values of z provided that $q<p+\Re(\alpha)$. Moreover if $q=p+\Re(\alpha)$, then $E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)$ converges for $|z|<1$.

Proof. Rewriting $E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)$ in the form of power series $E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)=\sum\limits_{n=0}^{\infty}b_nz^n$ where $b_n=\dfrac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_{pn}}$
and applying $\dfrac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\left[1+\dfrac{(a-b)(a+b-1)}{2z}+O\left(\dfrac{1}{z^2}\right)\right]$,
we get

$$\begin{aligned}
\left|\frac{c_{n+1}}{c_n}\right| &= \left|\frac{(\gamma)_{qn+q}}{(\gamma)_{qn}}\frac{(\delta)_{pn}}{(\delta)_{pn+p}}\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\alpha)}\frac{z^{n+1}}{z^n}\right| \\
&= (nq)^q\left[1+\frac{q(2q+\gamma-1)}{2qn}+O\left(\frac{1}{(nq)^2}\right)\right] \\
&\times (np)^{-p}\left[1+\frac{-p(2\delta+p-1)}{2pn}+O\left(\frac{1}{(np)^2}\right)\right] \\
&\times (\alpha n)^{-\alpha}\left[1+\frac{-\alpha(2\beta+\alpha-1)}{2\alpha n}+O\left(\frac{1}{(\alpha n)^2}\right)\right]|z| = \frac{q^q}{p^p\alpha^\alpha}\frac{n^q}{n^{p+\alpha}},
\end{aligned}$$

then $\left|\dfrac{c_{n+1}}{c_n}\right| \to 0$ as $n\to\infty$ and $q<p+\Re(\alpha)$,

which means that the function $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ converges for all z provided that $q < p+\Re(\alpha)$. Moreover if $q=p+\Re(\alpha)$, then $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ converges for $|z|<1$.

Theorem 2.2 If the condition (7) is satisfied, then

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) - E^{\gamma,\delta-1,q}_{\alpha,\beta,p}(z) = \frac{zp}{1-\delta}\frac{d}{dz}E^{\gamma,\delta,q}_{\alpha,\beta,p}(z); \qquad \delta\neq 1 \tag{24}$$

and

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(z) + az\frac{d}{dz}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(z) \tag{25}$$

Proof.

$$\begin{aligned}
E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) - E^{\gamma,\delta-1,q}_{\alpha,\beta,p}(z) &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\left[\frac{1}{(\delta)_{pn}}-\frac{1}{(\delta-1)_{pn}}\right]z^n \\
&= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{\Gamma(\delta)}{\Gamma(\delta+pn)}\left[\frac{pn}{1-\delta}\right] = \frac{zp}{1-\delta}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}nz^{n-1}}{\Gamma(\alpha n+\beta)(\delta)_{pn}} \\
&= \frac{zp}{1-\delta}\frac{d}{dz}E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)
\end{aligned}$$

hence (24) is proved.

$$\begin{aligned}
E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) &= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\frac{(\alpha n+\beta)}{(\alpha n+\beta)\Gamma(\alpha n+\beta)} \\
&= \sum_{n=0}^{\infty}\frac{\beta(\gamma)_{qn}z^n}{(\alpha n+\beta)\Gamma(\alpha n+\beta)(\delta)_{pn}} + \sum_{n=0}^{\infty}\frac{\alpha(\gamma)_{qn}z^n}{(\alpha n+\beta)\Gamma(\alpha n+\beta)(\delta)_{pn}} \\
&= \beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(z) + az\frac{d}{dz}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(z)
\end{aligned}$$

which is (25).

Theorem 2.3 If the condition (7) is satisfied, then for $m\in\mathbb{N}$

$$\left(\frac{d}{dz}\right)^m E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \frac{(\gamma)_{qn}}{(\delta)_{pn}}\sum_{n=0}^{\infty}\frac{(\gamma+qm)_{qn}}{(\delta+pm)_{pn}}\frac{(n+1)_m}{\Gamma(\alpha n+\alpha m+\beta)}z^n; \tag{26}$$

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(wz^\alpha)\right] = z^{\beta-m-1}E^{\gamma,\delta,q}_{\alpha,\beta-m,p}(wz^\alpha) \tag{27}$$

Proof.

$$\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} = \sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn+qm)}{\Gamma(\gamma)\Gamma(\delta+pn+pm)}\frac{\Gamma(\delta)(n+1)_m}{\Gamma(\alpha n+\alpha m+\beta)}z^n$$

$$= \frac{(\gamma)_{qn}}{(\delta)_{pn}} \sum_{n=0}^{\infty} \frac{(\gamma+qm)_{qn}}{(\delta+pm)_{pn}} \frac{(n+1)_m}{\Gamma(\alpha n+\alpha m+\beta)} z^n;$$

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(wz^\alpha)\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} w^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} \frac{d}{dz}(z^{\alpha+\beta-1})$$
$$= z^{\alpha-\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} w^n (z^\alpha)^n}{\Gamma(\alpha n+\beta-m)(\delta)_{pn}} \frac{d}{dz}(z^{\alpha+\beta-1}) = z^{\beta-m-1} E^{\gamma,\delta,q}_{\alpha,\beta-m,p}(wz^\alpha).$$

Theorem 2.4 If the condition (7) is satisfied, then

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\lambda(s-t)^\alpha) ds = (x-t)^{\delta+\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta+\delta,p}(\lambda(s-t)^\alpha) \tag{28}$$

Proof. Let $u = \dfrac{s-t}{x-t}$, then

$$\frac{1}{\Gamma(\delta)} \int_t^x (x-s)^{\delta-1}(s-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\lambda(s-t)^\alpha) ds$$
$$= \frac{1}{\Gamma(\delta)} \int_t^x (x-t)^{\delta-1}(1-u)^{\delta-1}(x-t)^{\beta-1}u^{\beta-1}(x-t) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\lambda^n (x-t)^{\alpha n} u^{\alpha n}}{\Gamma(\alpha n+\beta)(\delta)_{pn}} du$$
$$= \frac{(x-t)^{\delta+\beta-1}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(\lambda(x-t)^\alpha)^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} \frac{\Gamma(\alpha n+\beta)\Gamma(\delta)}{\Gamma(\alpha n+\beta+\delta)}$$
$$= (x-t)^{\delta+\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta+\delta,p}(\lambda(s-t)^\alpha).$$

In particular, setting $t = 0$ and $x = 1$ in (28), we get

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\beta-1}(1-u)^{\delta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(zu^\alpha) ds = E^{\gamma,\delta,q}_{\alpha,\beta+\delta,p}(z).$$

3. $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ in Terms of Other Functions

In this section we write $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ in terms of Wright generalized function, generalized hypergeometric function, Mellin-Barnes integral and Fox's H-function.

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)} \frac{\Gamma(\delta)}{\Gamma(\delta+pn)} \frac{\Gamma(n+1)}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$$

hence, we can write $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ in terms of the Wright generalized function as

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)}{\Gamma(\delta+pn)} \frac{\Gamma(n+1)}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \cdot {}_2\Psi_2 \left[\begin{array}{c} (\gamma,q),(1,1) \\ (\delta,p),(\beta,\alpha) \end{array} ;z \right] \tag{29}$$

Theorem 3.1 Let (7) be satisfied with $\alpha = k \in \mathbb{N}$, then $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ can be written in terms of the generalized hypergeometric function as

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \frac{1}{\Gamma(\beta)} \cdot {}_{q+1}F_{p+k} \left[\begin{array}{c} 1,\Delta(q,\gamma) \\ \Delta(k,\beta),\Delta(p,\delta) \end{array} ; \frac{zq^q}{p^p k^k} \right], \tag{30}$$

where $\Delta(k,n)$ is $k-$tuple $\frac{n}{k}, \frac{n+1}{k}, \dots, \frac{n+k-1}{k}$.

Proof. Let $\alpha = k \in \mathbb{N}$, then

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}} = \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{(\beta)_{\alpha n}(\delta)_{pn}}$$

$$= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{q^{qn} \prod_{i=1}^{q} \left(\frac{\gamma+i-1}{q}\right)_n (1)_n z^n}{p^{pn} \prod_{j=1}^{p} \left(\frac{\delta+j-1}{p}\right)_n k^{kn} \prod_{r=1}^{k} \left(\frac{\beta+r-1}{k}\right)_n n!}$$

$$= \frac{1}{\Gamma(\beta)} \cdot {}_{q+1}F_{p+k} \left[\begin{matrix} 1, \Delta(q,\gamma) \\ \Delta(k,\beta), \Delta(p,\delta) \end{matrix} ; \frac{zq^q}{p^p k^k} \right].$$

Now in order to write $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ in terms of Fox's H-function, we first express $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ as Mellin-Barnes type integral

Theorem 3.2 Let (7) be satisfied, then $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ is represented in the Mellin-Barnes type integral as

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-qs)(-z)^{-s}}{\Gamma(\beta-\alpha s)\Gamma(\delta-ps)} ds, \tag{31}$$

where $|\arg(z)| < \pi$; the contour of integration begins at $-i\infty$ and ending at $i\infty$, and intended to separate the poles of the integrand at $s = -n$ for all $n \in \mathbb{N}$ (to the left) from those at $s = n+1$ and at $s = \frac{\gamma+n}{q}$ for all $n \in \mathbb{N} \cup \{0\}$ (to the right).

Proof. Simply, by writing the Wright generalized function in (29) in terms of Mellin-Barnes integral, we get

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)}{\Gamma(\delta+pn)} \frac{\Gamma(n+1)}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = \frac{\Gamma(\delta)}{\Gamma(\gamma)} \cdot {}_2\Psi_2 \left[\begin{matrix} (\gamma,q),(1,1) \\ (\delta,p),(\beta,\alpha) \end{matrix} ; z \right]$$

$$= \frac{1}{2\pi i} \frac{\Gamma(\delta)}{\Gamma(\gamma)} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-qs)(-z)^{-s}}{\Gamma(\beta-\alpha s)\Gamma(\delta-ps)} ds$$

$$= \frac{\Gamma(\delta)}{\Gamma(\gamma)} H^{1,2}_{2,3} \left[-z \middle| \begin{matrix} (0,1),(1-\gamma,q) \\ (0,1),(1-\beta,\alpha),(1-\delta,p) \end{matrix} \right]. \tag{32}$$

The last equation is just a representation of $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ in terms of Fox's H-function.

4. Integral Transforms of $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$

In this section, the image of $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ under Beta, Laplace, Mellin and Whittaker transforms with some special cases are proved in the following theorems

Theorem 4.1 (Beta Transform)

$$B\left\{E^{\gamma,\delta,q}_{\alpha,\beta,p}(xz^\sigma); a, b\right\} = \frac{\Gamma(b)\Gamma(\delta)}{\Gamma(\gamma)} \cdot {}_3\Psi_3 \left[\begin{matrix} (\gamma,q),(a,\sigma),(1,1) \\ (\beta,\alpha),(\delta,p),(a+b,\sigma) \end{matrix} ; z \right], \tag{33}$$

where (7) is satisfied and $\Re(a) > 0, \Re(b) > 0$.

Proof.

$$B\left\{E_{\alpha,\beta,p}^{\gamma,\delta,q}(xz^{\sigma});a,b\right\}=\int_0^1 z^{a-1}(1-z)^{b-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}(xz^{\sigma})dz$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}x^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}B(a+\sigma n,b)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}x^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\frac{\Gamma(a+\sigma n)\Gamma(b)}{\Gamma(a+\sigma n+b)}$$
$$=\frac{\Gamma(b)\Gamma(\delta)}{\Gamma(\gamma)}.{}_3\Psi_3\left[\begin{array}{c}(\gamma,q),(a,\sigma),(1,1)\\(\beta,\alpha),(\delta,p),(a+b,\sigma)\end{array};x\right].$$

Theorem 4.2 (Laplace Transform)

$$\mathcal{L}\left\{z^{a-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}(xz^{\sigma});s\right\}=\frac{\Gamma(\delta)s^{-a}}{\Gamma(\gamma)}.{}_3\Psi_2\left[\begin{array}{c}(\gamma,q),(a,\sigma),(1,1)\\(\beta,\alpha),(\delta,p)\end{array};\frac{x}{s^{\sigma}}\right]\quad(34)$$

Proof.
$$\mathcal{L}\left\{z^{a-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}(xz^{\sigma});s\right\}=\int_0^{\infty}z^{a-1}e^{-sz}E_{\alpha,\beta,p}^{\gamma,\delta,q}(xz^{\sigma})dz$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}x^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\int_0^{\infty}z^{a+\sigma n-1}e^{-sz}dz$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}x^n\Gamma(a+\sigma n)}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\mathcal{L}\left\{\frac{z^{a+\sigma n-1}}{\Gamma(a+\sigma n)};s\right\}=\frac{\Gamma(\delta)s^{-a}}{\Gamma(\gamma)}.{}_3\Psi_2\left[\begin{array}{c}(\gamma,q),(a,\sigma),(1,1)\\(\beta,\alpha),(\delta,p)\end{array};\frac{x}{s^{\sigma}}\right].$$

Theorem 4.3 (Mellin Transform)

$$\mathcal{M}\left\{E_{\alpha,\beta,p}^{\gamma,\delta,q}(-wz);s\right\}=\frac{\Gamma(\delta)}{\Gamma(\gamma)}\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-qs)}{\Gamma(\beta-\alpha s)\Gamma(\delta-ps)}w^{-s}\quad(35)$$

Proof. According to Theorem 3.2 and using (31), $E_{\alpha,\beta,p}^{\gamma,\delta,q}(-wz)$ can be written as

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(-wz)=\frac{1}{2\pi i}\frac{\Gamma(\delta)}{\Gamma(\gamma)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-qs)}{\Gamma(\beta-\alpha s)\Gamma(\delta-ps)}(wz)^{-s}ds=\frac{1}{2\pi i}\frac{\Gamma(\delta)}{\Gamma(\gamma)}\int_L f^*(s)z^{-s}ds$$

where $f^*(s)=\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-qs)}{\Gamma(\beta-\alpha s)\Gamma(\delta-ps)w^s}$ and L is the contour of integration that begins at $c-i\infty$ and ends at $c-i\infty;\ c\in\mathbb{R}$.
Hence

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(-wz)=\frac{\Gamma(\delta)}{\Gamma(\gamma)}\mathcal{M}^{-1}\left\{f^*(s);z\right\}$$

Now applying Mellin transform to both sides, we obtain

$$\mathcal{M}\left\{E_{\alpha,\beta,p}^{\gamma,\delta,q}(-wz);s\right\}=\frac{\Gamma(\delta)}{\Gamma(\gamma)}\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-qs)}{\Gamma(\beta-\alpha s)\Gamma(\delta-ps)}w^{-s}$$

which proves (35).

Theorem 4.4 (Whittaker Transform)

$$\int_0^{\infty}e^{-\frac{1}{2}\phi t}t^{\zeta-1}W_{\lambda,\mu}(\phi t)E_{\alpha,\beta,p}^{\gamma,\delta,q}(wt^{\sigma})dt$$
$$=\frac{\Gamma(\delta)\phi^{-\zeta}}{\Gamma(\gamma)}.{}_4\Psi_3\left[\begin{array}{c}(\gamma,q),(1,1),(\frac{1}{2}+\mu+\zeta,\sigma),(\frac{1}{2}-\mu+\zeta,\sigma)\\(\beta,\alpha),(\delta,p),(1-\lambda+\zeta,\sigma)\end{array};\frac{w}{\phi^{\sigma}}\right]\quad(36)$$

where (7) is satisfied and $\Re(\zeta)>0,\Re(\phi)>0$.
Proof. Setting $v=\phi t$, then we get

$$\int_0^\infty e^{-\frac{1}{2}\phi t}t^{\zeta-1}W_{\lambda,\mu}(\phi t)E^{\gamma,\delta,q}_{\alpha,\beta,p}(wt^\sigma)dt = \int_0^\infty e^{-\frac{v}{2}}\left(\frac{v}{\phi}\right)^{\zeta-1}W_{\lambda,\mu}(v)\sum_{n=0}^\infty \frac{(\gamma)_{qn}w^n\left(\frac{v}{\phi}\right)^{\sigma n}}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\frac{1}{\phi}dv$$

$$= \frac{\Gamma(\delta)\phi^{-\zeta}}{\Gamma(\gamma)}\sum_{n=0}^\infty \frac{\Gamma(qn+\gamma)}{\Gamma(\alpha n+\beta)\Gamma(pn+\delta)}\left(\frac{w}{\phi^\sigma}\right)^n \int_0^\infty e^{-\frac{v}{2}}v^{\zeta+\sigma n-1}W_{\lambda,\mu}(v)dv$$

$$= \frac{\Gamma(\delta)\phi^{-\zeta}}{\Gamma(\gamma)}\sum_{n=0}^\infty \frac{\Gamma(qn+\gamma)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(pn+\delta)n!}\left(\frac{w}{\phi^\sigma}\right)^n \frac{\Gamma(\frac{1}{2}+\mu+\zeta+\sigma n)\Gamma(\frac{1}{2}-\mu+\zeta+\sigma n)}{\Gamma(1-\lambda+\zeta+\sigma n)}$$

$$= \frac{\Gamma(\delta)\phi^{-\zeta}}{\Gamma(\gamma)}\sum_{n=0}^\infty \frac{\Gamma(qn+\gamma)\Gamma(n+1)\Gamma(\frac{1}{2}+\mu+\zeta+\sigma n)\Gamma(\frac{1}{2}-\mu+\zeta+\sigma n)}{\Gamma(\alpha n+\beta)\Gamma(pn+\delta)\Gamma(1-\lambda+\zeta+\sigma n)}\frac{\left(\frac{w}{\phi^\sigma}\right)^n}{n!}$$

which directly yields (36).

5. Integral Operators with Generalized Mittag-Leffler Function in the Kernel

In this section, we consider composition of the Riemann-Liouville fractional integral and derivative and Hilfer's fractional derivative (9) - (11) with Mittag-Leffler function defined by (7).

Theorem 5.1 Let $a \in \mathbb{R}_+,\ \alpha,\beta,\gamma,\delta,\lambda,w \in \mathbb{C},\ \min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda)\} > 0$ and $p,q>0$, then for $x>a$ we have

$$D^\lambda_{a+}\left[(t-a)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(t-a)^\alpha]\right](x) = (x-a)^{\beta-\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta-\lambda,p}[w(x-a)^\alpha] \quad (37)$$

Proof. Beginning with $I^\lambda_{a+}\left[(t-a)^{\beta-1}\right](x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\lambda)}(x-a)^{\beta+\lambda-1}$, then

$$\begin{aligned} I^\lambda_{a+}\left[(t-a)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(t-a)^\alpha]\right](x) &= I^\lambda_{a+}\left[\sum_{n=0}^\infty \frac{(\gamma)_{qn}w^n(t-a)^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\right](x) \\ &= \sum_{n=0}^\infty \frac{(\gamma)_{qn}w^n(t-a)^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)(\delta)_{pn}}\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\lambda)}(x-a)^{\alpha n+\beta+\lambda-1} \quad (38) \\ &= (x-a)^{\beta+\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta+\lambda,p}[w(x-a)^\alpha] \end{aligned}$$

Now making use of (9), (27) and (38) yields

$$D^\lambda_{a+}\left[(t-a)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(t-a)^\alpha]\right](x) = \left(\frac{d}{dx}\right)^m\left[I^{m-\lambda}_{a+}(t-a)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(t-a)^\alpha]\right](x)$$

$$= \left(\frac{d}{dx}\right)^m\left[(x-a)^{\beta+m-\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(x-a)^\alpha]\right] = (x-a)^{\beta-\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta-\lambda,p}[w(x-a)^\alpha].$$

Now, making use of the formulas in (27) and (38), we can get the following result contained in

Theorem 5.2 Let $a \in \mathbb{R}_+,\ \alpha,\beta,\gamma,\delta,w \in \mathbb{C},\ \min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)\} > 0,\ 0 < u < 1,\ 0 \le v \le 0,\ \Re(\beta) > u+v-uv$ and $p,q>0$, then for $x>0$ we have

$$D^{u,v}_{a+}\left[(t-a)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(t-a)^{\alpha}]\right](x) = (x-a)^{\beta-u-1}E^{\gamma,\delta,q}_{\alpha,\beta-u,p}[w(x-a)^{\alpha}]. \quad (39)$$

Consider the integral operator defined in (8) containing the Mittag-Leffler function $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ in the kernel. First of all we will prove that the operator $\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a^+}$ is bounded on $L(a,b)$.

Theorem 5.3 Let $\alpha,\beta,\gamma,\delta,w \in \mathbb{C},\ \min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)\} > 0,\ b > a$ and $p,q>0$, then the operator $\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a^+}$ is bounded on $L(a,b)$ and

$$\left\|\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right\|_1 \le B\,\|\varphi\|_1 \quad (40)$$

where

$$B = (b-a)^{\Re(\beta)}\sum_{n=0}^{\infty}\frac{|(\gamma)_{qn}|\left|w(b-a)^{\Re(\alpha)}\right|^n}{|\Gamma(\alpha n+\beta)|\,|(\delta)_{pn}|\,|\Re(\alpha)n+\Re(\beta)|} \quad (41)$$

Proof. First of all, let C_n denote the n^{th} term of (41), then

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(\gamma)_{qn+q}}{(\gamma)_{qn}}\right|\left|\frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\alpha)}\right|\left|\frac{(\delta)_{pn}}{(\delta)_{pn+p}}\right|\left|\frac{\Re(\alpha)n+\Re(\beta)}{\Re(\alpha)n+\Re(\alpha)+\Re(\beta)}\right|\left|w(b-a)^{\Re(\alpha)}\right|$$

$\sim \dfrac{\left|w(b-a)^{\Re(\alpha)}\right|}{(|\alpha|\,n)^{\Re(\alpha)}}\dfrac{(qn)^q}{(pn)^p}$ as $n \to \infty$, provided that $q < p+\Re(\alpha)$. Hence $\left|\dfrac{c_{n+1}}{c_n}\right| \to 0$ as $n \to \infty$, which means that the right hand side of (41) is convergent and finite under the given condition.

Now according to (8) and (22)

$$\left\|\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right\|_1 = \int_a^b\left|\int_a^b(x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(x-t)^{\alpha}]\varphi(t)dt\right|dx$$

$$\le \int_a^b\left[\int_a^b(x-t)^{\beta-1}\left|E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(x-t)^{\alpha}]\right|dx\right]|\varphi(t)|\,dt = \int_a^b\left[\int_0^{b-t}u^{\Re(\beta)-1}\left|E^{\gamma,\delta,q}_{\alpha,\beta,p}[wu^{\alpha}]\right|du\right]|\varphi(t)|\,dt$$

$$\le \int_a^b\left[\int_0^{b-a}u^{\Re(\beta)-1}\left|E^{\gamma,\delta,q}_{\alpha,\beta,p}[wu^{\alpha}]\right|du\right]|\varphi(t)|\,dt.$$

But we have

$$\int_0^{b-a}u^{\Re(\beta)-1}\left|E^{\gamma,\delta,q}_{\alpha,\beta,p}[wu^{\alpha}]\right|du = \sum_{n=0}^{\infty}\frac{|(\gamma)_{qn}|\,|w|^n}{|\Gamma(\alpha n+\beta)|\,|(\delta)_{pn}|}\int_0^{b-a}u^{\Re(\alpha)n+\Re(\beta)-1}du = B$$

so that $B = (b-a)^{\Re(\beta)}\sum\limits_{n=0}^{\infty}\dfrac{|(\gamma)_{qn}|\left|w(b-a)^{\Re(\alpha)}\right|^n}{|\Gamma(\alpha n+\beta)|\,|(\delta)_{pn}|\,|\Re(\alpha)n+\Re(\beta)|}$

Hence

$$\left\|\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right\|_1 \le \int_a^b B\,|\varphi(t)|\,dt = B\,\|\varphi\|_1\,.$$

Equality (28) can simply be written by means of the operator $\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a^+}$ as

Corollary 5.4 Let $\alpha,\beta,\gamma,\delta,\zeta,w\in\mathbb{C},\ \min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda)\}>0$ and $p,q>0$, then

$$\left[\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}(t-a)^{\zeta-1}\right](x)=\Gamma(\zeta)(x-a)^{\beta+\zeta-1}E^{\gamma,\delta,q}_{\alpha,\beta+\zeta,p}[w(x-a)^{\alpha}]. \tag{42}$$

6. Composition of Fractional Calculus Operators and Integral Operator with Generalized Mittag-Leffler Function in the Kernel

We consider now composition of the Riemann-Liouville fractional integration operator $I^{\lambda}_{a^+}$ with the operator $\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}$

Theorem 6.1 Let $\alpha,\beta,\gamma,\delta,\lambda,w\in\mathbb{C},\ \min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\lambda)\}>0$ and $p,q>0$, then

$$I^{\lambda}_{a+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi=\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta+\lambda,p,w,a+}\varphi=\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}I^{\lambda}_{a+}\varphi \tag{43}$$

holds for any summable function $\varphi\in L(a,b)$.

Proof.

$$\left(I^{\lambda}_{a+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)=\frac{1}{\Gamma(\lambda)}\int_a^x(x-u)^{\lambda-1}\left[\int_a^u(u-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(u-t)^{\alpha}]\varphi(t)dt\right]du$$

$$=\int_a^x\left[\frac{1}{\Gamma(\lambda)}\int_t^x(x-u)^{\lambda-1}(u-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w(u-t)^{\alpha}]du\right]\varphi(t)dt$$

letting $\tau=u-t$ implies

$$\left(I^{\lambda}_{a+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)=\int_a^x\left[\frac{1}{\Gamma(\lambda)}\int_0^{x-t}(x-t-\tau)^{\lambda-1}\tau^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w\tau^{\alpha}]d\tau\right]\varphi(t)dt$$

$$=\int_a^x I^{\lambda}_0\left[\tau^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}[w\tau^{\alpha}]\right](x-t)\varphi(t)dt=\int_a^x\left[\tau^{\beta+\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta+\lambda,p}[w\tau^{\alpha}]\right]\varphi(t)dt$$

$$=\int_a^x\left[(x-t)^{\beta+\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta+\lambda,p}[w(x-t)^{\alpha}]\right]\varphi(t)dt=\left(\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta+\lambda,p,w,a+}\varphi\right)(x)$$

Similarly, we can prove the other side .

Theorem 6.2 If the conditions of Theorem 6.1 is satisfied, then

$$\left(D^{\lambda}_{a+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)=\left(\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta-\lambda,p,w,a+}\varphi\right)(x). \tag{44}$$

Proof. Let $n=[\Re(\lambda)]+1$ and using (9), we get

$$\left(D^{\lambda}_{a+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)=\left(\frac{d}{dx}\right)^n\left(I^{n-\lambda}_{a^+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)$$

$$=\left(\frac{d}{dx}\right)^n\left(\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta+n-\lambda,p,w,a+}\varphi\right)(x)$$

$$=\left(\frac{d}{dx}\right)^n\int_a^x\left[(x-t)^{\beta+n-\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta+n-\lambda,p}[w(x-t)^{\alpha}]\right]\varphi(t)dt$$

Since the integral is continuous, (23) yields

$$\left(D^{\lambda}_{a+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)=\left(\frac{d}{dx}\right)^{n-1}\int\limits_a^x\frac{\partial}{\partial x}\left[(x-t)^{\beta+n-\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta+n-\lambda,p}[w(x-t)^{\alpha}]\right]\varphi(t)dt$$
$$+\lim_{t\to x}(x-t)^{\beta+n-\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta+n-\lambda,p}[w(x-t)^{\alpha}]$$
$$=\left(\frac{d}{dx}\right)^{n-1}\int\limits_a^x(x-t)^{\beta+n-\lambda-2}\left(\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\,[w(x-t)^{\alpha}]^n}{\Gamma(\alpha n+\beta+n-\lambda-1)(\delta)_{pn}}\right)\varphi(t)dt$$
$$=\left(\frac{d}{dx}\right)^{n-1}\int\limits_a^x\left[(x-t)^{\beta+n-\lambda-2}E^{\gamma,\delta,q}_{\alpha,\beta+n-\lambda-1,p}[w(x-t)^{\alpha}]\right]\varphi(t)dt$$

Repeating this process $(n-1)$ times, then we get

$$\left(D^{\lambda}_{a+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)=\int\limits_a^x\left[(x-t)^{\beta-\lambda-1}E^{\gamma,\delta,q}_{\alpha,\beta-\lambda,p}[w(x-t)^{\alpha}]\right]\varphi(t)dt$$
$$=\left(\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta-\lambda,p,w,a+}\varphi\right)(x).$$

Theorem 6.3 Let $\alpha,\beta,\gamma,\delta,w\in\mathbb{C}$, $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta)\}>0$ $0<u<1$, $0\le v\le 1$, $\Re(\beta)>u+v-uv$ and $p,q>0$, then

$$\left(D^{u,v}_{a^+}\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta,p,w,a+}\varphi\right)(x)=\left(\mathcal{E}^{\gamma,\delta,q}_{\alpha,\beta-u,p,w,a+}\varphi\right)(x). \quad (45)$$

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