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Fractional Cauchy Problems for Degenerate Differential Equations

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Abstract: We consider fractional abstract Cauchy problems. Related inverse problems are discussed. Applications to partial differential equations are given. Required conditions on spaces and operators are given guaranteeing existence and uniqueness of solutions.

Keywords: Fractional derivative, evolution problems, inverse problem.

1 Introduction, motivation and preliminaries

Many papers and monographs concern the abstract equation

$$BMu - Lu = f\,, \tag{1}$$

where B, M, L are closed linear operators on the complex Banach space E with $D(L) \subseteq D(M)$, $0 \in \rho(L)$, the resolvent of L, $f \in E$ and u is the unknown. The first approach to handle existence and uniqueness of the solution u to (1) was given by Favini-Yagi [1], see in particular the monograph [2]. By using the real interpolation space $(E, D(B))_{\theta,\infty}$, $0 < \theta < 1$, see [3], [4], suitable assumptions on the operators B, M, L guarantee that (1) has a unique solution. Notice that different approaches are presented in literature, but in general only weak solutions are found, see Carroll-Showalter [5]. The quoted result was improved by Favini, Lorenzi, Tanabe in [6]. Very recently, an extension to interpolation spaces $(E, D(B))_{\theta,p}$, $1 < p < \infty$, was obtained by Al Horani et al., see [7], see also [8], [9]-[11].

Here we indicate the basic assumptions:

(H_1) Operator B has a resolvent $(z-B)^{-1}$ for any $z \in \mathbb{C}$, Re$z < a$, $a > 0$ satisfying

$$\|(z-B)^{-1}\|_{\mathscr{L}(E)} \le \frac{c}{|Re\,z|+1}, \quad Re\,z < a\,. \tag{2}$$

(H_2) Operators L, M satisfy the resolvent estimate

$$\|M(zM-L)^{-1}\|_{\mathscr{L}(E)} \le \frac{c}{(|z|+1)^{\beta}} \tag{3}$$

for any $z \in \Sigma_\alpha := \left\{z \in \mathbb{C} : Re\,z \ge -c(1+|Im\,z|)^{\alpha},\ c>0,\ 0<\beta\le\alpha\le 1\right\}$.

(H_3) Let A be the possibly multivalued linear operator $A = LM^{-1}$, $D(A) = M(D(L))$. Then A and B commute in the resolvent sense:

$$B^{-1}A^{-1} = A^{-1}B^{-1}\,.$$

Al Horani et al. [8] obtain the following:

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Theorem 1. *Let $\alpha+\beta>1$, $2-\alpha-\beta<\theta<1$. Then, under hypotheses (H_1)-(H_3), equation (1) admits a unique strict solution u such that Lu, $BMu \in (E,D(B))_{\omega,p}$, $\omega=\theta-2+\alpha+\beta$, $1<p\leq\infty$, provided that $f\in(E,D(B))_{\theta,p}$.*

Notice that if B generates a bounded c_0-group in E or $-B$ generates a bounded c_0-semigroup in E, then assumption (H_1) holds.

In order to introduce another important example of operator B, we recall that a linear operator B on the complex Banach space E is said to be positive of type ω, $\omega\in(0,\pi)$, if

$$\left\{z\in\mathbb{C}\backslash\{0\},\ |\mathrm{Arg}\,z|>\omega\right\}\cup\{0\}\subseteq\rho(B)$$

and for all $\varepsilon\in(0,\pi-\omega)$, there exists $\mu(\varepsilon)>0$ such that

$$\|z(z-B)^{-1}\|_{\mathscr{L}(E)}\leq\mu(\varepsilon) \tag{4}$$

for all $z\in\mathbb{C}\backslash\{0\}$, $|\mathrm{Arg}\,z|\geq\omega+\varepsilon$. If B is a positive operator on E of type ω and $\delta\in\mathbb{R}^+$, one sets

$$B^{-\delta}:=-\frac{1}{2\pi i}\int_{\Gamma(\Phi,R)}z^{-\delta}(z-B)^{-1}\,dz,$$

with $\Phi\in(\omega,\pi)$, $R>0$ such that $\{z\in\mathbb{C}:|z|\leq R\}\subseteq\rho(B)$, $\Gamma(\Phi,R)$ being a piecewise C^1 path describing

$$\left\{z\in\mathbb{C};|z|\geq R,\ |\mathrm{Arg}\,z|=\Phi\right\}\cup\left\{z\in\mathbb{C};|z|=R,\ |\mathrm{Arg}\,z|\leq\Phi\right\}$$

oriented from $\infty e^{-i\Phi}$ to $\infty e^{i\Phi}$. We shall write $z\in\Gamma(\Phi,R)$ to indicate that z belongs to the range of $\Gamma(\Phi,R)$.

If X is a complex Banach space, we introduce operator B_X by

$$\begin{aligned}&B_X:\left\{v\in C^1([0,T];X);\,v(0)=0\right\}\longrightarrow C([0,T];X)\\&B_Xv:=D_tv.\end{aligned} \tag{5}$$

Then $\rho(B_X)=\mathbb{C}$ and B_X is a positive operator in $C([0,T];X)$ of type $\pi/2$.

If $\delta>0$, for any $f\in C([0,T];X)$ and any $t\in[0,T]$, operator $B_X^{-\delta}$ is given by

$$B_X^{-\delta}f(t)=\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}f(s)\,ds.$$

One sees easily that for all $\delta\in R^+$, $B_X^{-\delta}$ is injective, so that we can define for all $\delta\in R^+$

$$B_X^{\delta}=(B_X^{-\delta})^{-1}.$$

The following well-known result holds.

Proposition 1. *If $\delta\in(0,2)$, B_X^{δ} is positive of type $\dfrac{\delta\pi}{2}$.*

Notice that for every $c\in\mathbb{C}$, operator $c+B_X^{\delta}$, $\delta\in(0,2)$, is positive of type $\dfrac{\delta\pi}{2}$, too.

Proposition 2. *Let $0\leq\alpha_0<\alpha_1$, $\xi\in(0,1)$, $(1-\xi)\alpha_0+\xi\alpha_1\notin\mathbb{N}$; then*

$$\begin{aligned}\left(D(B_X^{\alpha_0}),D(B_X^{\alpha_1})\right)_{\xi,\infty}=\Big\{&f\in C^{(1-\xi)\alpha_0+\xi\alpha_1}([0,T];X),\ f^{(k)}(0)=0,\ \text{for all } k\in\mathbb{N}_0,\\&k<(1-\xi)\alpha_0+\xi\alpha_1,\ \mathbb{N}_0=\mathbb{N}\cup\{0\}\Big\}\end{aligned}$$

with equivalent norms. Of course, if $\alpha_0=0$, then $D(B_X^0)=D(I)=C([0,T];X)$. In particular,

$$\left(C([0,T];X),D(B_X^{\alpha_1})\right)_{\xi,\infty}=\left\{f\in C^{\xi\alpha_1}([0,T];X),\ f^{(k)}(0)=0,\ \text{for all } k\in\mathbb{N}_0,\ k<\xi\alpha_1\right\}.$$

Let now $\tilde{\alpha} \in (0,1]$. Since $B_X^{\tilde{\alpha}}$ is positive of type $\frac{\tilde{\alpha}\pi}{2} \leq \frac{\pi}{2}$, we deduce that it satisfies assumption (H_1). Therefore, if M, L are two operators satisfying (H_2) in X and $B_X^{-1}ML^{-1} = ML^{-1}B_X^{-1}$, we can apply Theorem 1 to the equation

$$B_X^{\tilde{\alpha}}Mu - Lu = f \in \left(C([0,T];X), D(B_X^{\tilde{\alpha}})\right)_{\theta,\infty}, \tag{6}$$

$\alpha+\beta > 1$, $2-\alpha-\beta < \theta < 1$. We shall establish precisely this result and focus on some examples. Then we apply the previous general result to

$$\mathbb{D}_X^{\tilde{\alpha}}Mu - Lu = f, \tag{7}$$

where $\mathbb{D}_X^{\tilde{\alpha}}$ denotes the Caputo derivative of order $\tilde{\alpha} > 0$, i.e.,

$$\mathbb{D}_X^{\tilde{\alpha}}u(t) := \frac{1}{\Gamma(n-\tilde{\alpha})}\int_0^t (t-s)^{n-\tilde{\alpha}-1}u^{(n)}(s)\,ds, \quad t \in [0,T],$$

where $n := \lceil \tilde{\alpha} \rceil$ is the smallest integer greater than or equal to $\tilde{\alpha}$, see [12]. We shall also study some related inverse problems. We refer to Guidetti [13] and Bazhlekova [14] for results concerning nondegenerate problems. In particular, we recall that, see Guidetti [13], Definition 2.2, if $\tilde{\alpha} \in \mathbb{R}^+$, $u \in C^{\lceil \tilde{\alpha} \rceil}([0,T];X)$, then Caputo derivative $\mathbb{D}_X^{\tilde{\alpha}}$ of order $\tilde{\alpha}$ exists if $u - \sum_{k<\tilde{\alpha}} \frac{t^k}{k!}u^{(k)}(0)$ belongs to $D\left(B_X^{\tilde{\alpha}}\right)$ and

$$\mathbb{D}_X^{\tilde{\alpha}}u := B_X^{\tilde{\alpha}}\left(u - \sum_{k<\tilde{\alpha}} \frac{t^k}{k!}u^{(k)}(0)\right).$$

At last, we consider an abstract equation, generalizing second-order equation in time.

The paper is organized as follows: In Section 2 we apply Theorem 1 to equation (6); some examples illustrating our abstract results are also given. In Section 3 we apply the previous general result to equation (7) and give some examples. In Section 4 we study some related inverse problems. Section 5 is devoted to an abstract equation, generalizing second-order equation in time; an example illustrates our results is also given.

2 Equation: $B_X^{\tilde{\alpha}}My - Ly = f$

Taking into account the previous results in Section 1, we can consider and solve the initial value problem

$$\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}My(t) = B_X^{\tilde{\alpha}}My(t) = Ly(t) + f(t), \quad 0 \leq t \leq T, \tag{8}$$

$$(My)(0) = My_0, \quad y_0 \in D(L). \tag{9}$$

We write

$$\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}My(t) = \frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}[My(t) - My_0] + \frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}My_0,$$

so that, if $\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}My_0 = 0$, then $\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}[My(t) - My_0] = \frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}My(t)$. Hence (8) reads

$$\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}\left[M(y(t) - y_0)\right] = L\big[y(t) - y_0\big] + f(t) + Ly_0, \quad 0 \leq t \leq T.$$

Applying Theorem 1, we conclude that if $f(\cdot) + Ly_0 \in \left(C([0,T];X), D(B_X^{\tilde{\alpha}})\right)_{\theta,\infty}$, i.e., $f \in C^{\theta\tilde{\alpha}}([0,T];X)$, $f(0) + Ly_0 = 0$, then problem (8)-(9) admits a unique strict solution $y(\cdot)$ such that $Ly(\cdot)$, $\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}(My)(\cdot) \in C^{\omega\tilde{\alpha}}([0,T];X)$, $\omega = \theta - 2 + \alpha + \beta$.

This result can be written as follows:

Theorem 2. *Let operators L, M satisfying (H_2), $\alpha+\beta > 1$, $0 < \beta \leq \alpha \leq 1$. Take $0 < \tilde{\alpha} \leq 1$, $y_0 \in D(L)$, $\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}My_0 = 0$. Then problem (8)-(9) admits a unique strict solution $y(\cdot)$ such that $Ly(\cdot)$, $\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}(My)(\cdot) \in C^{\omega\tilde{\alpha}}([0,T];X)$, $\omega = \theta - 2 + \alpha + \beta$, $2-\alpha-\beta < \theta < 1$, provided that $f \in C^{\theta\tilde{\alpha}}([0,T];X)$ and $f(0) + Ly_0 = 0$.*

*Example 1.*Consider

$$\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}} m(x)v = \Delta v - cv + f(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T],$$
$$m(x)v(x,0) = m(x)v_0(x), \quad x \in \mathbb{R}^n,$$

where $m(x) \geq 0$ in $\mathbb{R}^n$ with $m \in L^\infty(\mathbb{R}^n)$, $c > 0$ is a constant. Taking $X = H^{-1}(\mathbb{R}^n)$, M the multiplication operator by the function $m(x)$, $L = \Delta - c : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$, it is seen in Favini-Yagi [2], p. 81, that assumption (H_2) holds for $\alpha = \beta = 1$. If we take $X = L^2(\mathbb{R}^n)$, then it is shown that (H_2) holds for $\alpha = 1$, $\beta = 1/2$, see [2], p. 81, see also [1].

Operator $B_X^{\tilde{\alpha}}$, $0 < \tilde{\alpha} \leq 1$, satisfies ($H_1$) and clearly ($H_3$) holds, so Theorem 2 can be applied in both cases.

*Example 2.*Consider for $0 < \tilde{\alpha} \leq 1$,

$$\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}} \left(\left(1 - \frac{\partial^2}{\partial x^2}\right) v \right) = -\frac{\partial^4 v}{\partial x^4} + f(x,t), \quad x \in [0,1], \quad t \in [0,T],$$
$$v(0,t) = v(1,t) = \frac{\partial^2 v}{\partial x^2}(0,t) = \frac{\partial^2 v}{\partial x^2}(1,t) = 0, \quad 0 < t < T,$$
$$\left(1 - \frac{\partial^2}{\partial x^2}\right) v(x,0) = u_0(x).$$

Take $X = L^2(0,1)$. If $P = -\frac{d^2}{dx^2}$ with $D(P) = H^2(0,1) \cap H_0^1(0,1)$, P is a positive self- adjoint operator in X and $M = P+1$, $L = -P^2$. One can easily show that all hypotheses (H_1)-(H_3) hold with $\alpha = \beta = 1$. Moreover, existence, uniqueness and regularity of solution are guaranteed provided that $u_0(x) = \left(1 - \frac{\partial^2}{\partial x^2}\right) v_0(x)$, $v_0 \in H^2(0,1) \cap H_0^1(0,1)$, see Favini-Yagi [2], p. 73.

If we take $X = L^2(0,1)$, $M = 1 - \frac{d^2}{dx^2}$ with $D(M) = H^2(0,1) \cap H_0^1(0,1)$ and $L = -\frac{d^4}{dx^4}$ with $D(L) = H^4(0,1) \cap H_0^2(0,1)$, then one can see that hypothesis (H_2) holds for $\alpha = 1$, $\beta = 1/2$. Therefore, Theorem 2 applies again.

*Example 3.*Consider the fractional Poisson-heat equation

$$\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}} (m(x)v) = \Delta v + f(x,t), \quad x \in \Omega, \quad t \in [0,T],$$
$$v = 0, \quad x \in \partial\Omega, \quad t \in [0,T],$$
$$m(x)v(x,0) = m(x)v_0(x) \ \ forsome\ v_0 \in H^2(\Omega) \cap H_0^1(\Omega),$$

Ω being a bounded region in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Here $m(x) \geq 0$ in Ω is a given function in $L^\infty(\Omega)$. It is shown in Favini-Yagi [2], p. 76, formula (3.41), that multiplication operator M by $m(\cdot)$ and operator $L = \Delta$, with $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$ satisfy hypothesis (H_2) in $L^2(\Omega)$ for $\alpha = 1$, $\beta = 1/2$. Remember that $0 < \tilde{\alpha} \leq 1$. Suppose that $0 < \beta \leq \alpha \leq 1$, $\alpha + \beta > 1$, $\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}(Mv_0) = 0$. Then the above problem admits a unique strict solution v such that $\frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}(Mv)$, $\Delta v(\cdot) \in C^{\omega\tilde{\alpha}}\big([0,T]; L^2(\Omega)\big)$, $\omega = \theta - 1/2$, $1/2 < \theta < 1$, provided that $f(x,0) + \Delta v_0(x) = 0$, $f \in C^{\theta\tilde{\alpha}}\big([0,T]; L^2(\Omega)\big)$.

3 Caputo fractional derivative

In this section we consider the initial value problem

$$\mathbb{D}_X^{\tilde{\alpha}}(Mu(t)) - Lu(t) = f(t), \quad t \in [0,T], \tag{10}$$
$$(Mu)(0) = Mu_0, \quad u_0 \in D(L), \tag{11}$$

where $0 < \tilde{\alpha} \leq 1$ and $\mathbb{D}_X^{\tilde{\alpha}}$ denotes the Caputo derivative of order $\tilde{\alpha}$. Since $0 < \tilde{\alpha} \leq 1$, we know that $\mathbb{D}_X^{\tilde{\alpha}} u(t) = B_X^{\tilde{\alpha}}\big(u(t) - u(0)\big) = B_X^{\tilde{\alpha}}\big(u(t) - u_0\big)$, and thus problem (10)-(11) is reduced, equivalently, to

$$B_X^{\tilde{\alpha}}(Mu(t) - Mu_0) = Lu(t) + f(t), \quad t \in [0,T], \tag{12}$$
$$M(u(t) - u_0)|_{t=0} = 0. \tag{13}$$

Since $u_0 \in D(L)$, we can represent (12)-(13) in the form

$$B_X^{\tilde{\alpha}}\big(M(u(t)-u_0)\big) = L\big(u(t)-u_0\big) + Lu_0 + f(t), \quad t \in [0,T],$$
$$M(u(t)-u_0)|_{t=0} = 0.$$

Call $u(t)-u_0 = v(t)$, $t \in [0,T]$. Then we get the problem

$$B_X^{\tilde{\alpha}}(Mv) = Lv + Lu_0 + f(t), \quad t \in [0,T],$$
$$(Mv)(0) = 0.$$

Therefore, we can immediately apply Theorem 2 and obtain

Theorem 3. *Let operators L, M satisfying (H_2), $\alpha+\beta>1$, $0<\beta\leq\alpha\leq 1$, $0<\tilde{\alpha}\leq 1$, $u_0 \in D(L)$. Then problem (10)-(11) admits a unique strict solution $u(\cdot)$ such that $L\big(u(\cdot)-u_0\big)$, $B_X^{\tilde{\alpha}}\big(M(u(\cdot)-u_0)\big) = \mathbb{D}_X^{\tilde{\alpha}}Mu \in C^{\omega\tilde{\alpha}}([0,T];X)$, $\omega = \theta-2+\alpha+\beta$, $2-\alpha-\beta<\theta<1$, provided that $f \in C^{\theta\tilde{\alpha}}([0,T];X)$, $f(0)+Lu_0 = 0$.*

Of course, Example 1, Example 2 and Example 3 can be adapted to handle Caputo fractional derivative instead of $D_X^{\tilde{\alpha}}$, i.e., $\dfrac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}$.

*Example 4.*Consider the problem in $L^p(\Omega)$, $1<p<\infty$,

$$\mathbb{D}_{L^p(\Omega)}^{\tilde{\alpha}} m(x)v = \Delta v - cv + f(x,t), \quad (x,t) \in \Omega \times [0,T],$$
$$v = 0, \quad (x,t) \in \partial\Omega \times [0,T],$$
$$m(x)v(x,0) = m(x)v_0(x), \quad x \in \Omega,$$

in a bounded region $\Omega \subseteq \mathbb{R}^n$, where $m(x) \geq 0$ and $m \in L^\infty(\Omega)$, c is a positive constant, M is the multiplication operator by the function $m(x)$ in $L^p(\Omega)$, $L = \Delta - c : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$, $1<p<\infty$. Then one sees, cfr. Favini-Yagi [2], pp. 79-80, that assumption (H_2) holds for $\alpha = 1$, $\beta = 1/p$.

If $2<p<\infty$, and the stronger condition $m \in C^1(\overline{\Omega})$, with

$$|\nabla m(x)| \leq c\,(m(x))^\rho, \quad x \in \overline{\Omega}, \ 0<\rho\leq 1$$

holds, then we obtain a better exponent $\beta = \dfrac{2}{p(2-\rho)} \ \left(> \dfrac{1}{p}\right)$.

*Example 5.*Consider the problem

$$\mathbb{D}_X^{\tilde{\alpha}}\left(1+\frac{\partial^2}{\partial x^2}\right)v = \frac{\partial^2 v}{\partial x^2} + f(x,t), \quad x \in [0,l\pi], \quad t \in [0,T],$$
$$v(0,t) = v(l\pi,t) = 0, \quad t \in [0,T],$$
$$\left(1+\frac{\partial^2}{\partial x^2}\right)v(x,0) = \left(1+\frac{\partial^2}{\partial x^2}\right)v_0(x), \quad x \in [0,l\pi],$$

where l is a positive integer.

Various choices of the space X are possible, but we take

$$X = \{f \in C([0,l\pi];\mathbb{C}) : \ f(0) = f(l\pi) = 0\}.$$

Operators M and L are introduced correspondingly, for example

$$D(L) = \left\{v \in C^2([0,l\pi];\mathbb{C}) : \ v(0) = v(l\pi) = v''(0) = v''(l\pi) = 0\right\},$$
$$Lv = \frac{\partial^2 v}{\partial x^2}, \quad v \in D(L).$$

Then one can see that hypothesis (H_2) holds for $\alpha = \beta = 1$, cfr. Favini-Yagi [2], p. 85.

4 Inverse problems

Following previous methods, we could consider an inverse problem as follows. To find a pair $(y, f) \in C([0,T]; D(L)) \times C([0,T]; \mathbb{C})$ satisfying the inverse problem

$$B_X^{\tilde{\alpha}} My(t) = Ly(t) + f(t)z + h(t), \quad 0 \le t \le T, \tag{14}$$

$$(My)(0) = My_0, \tag{15}$$

$$\Phi[My(t)] = g(t), \quad 0 \le t \le T, \tag{16}$$

where M, L are closed linear operators on the complex Banach space X with $D(L) \subseteq D(M)$, $0 \in \rho(L)$, $z \in X$, $h \in C([0,T]; X)$, $y_0 \in D(L)$, $\Phi \in X^*$, $g \in C([0,T]; \mathbb{C})$. Clearly, the compatibility relation

$$\Phi[My_0] = g(0)$$

must hold. To reduce such an inverse problem to a direct one, we need the following important step:

$$B_{\mathbb{C}}^{\tilde{\alpha}} \Phi[u] = \Phi[B_X^{\tilde{\alpha}} u].$$

Proof. Using [15], one obtains that for $u \in D(B_X)$and $f \in D(B_{\mathbb{C}})$

$$B_X^{\tilde{\alpha}} u = \frac{\sin \pi\tilde{\alpha}}{\pi} \int_0^\infty l^{\tilde{\alpha}-1} B_X (l + B_X)^{-1} u dl, \tag{17}$$

$$B_{\mathbb{C}}^{\tilde{\alpha}} f = \frac{\sin \pi\tilde{\alpha}}{\pi} \int_0^\infty l^{\tilde{\alpha}-1} B_{\mathbb{C}} (l + B_{\mathbb{C}})^{-1} f dl, \tag{18}$$

respectively.

If u is a function with values in X defined in $[0,T]$, $\Phi[u]$ denotes the function defined by $\Phi[u](t) = \Phi[u(t)]$, $0 \le t \le T$. It is evident that if $u \in D(B_X)$, then $\Phi[u] \in D(B_{\mathbb{C}})$.

Let $u \in D(B_X)$. Then, in view of (17) one has

$$\Phi[B_X^{\tilde{\alpha}} u] = \frac{\sin \pi\tilde{\alpha}}{\pi} \int_0^\infty l^{\tilde{\alpha}-1} \Phi[B_X (l + B_X)^{-1} u] dl. \tag{19}$$

Since

$$\Phi[B_X (l + B_X)^{-1} u](t) = \Phi[B_X (l + B_X)^{-1} u(t)] = \Phi\left[D_t \int_0^t e^{-l(t-s)} u(s) ds \right]$$

$$= D_t \Phi\left[\int_0^t e^{-l(t-s)} u(s) ds \right] = D_t \int_0^t e^{-l(t-s)} \Phi[u(s)] ds = D_t \int_0^t e^{-l(t-s)} \Phi[u](s) ds$$

$$= D_t (B_{\mathbb{C}} + l)^{-1} \Phi[u](t) = B_{\mathbb{C}} (B_{\mathbb{C}} + l)^{-1} \Phi[u](t),$$

one has with the aid of (19) and (18)

$$\Phi[B_X^{\tilde{\alpha}} u] = \frac{\sin \pi\tilde{\alpha}}{\pi} \int_0^\infty l^{\tilde{\alpha}-1} B_{\mathbb{C}} (B_{\mathbb{C}} + l)^{-1} \Phi[u] dl = B_{\mathbb{C}}^{\tilde{\alpha}} \Phi[u]. \tag{20}$$

Suppose $u \in D(B_X^{\tilde{\alpha}})$. Set $u_n = n(n + B_X)^{-1} u$ for $n = 1, 2, \cdots$. Then $u_n \in D(B_X)$, and

$$u_n \to u$$

$$B_X^{\tilde{\alpha}} u_n = n(n + B_X)^{-1} B_X^{\tilde{\alpha}} u \to B_X^{\tilde{\alpha}} u.$$

Furthermore applying (20) to u_n

$$\Phi[B_X^{\tilde{\alpha}} u_n] = B_{\mathbb{C}}^{\tilde{\alpha}} \Phi[u_n].$$

Since

$$\|\Phi[B_X^{\tilde{\alpha}} u_n] - \Phi[B_X^{\tilde{\alpha}} u]\| = \|\Phi[B_X^{\tilde{\alpha}} u_n - B_X^{\tilde{\alpha}} u]\| = \sup_{0 \le t < \infty} |\Phi[B_X^{\tilde{\alpha}} u_n(t) - B_X^{\tilde{\alpha}} u(t)]|$$

$$\le \|\Phi\|_{X^*} \sup_{0 \le t < \infty} \|B_X^{\tilde{\alpha}} u_n(t) - B_X^{\tilde{\alpha}} u(t)\|_X = \|\Phi\|_{X^*} \|B_X^{\tilde{\alpha}} u_n - B_X^{\tilde{\alpha}} u\| \to 0,$$

one has

$$\Phi[u_n] \to \Phi[u],\ \ \Phi[B_X^{\tilde{\alpha}} u_n] \to \Phi[B_X^{\tilde{\alpha}} u].$$

Therefore $\Phi[u] \in D(B_{\mathbb{C}}^{\tilde{\alpha}})$ and $B_{\mathbb{C}}^{\tilde{\alpha}} \Phi[u] = \Phi[B_X^{\tilde{\alpha}} u]\ \forall u \in D(B_X^{\tilde{\alpha}})$. $\square$

Now, by applying Φ to both sides of equation (14), we get

$$B_{\mathbb{C}}^{\tilde{\alpha}} g(t) = \Phi[Ly(t)] + \Phi[h(t)] + f(t)\Phi[z].$$

If $\Phi[z] \neq 0$, then necessarily

$$f(t) = \frac{1}{\Phi[z]} \left\{ B_{\mathbb{C}}^{\tilde{\alpha}} g(t) - \Phi[Ly(t)] - \Phi[h(t)] \right\}.$$

Therefore, we obtain the direct problem

$$B_X^{\tilde{\alpha}} My(t) = Ly(t) - \frac{\Phi[Ly(t)]}{\Phi[z]} z + h(t) - \frac{\Phi[h(t)]}{\Phi[z]} z + \frac{B_{\mathbb{C}}^{\tilde{\alpha}} g(t)}{\Phi[z]} z, \ \ t \in [0,T] \tag{21}$$

$$(My)(0) = My_0. \tag{22}$$

Let L_1 be the operator defined by

$$D(L_1) = D(L), \quad L_1 y = -\frac{\Phi[Ly]}{\Phi[z]} z.$$

Then Problem (21)-(22) reduces to

$$B_X^{\tilde{\alpha}} My(t) = (L + L_1)y(t) + h(t) - \frac{\Phi[h(t)]}{\Phi[z]} z + \frac{B_{\mathbb{C}}^{\tilde{\alpha}} g(t)}{\Phi[z]} z, \ \ t \in [0,T] \tag{23}$$

$$(My)(0) = My_0. \tag{24}$$

Introduce the multivalued linear operator $A := LM^{-1}$, $D(A) = M(D(L))$, so that (H_2) holds. This means that $\|(zI - A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{c}{(1+|z|)^{\beta}}$, $z \in \Sigma_{\alpha}$. Theorem 1 in [16], pp. 148-149, see also Lorenzi [17], affirms that if (H_2) holds and $L_1 \in \mathscr{L}\left(D(L), X_A^{\theta_1}\right)$, $1-\beta < \theta_1 < 1$, where

$$X_A^{\theta_1} = \left\{ u \in X,\ \sup_{t>0} t^{\theta_1} \|A^0 (t-A)^{-1} u\|_X < \infty \right\},$$

with $A^0(t-A)^{-1} = -I + t(t-A)^{-1}$, then

$$\|M(zM - L - L_1)^{-1}\|_{\mathscr{L}(X)} \leq c(1+|z|)^{-\beta}, \quad \forall z \in \Sigma_{\alpha},\ |z|\ large.$$

In order to apply this result to our case, we must assume that $z \in X_A^{\theta_1}$ for some $\theta_1 \in (1-\beta, 1)$. Then problem (23)-(24) admits a unique strict solution y such that $Ly, \frac{\partial^{\tilde{\alpha}}}{\partial t^{\tilde{\alpha}}}(My) \in C^{\omega\tilde{\alpha}}([0,T];X)$, $\omega = \theta - 2 + \alpha + \beta$, provided that $\alpha + \beta > 1$, $0 < \beta \leq \alpha \leq 1, 0 < \tilde{\alpha} \leq 1, B_{\mathbb{C}}^{\tilde{\alpha}} g(t) \in C^{\theta\tilde{\alpha}}([0,T];\mathbb{C}), h \in C^{\theta\tilde{\alpha}}([0,T];X), Ly_0 - \frac{\Phi[Ly_0]}{\Phi[z]} z + \frac{B_{\mathbb{C}}^{\tilde{\alpha}} g(0)}{\Phi[z]} z - \frac{\Phi[h(0)]}{\Phi[z]} z + h(0) = 0.$

The construction of the solution tells us that the corresponding function $f(\cdot)$ belongs to $C([0,T];\mathbb{C})$. Indeed, $B_{\mathbb{C}}^{\tilde{\alpha}} g(\cdot) \in C^{\theta\tilde{\alpha}}([0,T];\mathbb{C})$, $h \in C^{\theta\tilde{\alpha}}([0,T];X)$, $Ly \in C^{\omega\tilde{\alpha}}([0,T];X)$, so that $f \in C^{\omega\tilde{\alpha}}([0,T];\mathbb{C})$. The same argument applies to the corresponding inverse problem related to the Caputo fractional derivative

$$\mathbb{D}_X^{\tilde{\alpha}} My(t) = Ly(t) + f(t)z + h(t), \quad 0 \leq t \leq T.$$

5 Generalized higher-order abstract equations

Let us consider the abstract equation, generalizing second-order equation in time,

$$B_2 C B_1 u + B B_1 u + Au = f,$$

where A, B, C are some closed linear operators in the complex Banach space X, while B_1 and B_2 are suitable operators defined on suitable Banach spaces. The change of variables $B_1 u = v$ transforms the given equation to the system

$B_1 u = v\,,$
$B_2 C v + B v + A u = f\,,$

which is written in the matrix form

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & -I \\ A & B \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

with a convenient product space and related domains of operator matrices. Noting

$$\mathbb{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad M = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}, \quad L = \begin{bmatrix} 0 & I \\ -A & -B \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

it assumes the form

$$(\mathbb{B}M - L)U = F\,, \quad U = (u,v)^T\,.$$

Here we do not use the assumption $D(B) \subseteq D(A) \cap D(C)$. Moreover, it is not restrictive to assume that $D(B) \subseteq D(C)$ and this assumption shall be maintained in the sequel. Of course the essential step is to invert $P(z) = z^2 C + zB + A$ under the possibility that AB^{-1} is not a bounded operator. To this end, we introduce the multivalued linear operator

$$V = BC^{-1}\,, \quad D(V) = \{Cx;\ x \in D(B)\}$$

and the closed linear operator $U = AB^{-1}$ (assuming that $0 \in \rho(A)$ guarantees that this is true). Here we assume

Assumption 1: The C modified resolvent set $\rho_C(-B)$ contains

$$\Sigma_\alpha := \left\{ z \in \mathbb{C} : Re\,z \geq -c(1 + |Im\,z|)^\alpha,\ c > 0,\ 0 < \beta \leq \alpha \leq 1 \right\},$$

such that

$$\|C(zC + B)^{-1}\|_{\mathscr{L}(X)} \leq \frac{c_1}{(1+|z|)^\beta}\,, \quad z \in \Sigma_\alpha\,. \tag{25}$$

Assumption 2:

$$\|B(zB + A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{c}{(1+|z|)^\gamma}\,, \quad \forall z \in \Sigma_\alpha, \tag{26}$$

where $0 < \gamma \leq 1$.

Assumption 3: $D(V) \subseteq D(U)$ and there is $\delta > 0$ such that

$$\|U(z + V)^{-1}\|_{\mathscr{L}(X)} \leq \frac{c}{(1+|z|)^\delta}\,, \quad \forall z \in \Sigma_\alpha, \quad |z| \text{ large}\,. \tag{27}$$

Assumption 3 is essential and crucial in the arguments. Taking into account Assumptions: 1-3, one recognizes that the following improvement of Favini-Yagi, Theorem 6.3, pp. 186-187, holds. Indeed, one takes

$$D(M) = D(B) \times D(B)\,, \quad M(u,v)^T = (u, Cv)^T\,,$$

$$D(L) = D(A) \times D(B)\,, \quad L(u,v)^T = (v, -Au - Bv)^T$$

and $D(B) \times X$ as a pivot space. Then, see Favini-Yagi, p. 187,

$$\|M(zM - L)^{-1}\|_{\mathscr{L}(D(B)\times X)} \leq c\,|z|^{1-(\gamma+\beta)}\,, \quad \forall z \in \Sigma_\alpha\,.$$

Now, we take B_1 as $B^{\tilde{\alpha}}_{D(B)}$ and B_2 as $B^{\tilde{\beta}}_X$, $0 < \tilde{\alpha},\ \tilde{\beta} \leq 1$. Thus

$$\left(C([0,T];D(B)),\, D(B^{\tilde{\alpha}}_{D(B)}) \right)_{\xi,\infty} = \left\{ f \in C^{\xi\tilde{\alpha}}([0,T];D(B)) :\ f(0) = 0 \right\}$$

while

$$\left(C([0,T];X),\, D(B^{\tilde{\beta}}_X) \right)_{\xi,\infty} = \left\{ f \in C^{\xi\tilde{\beta}}([0,T];X) :\ f(0) = 0 \right\}.$$

Therefore, applying Theorem 1, we get that

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}+\begin{bmatrix} 0 & -I \\ A & B \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 0 \\ f \end{bmatrix}\in$$

$$\left(C([0,T];D(B)), D(B^{\tilde{\alpha}}_{D(B)})\right)_{\theta,\infty}\times\left(C([0,T];X), D(B^{\tilde{\beta}}_{X})\right)_{\theta,\infty}$$

$= \left\{w\in C^{\theta\tilde{\alpha}}([0,T];D(B)):\ w(0)=0\right\}\times\left\{v\in C^{\theta\tilde{\beta}}([0,T];X):\ v(0)=0\right\}$ admits a unique strict solution (u,v) such that

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix},\quad \begin{bmatrix} 0 & -I \\ A & B \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}\in$$

$= \left\{w\in C^{\omega\tilde{\alpha}}([0,T];D(B)):\ w(0)=0\right\}\times\left\{v\in C^{\omega\tilde{\beta}}([0,T];X):\ v(0)=0\right\}$ provided that $\alpha+\beta+\gamma>2,\ 3-\alpha-\beta-\gamma<\theta<1,\ \omega=\theta+\alpha+\beta+\gamma-3$, i.e., $B^{\tilde{\alpha}}_{D(B)}u=v\in C^{\omega\tilde{\alpha}}([0,T];D(B)),\ v(0)=0,\ B^{\tilde{\beta}}_{X}Cv=B^{\tilde{\beta}}_{X}CB^{\tilde{\alpha}}_{D(B)}u\in C^{\omega\tilde{\beta}}([0,T];X)$,
$\left(B^{\tilde{\beta}}_{X}Cv\right)(0)=0,\ Au+BB^{\tilde{\alpha}}_{D(B)}u\in C^{\omega\tilde{\beta}}([0,T];X),\ \left(Au+BB^{\tilde{\alpha}}_{D(B)}u\right)(0)=0$. Therefore, for all $f\in C^{\theta\tilde{\beta}}([0,T];X),\ f(0)=0,\ \alpha+\beta+\gamma>2,\ 3-\alpha-\beta-\gamma<\theta<1$, the problem

$$B^{\tilde{\beta}}_{X}CB^{\tilde{\alpha}}_{D(B)}u(t)+BD^{\tilde{\alpha}}_{D(B)}u(t)+Au(t)=f(t),\quad 0\le t\le T,$$
$$u(0)=0,\quad CB^{\tilde{\alpha}}_{D(B)}u(0)=0$$

admits a unique strict solution u such that $v=B^{\tilde{\alpha}}_{D(B)}u\in C^{\omega\tilde{\alpha}}([0,T];D(B)),\ Au+BB^{\tilde{\alpha}}_{D(B)}u\in C^{\omega\tilde{\beta}}([0,T];X)$, $\left(Au+BB^{\tilde{\alpha}}_{D(B)}u\right)(0)=0,\ \left(B^{\tilde{\beta}}_{X}CB^{\tilde{\alpha}}_{D(B)}u\right)(0)=0$.

Let us consider the inverse problem to find $(u,f)\in C([0,T];D(A))\times C([0,T];\mathbb{C})$ such that

$$B^{\tilde{\beta}}_{X}CB^{\tilde{\alpha}}_{D(B)}u(t)+BD^{\tilde{\alpha}}_{D(B)}u(t)+Au(t)=f(t)z+h(t),\quad 0\le t\le T, \tag{28}$$
$$u(0)=D^{\tilde{\alpha}}_{D(B)}u(0)=0, \tag{29}$$
$$\Phi[Cu(t)]=g(t), \tag{30}$$

where z fixed in $X,\ g\in C^{\tilde{\alpha}+\tilde{\beta}}([0,T];\mathbb{C})$. Applying Φ to both sides of equation (28), we get

$$B^{\tilde{\alpha}+\tilde{\beta}}_{\mathbb{C}}g(t)+\Phi[Bv(t)]+\Phi[Au(t)]=f(t)\Phi[z]+\Phi[h(t)].$$

Assume $\Phi[z]\neq 0$ and recall that $v(t)=D^{\tilde{\alpha}}_{D(B)}u(t)$, so that necessarily

$$f(t)=(\Phi[z])^{-1}\left\{B^{\tilde{\alpha}+\tilde{\beta}}_{\mathbb{C}}g(t)+\Phi[Bv(t)]+\Phi[Au(t)]-\Phi[h(t)]\right\}. \tag{31}$$

After substituting such a value of f in (28), we obtain the direct problem

$$B^{\tilde{\beta}}_{X}CB^{\tilde{\alpha}}_{D(B)}u(t)+BD^{\tilde{\alpha}}_{D(B)}u(t)-\frac{\Phi[BD^{\tilde{\alpha}}_{D(B)}u(t)]}{\Phi[z]}z+Au(t)-\frac{\Phi[Au(t)]}{\Phi[z]}z$$
$$=h(t)-\frac{\Phi[h(t)]}{\Phi[z]}z+\frac{B^{\tilde{\alpha}+\tilde{\beta}}_{\mathbb{C}}g(t)}{\Phi[z]}z\quad 0\le t\le T,$$
$$u(0)=0=CB^{\tilde{\alpha}}_{D(B)}u(0).$$

Introducing the new variable $B^{\tilde{\alpha}}_{D(B)}u=v$, we get

$$B^{\tilde{\beta}}_{X}Cv(t)+Bv(t)+B_1v(t)+Au(t)+B_0u(t)$$
$$=h(t)-\frac{\Phi[h(t)]}{\Phi[z]}z+\frac{B^{\tilde{\alpha}+\tilde{\beta}}_{\mathbb{C}}g(t)}{\Phi[z]}z\quad 0\le t\le T,$$

where

$$D(B_1)=D(B),\quad B_1v=-\frac{\Phi[Bv]}{\Phi[z]}z,\ \ v\in D(B_1),$$

$$D(B_0)=D(A),\quad B_0u=-\frac{\Phi[Au]}{\Phi[z]}z,\ \ u\in D(B_0).$$

Therefore, we get the direct problem

$$B^{\tilde{\alpha}}_{D(B)}u(t)=v(t), \tag{32}$$

$$B^{\tilde{\beta}}_X Cv(t)+(A+B_0)u(t)+(B+B_1)v(t)=h(t)-\frac{\Phi[h(t)]}{\Phi[z]}z+\frac{B^{\tilde{\alpha}+\tilde{\beta}}_{\mathbb{C}}g(t)}{\Phi[z]}z. \tag{33}$$

We know that :

$$\|M(zM-L_1)^{-1}\|_{\mathscr{L}(D(B)\times X)}\leq c|z|^{1-(\gamma+\beta)},$$

where

$$M=\begin{bmatrix}1&0\\0&C\end{bmatrix},\ \ D(M)=D(B)\times D(B),$$

$$L_1=\begin{bmatrix}0&0\\-B_0&-B_1\end{bmatrix},\ \ D(L_1)=D(L)=D(A)\times D(B).$$

It follows that the modified resolvent for the system (32)-(33) is

$$\|M(zM-L-L_1)^{-1}\|_{\mathscr{L}(D(B)\times X)}$$

and the resolvent estimate is equal to the previous one, provided that, see [18],

$$\begin{bmatrix}0&0\\-B_0&-B_1\end{bmatrix}\in\mathscr{L}\left(D(A)\times D(B),X^{\delta}_{LM^{-1}}\right),\ \ \delta>\beta+\gamma-1,$$

where $X^{\delta}_{LM^{-1}}=\left\{(u,v)\in D(B)\times X;\ \sup\limits_{0<k<t<\infty}t^{\delta}\|L(tM-L)^{-1}(u,v)^T\|_{D(B)\times X}<\infty\right\}$

$$=\sup_{0<k<t<\infty}t^{\delta}\left\|\begin{bmatrix}0&1\\-A&-B\end{bmatrix}\left(t\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&-1\\A&B\end{bmatrix}\right)^{-1}\begin{bmatrix}u\\v\end{bmatrix}\right\|_{D(B)\times X}$$

$$=\sup_{0<k<t<\infty}t^{\delta}\left\|\begin{bmatrix}0&1\\-A&-B\end{bmatrix}\begin{bmatrix}t&-1\\A&tC+B\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}\right\|_{D(B)\times X}.$$

On the other hand, it is very difficult to characterize such a space, but we observe that $MD(L)=D(LM^{-1})\subseteq X^{\delta}_{LM^{-1}}$ and thus it suffices to suppose that

$$\begin{bmatrix}0&0\\-B_0&-B_1\end{bmatrix}\in\mathscr{L}\left(D(A)\times D(B),\overline{M(D(L))}\right).$$

Now, $M(D(L))=\begin{bmatrix}1&0\\0&C\end{bmatrix}\begin{bmatrix}D(A)\\D(B)\end{bmatrix}=D(A)\times C(D(B))$.

Since

$$B_0u+B_1v=\frac{\Phi[Au]}{\Phi[z]}z+\frac{\Phi[Bv]}{\Phi[z]}z,$$

$$\begin{bmatrix}0&0\\-B_0&-B_1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\-B_0u-B_1v\end{bmatrix}=\begin{bmatrix}0\\-\frac{\Phi[Au]}{\Phi[z]}z-\frac{\Phi[Bv]}{\Phi[z]}z\end{bmatrix}.$$

If $z=C\bar{z}$, with $\bar{z}\in D(B)$,

$$\begin{bmatrix}0\\C\left(-\frac{\Phi[Au]}{\Phi[C\bar{z}]}-\frac{\Phi[Bv]}{\Phi[C\bar{z}]}\right)\bar{z}\end{bmatrix}=\begin{bmatrix}1&0\\0&C\end{bmatrix}\begin{bmatrix}0\\\left(-\frac{\Phi[Au]}{\Phi[C\bar{z}]}-\frac{\Phi[Bv]}{\Phi[C\bar{z}]}\right)\bar{z}\end{bmatrix}.$$

Thus, we can apply the previous result assuming equations in Sections 2-4. The details are left to the interested readers.

Example 6.(see Favini-Yagi p. 203)
Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $X = L^p(\Omega)$, $1 < p < \infty$, and let Y denote the space $W^{2,p}(\Omega) \cap W_0^{1,p}$. If Δ is the Laplacian with $D(\Delta) = Y$ and m is a positive integer, let $D(\Delta^m) = \{u \in D(\Delta^{m-1});\ \Delta^{m-1}u \in Y\}$. Let m, and k be two fixed positive integers such that $k < m$ and $m+1 < 2k$. Introduce the operators A, B, C on the space X by

$$D(A) = D(\Delta^m),\ \ Au = (-1)^m\Delta^m u,\ \ u \in D(A),$$

$$D(B) = D(\Delta^k),\ \ Bu = (-1)^k\Delta^k u,\ \ u \in D(B),$$

$$D(C) = Y,\ \ Cu = (1-\Delta)u,\ \ u \in Y.$$

It is well known that $-AB^{-1} = -U$ generates an analytic semigroup in X. It is easy to verify that $-BC^{-1}$ generates another analytic semigroup in X, so that $\alpha = \beta = \gamma = 1$. One may also observe that Assumption 3 is verified with $\delta = \big(2k-(m+1)\big)(k-1)^{-1}$, $k > 1$.

6 Conclusion

In conclusion, we have shown that initial value problems for degenerate evolution equations including Caputo fractional derivative can be handled by means of a general abstract equation. Various examples of partial differential equations of interest in applied mathematics clarifying our abstract results are given.

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