Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/090111

Sequential Testing Procedure for the Parameter of Left Truncated Exponential Distribution

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Received: 3 Apr. 2019, Revised: 2 Jul. 2019, Accepted: 18 Jul. 2019
Published online: 1 Mar. 2020

Abstract: In this paper, we have explored the sequential testing procedure for the left truncated exponential distribution (LTED). Also we have obtained the expressions of operating characteristic (OC) and average sample number (ASN) functions for left truncated exponential distribution.These results are presented through tables and graphs. The sequential probability ratio test (SPRT) is used for testing the hypothesis regarding parameter.

Keywords: Left truncated exponential distribution, Sequential probability ratio test, Simulation method, Newton-Raphson method.

1 Introduction

The theory of sequential analysis was began with the work of Wald and Barnard (1945). The Sequential probability ratio test (SPRT) for testing simple null hypothesis against simple alternative was first time introduced by Wald.[1] Wald and his collaborators systematically developed the theory and methodology of sequential tests in early 1940s to reduce the numbers of sampling inspections without compromising the reliability of the termination decisions. In sequential analysis, an experimenter gathers information regarding an unknown parameter by observing random sample in successive steps. One may take one observation at a time or a few at a time, but a common characteristic among such sampling designs is that the total number of observations collected at termination is a positive integer valued random variable usually denoted by N.[2] SPRT is used to test the simple hypothesis regarding the mean of exponential distribution.[3] The problem of the testing hypothesis regarding the scale parameter of the Weibull distribution when the shape parameter is known was developed by Johnson.[4] Phatarfod used the SPRT for testing composite hypothesis for shape parameter of gamma distribution when the scale parameter is known.[5] SPRT is used to test a simple hypothesis (against a simple alternative) for the mean of an inverse Gaussian distribution, assuming the coefficient of variation (CV) to be known.[6] Sequential test is used to test simple and composite hypothesis regarding the parameters of class of distributions representing various life testing models [7] and study of robustness through sequential test for the parameters of an Inverse Gaussian distribution with known coefficient of variation.[8] SPRT is used to test the parameters of Pareto distribution[9] and the robustness behaviour of Power function distribution respectively.[10] Truncated distribution is a restricted distribution that results from controlling the domain of some other probability distribution. The occurrences are limited to values which lie above or below a given threshold or within a definite range. Truncated distributions are categorised in two types: left and right truncation each with different objectives. Truncation process is also applicable in Type I and Type II censoring i.e. time and failure respectively. However, with censored sample, the data remain incomplete. LTED is mostly used in the field of reliability and life testing. The LTED is generally applied for modeling of lifetime data, and stochastic activity network models for right truncated exponential distribution (RTED).[11] This paper proposes sequential testing procedure for the parameter of LTED. In section 2, setting of problem is described. In section 3, SPRT is used and we obtain the expressions for the OC and ASN functions by three approaches. It is illustrated with implementation of SPRT for LTED in section 4. Conclusion and applications are discussed in section 5 and section 6 respectively.

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2 Set-up of the Problem

Let the random variable X follow the exponential distribution with two parameters presented by the probability density function(pdf)

$$f(x;\theta,\lambda) = \lambda \exp -\lambda(x-\theta) \qquad ;\theta \leq x < \infty, \theta > 0, \lambda > 0 \tag{1}$$

Now the pdf of LTED with two parameters is

$$f(x;\theta,\lambda) = \lambda \exp -\lambda(x-\theta)/\exp\lambda(\theta-1) \qquad ;1 \leq x < \infty, \theta \geq 0, \lambda \geq 0 \tag{2}$$

Given a sequence of observations $X_1, X_2, ..., X_n$ from (2), we have to test the simple null hypothesis $H_0 : \lambda = \lambda_0$ against simple alternative $H_1 : \lambda = \lambda_1$ The expression for the OC and ASN functions are obtained and their behavior is studied numerically and results are presented by graphs and tables.

3 SPRT for the Parameter λ :

The SPRT for testing $H_0$is defined as follows

$$Z_i = ln[\frac{f(x_i : \lambda_1, \theta)}{f(x_i : \lambda_0, \theta}] \tag{3}$$

$$Z_i = ln(\frac{\lambda_1}{\lambda_0}) + (\lambda_1 - \lambda_0) + (\lambda_0 - \lambda_1)(x_i) \tag{4}$$

Then we have

$$e^{Z_i} = (\frac{\lambda_1}{\lambda_0})\exp -(\lambda_0 - \lambda_1)\exp(\lambda_0 - \lambda_1)(x_i) \tag{5}$$

We choose two numbers A and B such that $0 < B < 1 < A$ at the n^{th}stage, accept H_0 if $\sum_{i=1}^{n} Z_i \leq lnB$, reject H_0 if $\sum_{i=1}^{n} Z_i \geq lnA$, otherwise continue sampling by taking the $(n+1)^{th}$ observation. If$\alpha \in (0,1)$and $\beta \in (0,1)$ are probability of type I^{st} and type II^{nd} errors respectively, then according to Wald, A and B are approximately given by

$$A = \frac{1-\beta}{\alpha} \qquad and \qquad B = \frac{\beta}{1-\alpha} \tag{6}$$

The OC function of the SPRT is given by

$$L(\lambda) = \frac{A^h - 1}{A^h - B^h} \tag{7}$$

where h is the non zero solution of the equation
or, $E(e^{Z_i})^h = 1$

$$\int_1^\infty f[\frac{(x_i : \lambda_1, \theta)}{(x_i : \lambda_0, \theta)}]^h f(x_i, \lambda, \theta) dx_i = 1 \tag{8}$$

we obtain

$$E(e^{Z_i})^h = \frac{\lambda(\frac{\lambda_1}{\lambda_0})^h}{\lambda + h(\lambda_1 - \lambda_0)} \tag{9}$$

The ASN function is approximately given by

$$E(N) = \lfloor \frac{L(\lambda)(lnB) + (1 - L(\lambda))lnA}{E(Z)} \rfloor \tag{10}$$

provided $E(Z) \neq 0$,

$$E(Z) = ln(\frac{\lambda_1}{\lambda_0}) + (\lambda_1 - \lambda_0) + (\lambda_0 - \lambda_1)(1 + \frac{1}{\lambda}) \tag{11}$$

Now in order to study OC and ASN functions numerically the following methods are suggested.

(a) Simulation Method

Consider the equations (7) and (9). For any arbitrary values of 'h' the point $\lfloor \lambda, L(\lambda) \rfloor$ is computed from (7). The OC function can be graphed by plotting a sufficiently large number of points $\lfloor \lambda, L(\lambda) \rfloor$ corresponding to various values of 'h'. Remark 1: The problem of testing is considered for $H_0 : \lambda_0 = 20$ versus $H_1 : \lambda_1 = 25$ fixed $\alpha = \beta = 0.05$.For equation (7) and (9) various pairs of $\lfloor \lambda, L(\lambda) \rfloor$ are derived through varying 'h' between the intervals (-2, 2). The interval is chosen in such a ways so that the parameter λ lies within the parametric space. The results are presented in Table 1. It is marked from the table the values of $L(\lambda_0)$ and $L(\lambda_1)$ are quite close to their theoretical values i.e. 0.95 and 0.05 respectively. The OC and ASN functions are plotted in Fig 1.

(b) Approximation Method

Consider the equation (9) by taking the logarithms of both sides and solving, we get

$$\lambda(logb)^3h^2 + 3\lambda h(logb)^2 + 6\lambda(logb) - 6c = 0 \tag{12}$$

where,
$b = \frac{\lambda_1}{\lambda_0}$, $c = (\lambda_1 - \lambda_0)$ and root are given by

$$h = \frac{-3\lambda(logb)^2 \pm \sqrt{9\lambda^2(logb)^4 - 24\lambda(logb)^3(\lambda logb + c)}}{2\lambda(logb)^3} \tag{13}$$

Remark 2: For testing the hypothesis $H_0 : \lambda_0 = 20$ versus $H_1 : \lambda_1 = 25$ and fixed $\alpha = \beta = 0.05$ for different value of λ, the real roots of 'h' are obtained from (7). The OC and ASN functions are evaluated with the help of (7) and (10) respectively. It is interesting to note that the value of 'h' obtained through approximation gives satisfactory results in Table 1. The OC and ASN functions are plotted in Fig.2.

(c) Newton-Raphson Method

The values of 'h' are given by (13) as initial values for solving (9) through N-R method. To apply the N-R method for solving the (8), it has shown in (9)
$\lambda b^h = hc + \lambda \Rightarrow b^h = \frac{hc}{\lambda} + 1$
Taking logarithm in both sides we obtain
$hlogb - log(\frac{hc}{\lambda} + 1) = Fx$
$FD = log(b) - (\dfrac{c/\lambda}{1 + \frac{hb}{\lambda}})$
Where, FD (first derivative)of Fx. The ASN function is calculated by (10). It is interesting to note the value of 'h' is obtained through approximation and though N-R method found to be approximately closed in Table 1. The OC and ASN functions are plotted in Fig.3.

4 Illustration of SPRT for LTED:

The nature of SPRT in case of pdf is described as, let $X_1, X_2, X_3, ...$be independent and identically distributed random variables from LTED in equation (2), where $\lambda > 0$, to test the simple null hypothesis $H_0 : \lambda = \lambda_0$ versus the simple alternative hypothesis $H_1 : \lambda = \lambda_1 (> \lambda_0)$ having pre-assigned $0 < \alpha, \beta < 1$. Let A and B be defined $A \approx \frac{1-\beta}{\alpha}$, $B \approx \frac{\beta}{1-\alpha}$ and Z_i be defined

$$Z_i = ln[\frac{f(X_i|\lambda_1)}{f(X_i|\lambda_0)}\rfloor = ln(\frac{\lambda_1}{\lambda_0}) + (\lambda_0 - \lambda_1)x_i + (\lambda_1 - \lambda_0)] \tag{14}$$

Let $n(\geq 1) i = 1, 2, 3, ...$ the SPRT is summarized and simplified to the following.
Let $Y(n) = \sum_{i=1}^{n} X_i$ and N=first integer $n(\geq 1)$ for which inequality $Y(n) \leq c_1 + dn$ and $Y(n) \geq c_2 + dn$ holds with the constants.

$$c_1 = \frac{ln(B)}{(\lambda_1 - \lambda_0)}; \qquad c_2 = \frac{ln(A)}{(\lambda_1 - \lambda_0)}; \qquad d = \frac{\lfloor ln(\frac{\lambda_1}{\lambda_0}) + (\lambda_1 - \lambda_0) \rfloor}{(\lambda_1 - \lambda_0)} \tag{15}$$

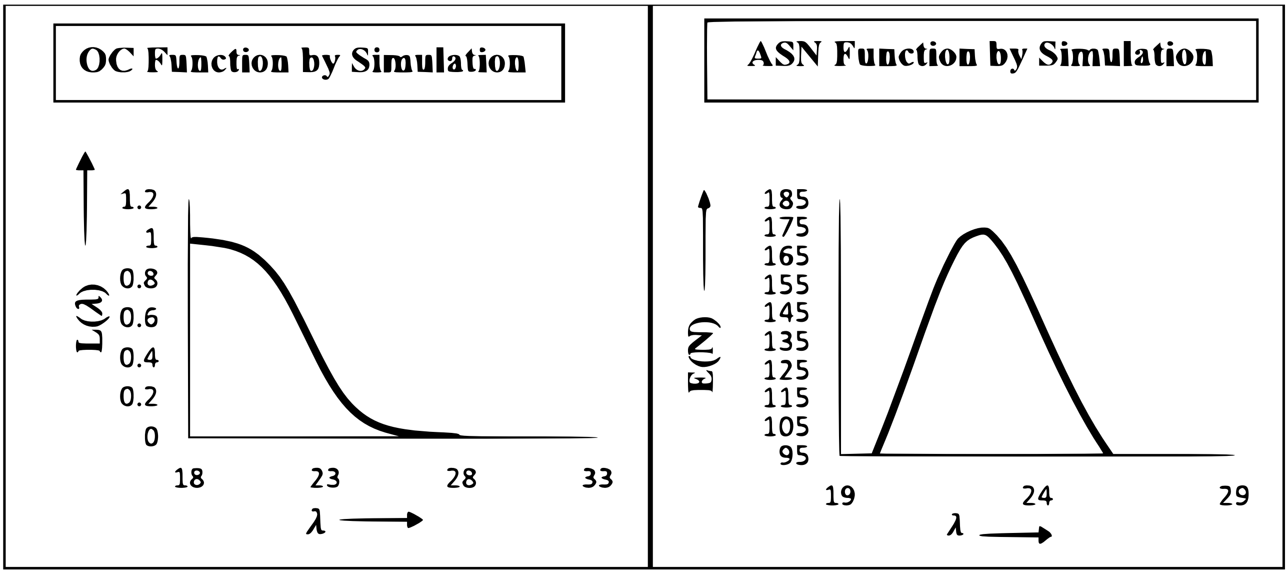

Fig: 1. Representation of OC and ASN function by simulation method.

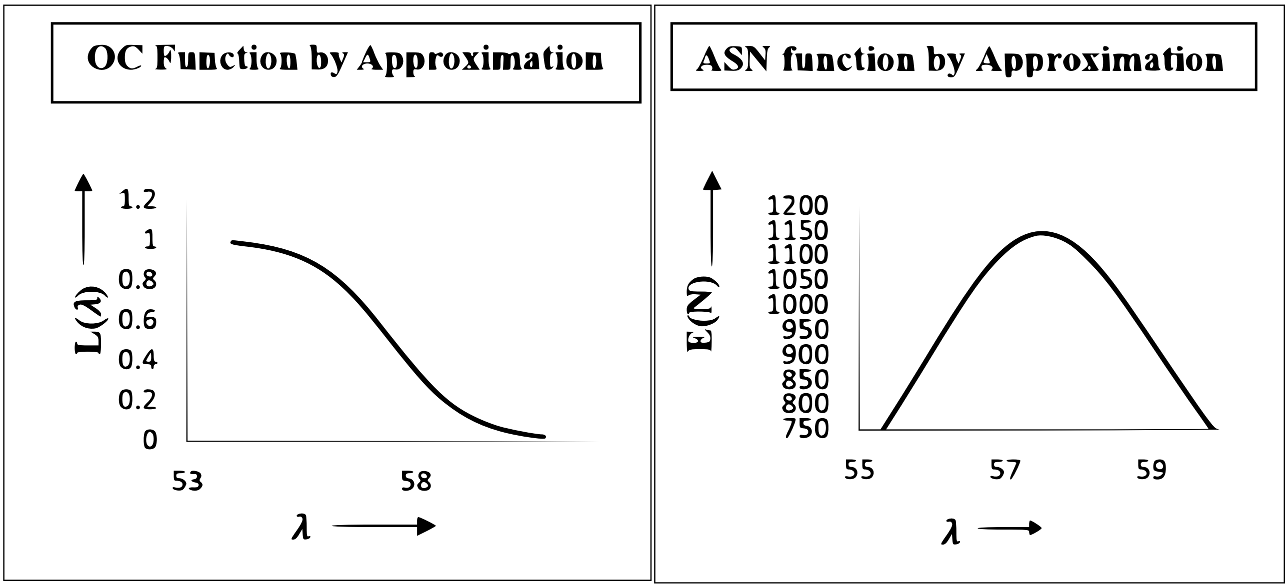

Fig: 2. Representation of OC and ASN function by approximation method

At the stopped stage, if $Y(N) \leq c_1 + dN$ we accept H_0 and if $Y(N) \geq c_1 + dN$ we reject H_0 for different values of N where A and B are the fixed quantities. The acceptance and rejection region may be for H_0 under the case when $H_0 : \lambda = \lambda_0 = 20, H_1 : \lambda = \lambda_1 = 25, b = 1, \alpha = \beta = 0.05$. The calculated values of constants are $c_1 = -0.5889, c_2 = 0.5889$ and $d = 1.0446$. If $Y(n) \leq 1.0446N + 0.5889$, we accept H_0, and if $Y(N) \geq 1.0446N - 0.5889$, we accept H_1. At the intermediate stage, we continue sampling.

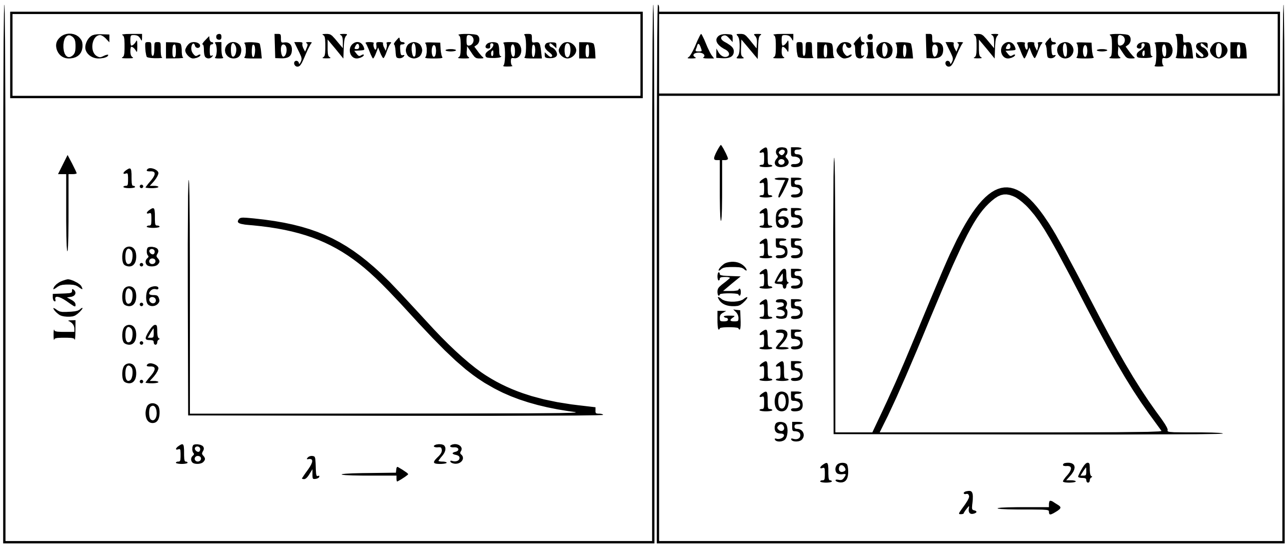

Fig. 3. Representation of OC and ASN function by Newton-Raphson method

5 Conclusion:

This study has been illustrated by three different approaches namely simulation, approximation and Newton- Raphson. It is found that all three methods are significant. The numerical values for OC and ASN functions are presented by tables and graphs. The results obtained by different methods are appropriate.

6 Application:

Sequential analysis is used to solve a wide range of practical problems from inventory, queuing theory, reliability, life tests, quality control, design of experiments and multiple comparisons. The area of clinical trials continues to be an important beneficiary of some of the basic research in sequential methodologies. The basic research in clinical trials has also enriched the area of sequential sampling deigns. This is essential in handling contemporary statistical challenges in agriculture, clinical trials, data mining, finance, gene mapping, micro-arrays, multiple comparisons.

7 Table formation

(a) Values of OC and ASN function obtained by simulation method and Newton-Raphson method for testing
$H_0 : \lambda_0 = 20$ versus $H_1 : \lambda_1 = 25$, $\alpha = \beta = 0.005$
(b) Values of OC and ASN function obtained by approximation method for testing
$H_0 : \lambda_0 = 55$ versus $H_1 : \lambda_1 = 60$, $\alpha = \beta = 0.005$

Acknowledgement

The authors would like to thank the referees and editor for their helpful comments.

Table 1: Computation and comparison of OC and ASN functions

Simulation Method				Approximation Method			Newton-Raphson Method		
h	$L(\lambda)$	λ	$E(N)$	λ	$L(\lambda)$	$E(N)$	λ	$L(\lambda)$	$E(N)$
-1.9	0.003705	27.4917	70.8158	54.2	0.98061	540.17	19.2	0.9893	0.9893
-1.8	0.004967	27.2075	74.0442	54.4	0.97535	571.24	19.4	0.9979	0.9979
-1.7	0.006656	26.9252	77.5898	54.6	0.96873	604.84	19.6	0.9788	0.9788
-1.6	0.008915	26.6446	81.4898	54.8	0.96044	641.08	19.8	0.9563	0.9563
-1.5	0.011930	26.3659	85.7844	55.0	0.95008	680.01	20.0	0.9552	0.9552
-1.4	0.015950	26.0890	90.5149	55.2	0.93723	721.57	20.2	0.9403	0.9403
-1.3	0.0212950	25.814	95.7219	55.4	0.92139	765.59	20.4	0.9240	0.9240
-1.2	0.028379	25.5408	101.4415	55.6	0.90204	811.73	20.6	0.9129	0.9129
-1.1	0.037728	25.2695	107.6994	55.8	0.87861	859.40	20.8	0.8874	0.8874
-1.0	0.051421	25.00	114.5025	56.0	0.85058	907.77	21.0	0.8850	0.8850
-0.9	0.065989	24.7324	121.8267	56.2	0.81753	955.70	21.2	0.8313	0.8313
-0.8	0.086626	24.4666	129.6012	56.4	0.77916	1001.75	21.4	0.7904	0.7904
-0.7	0.112935	24.2026	137.6892	56.6	0.73546	1044.26	21.6	0.7342	0.7342
-0.6	0.145958	23.9406	145.8697	56.8	0.68671	1081.39	21.8	0.6832	0.6832
-0.5	0.186606	23.6803	153.8217	57.0	0.63359	1111.33	22.0	0.6184	0.6184
-0.4	0.235452	23.422	161.1252	57.2	0.57715	1132.50	22.2	0.5759	0.5759
-0.3	0.292488	23.1655	167.2835	57.4	0.51876	1143.74	22.4	0.5351	0.5351
-0.2	0.35689	22.9108	171.7789	57.6	0.45999	1144.48	22.6	0.4436	0.4436
-0.1	0.426916	22.658	174.1567	57.8	0.40245	1134.86	22.8	0.3969	0.3969
0.1	0.573084	22.158	171.5841	58.0	0.34759	1115.65	23.0	0.3483	0.3483
0.2	0.64311	21.9108	166.7437	58.2	0.29660	1088.15	23.2	0.2989	0.2989
0.3	0.707512	21.6655	159.982	58.4	0.25032	1054.01	23.4	0.2563	0.2563
0.4	0.764548	21.422	151.8167	58.6	0.20920	1014.99	23.6	0.1908	0.1908
0.5	0.813395	21.1803	142.7949	58.8	0.17334	972.81	23.8	0.1785	0.1785
0.6	0.854042	20.9406	133.4129	59.0	0.14258	928.99	24.0	0.1461	0.1461
0.7	0.887065	20.7026	124.0713	59.2	0.11656	884.81	24.2	0.1212	0.1212
0.8	0.913374	20.4666	115.0581	59.4	0.09480	841.28	24.4	0.0904	0.0904
0.9	0.934011	20.2324	106.5585	59.6	0.07679	799.10	24.6	0.0782	0.0782
1	0.95013	20.0000	98.6726	59.8	0.06199	758.79	24.8	0.056	0.056
1.1	0.962272	19.7695	91.4387	60.0	0.04990	720.63	25.0	0.0464	0.0464
1.2	0.971621	19.5408	84.8529	60.2	0.04009	684.80	25.2	0.048	0.048
1.3	0.978705	19.314	78.8853	60.4	0.03216	651.32	25.4	0.0314	0.0314
1.4	0.98405	19.089	73.4916	60.6	0.02576	620.17	25.6	0.0253	0.0253
1.5	0.98807	18.8659	68.6212	60.8	0.02062	591.26	25.8	0.0204	0.0204
1.6	0.991085	18.6446	64.222	54.0	0.98478	511.50	19.0	0.9925	0.9925
1.7	0.993344	18.4252	60.2441	54.2	0.98061	540.17	19.2	0.9893	0.9893
1.8	0.995033	18.2075	56.6408	54.4	0.97535	571.24	19.4	0.9979	0.9979
1.9	0.996295	17.9917	53.3698	54.6	0.96873	604.84	19.6	0.9788	0.9788

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