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A Transmission Problem with Nonlinear Damping and Source Terms

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In this article we study the wave propagation in materials consisting of two components: one component is simple elastic while, the other has a nonlinear internal damping with elastic coefficients dependent on time. Both components have source terms. By using the potential well method we obtain the global existence. We also show that the energy of the system decays uniformly to zero.

Keywords: Stability, transmission problem, internal damping.

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1 Introduction

The main purpose of this work is to study the asymptotic behavior of solutions of the following nonlinear transmission problem

$$\begin{aligned}
&\rho_1 u_{tt} - b u_{xx} = \mu_1 f_1(u) && \text{in }]0, L_0[\times \mathbb{R}^+, && (1.1)\\
&\rho_2 v_{tt} - a(t) v_{xx} + g(v_t) = \mu_2 f_2(v) && \text{in }]L_0, L[\times \mathbb{R}^+, && (1.2)\\
&u(0,t) = 0 = v(L,t) && t > 0, && (1.3)\\
&u(L_0,t) = v(L_0,t) && t > 0, && (1.4)\\
&b u_x(L_0,t) = a(t) v_x(L_0,t) && t > 0, && (1.5)\\
&u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) && x \epsilon\,]0, L_0[\,, && (1.6)\\
&v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x) && x \epsilon\,]L_0, L[\,, && (1.7)
\end{aligned}$$

where ρ_1, ρ_2 are different densities of the material, $\mu_i \in \mathbb{R}$, $i = 1, 2$, $b > 0$, g is a non-decreasing C^1 function, a is an elastic coefficient dependent on time and f_i is a function like $-|u|^{p_i-1} u$, $p_i \geq 1$, $i = 1, 2$.

Transmission problem or diffraction problems arise in several applications in physics and biology. The stability of conservative system by means of a internal damping has been studied by many authors, see [1,9,12,14], among others. For the transmission problem there exist several works about controllability and stabilization by means of feedback functions on a part of the boundary [2,5,8,11].

When the coefficients depend on time and $f_i(s)\, s \geq 0,\ \mu_i = -1$, Muñoz Rivera and Cabanillas L. [10] showed that solutions converge to zero exponentially. In our case we have $\mu_i \epsilon \mathbb{R}$, $|f_i(s)| \leq |s|^{p_i}$, $\forall\ s\ \epsilon\ \mathbb{R}$, with non-linear damping g.

The first part of this paper is to study the global existence of regular and weak solution to problem (1.1)-(1.7), where we have some theoretical difficulties that we need over come because of that the coefficients depend on time and the source term. Semigroup arguments are not suitable for finding solutions to (1.1)-(1.7) and the method in $[10]$ does not seem to be directly applicable to the function f_i, therefore, we make use of a Galerkin approximation and the potential well method.

The second part is to give energy decay estimates of the solution of (1.1)-(1.7) for a general non-linear damping g. We found that the rate of decay of the solutions depend on behavior of the dissipative term in a neighborhood of zero, that is, for a linear dissipation we obtain exponential decay while for a polynomial dissipation we obtain polynomial decay.

In section 2, we present the notations and statement of results. In section 3, we prove solvability of (1.1)-(1.7) while section 4 deals with the asymptotic behavior of the solutions obtained in section 3.

2 Preliminaries

We denote

$$(w,z) = \int_I w(x)\, z(x)\, dx, \qquad |z|^2 = \int_I |z(x)|^2\, dx,$$

where $I =]0, L_0[$ or $]L_0, L[$ for $u's$ and $v's$ respectively.

We assume that

(A1) We take $f_i \in C^1(\mathbb{R}),\ i = 1,2,\ f_i(0) = 0,$

$$|f_i'(s)| \leq |s|^{p_i - 1} \qquad\qquad 1 \leq p_1, p_2 < \infty$$

and without lost of generality, we assume $p_1 \geq p_2$ and

$$F_i(s) = \mu_i \int_0^S f_i(\xi)\, d\xi$$

(A2) Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that

$$g(s) \cdot s > 0 \qquad \text{for all } s \neq 0,$$

and there exist $c_i > 0,\ i = 1,2,3,4$ such that

$$\begin{cases} c_3 |s|^p \le |g(s)| \le c_4 |s|^{1/p} & \text{if } |s| \le 1, \\ c_1 |s| \le |g(s)| \le c_2 |s| & \text{if } |s| > 1, \end{cases}$$

where $p \ge 1$.

A3) $a \in W^{1,1}_{Loc}(0,\infty),\ a(t) \ge a_0 > 0$, for some $a_0 > 0$.

By V we denote the Hilbert Space

$$V = \left\{(w,z) \in H^1(0,L_0) \times H^1(L_0,L) : w(0) = z(L) = 0;\ \ w(L_0) = z(L_0)\right\}.$$

By E_1 and E_2 we denote the first order energy associated to each equation

$$E_1(t,u) = \frac{\rho_1}{2}|u_t(t)|^2 + \frac{b}{2}|u_x(t)|^2 - \int_0^{L_0} F_1(u)\,dx$$

$$E_2(t,v) = \frac{\rho_2}{2}|v_t(t)|^2 + \frac{a(t)}{2}|v_x(t)|^2 - \int_{L_0}^{L} F_2(v)\,dx$$

and we define

$$J_1(u) = \frac{b}{2}|u_x|^2 - \frac{\mu_1}{p_1+1}|u|^{p_1+1}_{p_1+1},$$

$$J(u,v) = \frac{b}{2}|u_x|^2 - \int_0^{L_0} F_1(u)\,dx + \frac{a(t)}{2}|v_x|^2 - \int_{L_0}^{L} F_2(v)\,dx,$$

$$J_2(v) = \frac{a_0}{2}|v_x|^2 - \frac{\mu_2}{p_2+1}|v|^{p_2+1}_{p_2+1},$$

$$I(u,v) = b|u_x|^2 + a_0|v_x|^2 - \mu_1|u|^{p_1+1}_{p_1+1} - \mu_2|v|^{p_2+1}_{p_2+1},$$

$$E(t) \equiv E(t,u,v) = E_1(t,u) + E_2(t,v).$$

We also define the stable set as $W = \{(u,v) \in V :\ \ I(u,v) > 0\ \} \cup \{\theta\}$.

In order to show the decay property we will need the following lemma.

Lemma 2.1. [3, Lemma 9.1] *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function, and assume that there exist two constants $p > 0$ and $c > 0$ such that*

$$\int_s^{+\infty} E^{\frac{p+1}{2}}(t)\,dt \le CE(s) \qquad 0 \le s < +\infty.$$

Then, we have

$$E(t) \le CE(0)(1+t)^{-2/(p-1)} \ \textit{for all } t \ge 0 \textit{ and } p > 1,$$

$$E(t) \leq CE(0)\, e^{1-wt} \textit{ for all } t \geq 0 \;\; \textit{if} \;\; p = 1,$$

where c and w are positive constants.

3 Existence and Uniqueness of Solution

We begin this section with defining what we mean by weak solution to the system (1.1)-(1.7) .

Definition 3.1. We say that the couple $\{u, v\}$ is a weak solution of (1.1)-(1.7) if

$$\{u, v\} \in L^{\infty}(0, T; V) \cap W^{1,\infty}\left(0, T, L^2(0, L_0) \times L^2(L_0, L)\right)$$

and

$$-\rho_1 \int_0^{L_0} u^1(x)\, \varphi(x, 0)\, dx - \rho_2 \int_{L_0}^{L} v^1(x)\, \psi(x, 0)\, dx - \rho_1 \int_0^T \int_0^{L_0} u_t \varphi_t dxdt$$
$$-\rho_2 \int_0^T \int_{L_0}^{L} v_t \psi_t dxdt + b \int_0^T \int_0^{L_0} u_x \varphi_x dxdt + \int_0^T \int_0^{L_0} f_1(u)\, \varphi dxdt$$
$$+ \int_0^T a(t) \int_{L_0}^{L} v_x \psi_x dxdt + \int_0^T \int_{L_0}^{L} f_2(v)\, \psi dxdt + \int_0^T \int_{L_0}^{L} g(v_t)\, \psi dxdt = 0$$

for any $\{\varphi, \psi\} \in C^2(0, T; V)$ such that

$$\varphi(T) = \varphi_t(T) = \psi(T) = \psi_t(T) = 0.$$

In order to show the existence of strong solutions we need a regularity result for the elliptic system associated to the problem (1.1)-(1.7), whose proof can be obtained with little modifications, from the book of Ladyzhenkaya and Ural'tseva [4, theorem 16.2].

Lemma 3.2. *For any given functions $F \in L^2(0, L_0)$, $G \in L^2(L_0, L)$ there exists only one solution $\{u, v\}$ of*

$$\left| \begin{array}{ll} -bu_{xx} = F & \textit{in }]0, L_0[\,, \\ -a(t)\, v_{xx} = G & \textit{in }]L_0, L[\,, \\ u(0) = v(L) = 0, & \\ u(L_0) = v(L_0)\,, \quad bu_x(L_0) = a(t)\, v_x(L_0) & \end{array} \right.$$

with t being a fixed value in $[0, T]$ satisfying

$$u \in H^2(0, L_0) \text{ and } v \in H^2(L_0, L)\,.$$

Now we are in a position to state the global existence results.

Theorem 3.3. *Suppose that assumptions (A1)-(A4) holds. If* $\{u^0, v^0\} \in W$, $\{u^1, v^1\} \in L^2(0, L_0) \times L^2(L_0, L)$ *and*

$$\max\left\{ |\mu_1|\, b^{-1} c_*^{p_1+1} \left[\frac{2}{b} \frac{p_2+1}{p_2-1} E(0) \exp \int_0^T \frac{|a'(s)|}{a(s)} ds \right]^{(p_1-1)/2}, \right.$$
$$\left. |\mu_2|\, a_0^{-1} c_*^{p_2+1} \left[\frac{2}{a_0} \frac{p_2+1}{p_2-1} E(0) \exp \int_0^T \frac{|a'(s)|}{a(s)} ds \right]^{(p_2-1)/2} \right\} < 1, \tag{3.1}$$

where c_* *is the constant of the Sobolev´s Imbedding, then there exists a unique weak solution of* (1.1)-(1.7) *satisfying*

$$\{u, v\} \in C(0, T; V) \cap C^1\left(0, T; L^2(0, L_0) \times L^2(L_0, L)\right).$$

In addition, if $\{u^0, v^0\} \in W \cap \left(H^2(0, L_0) \times H^2(L_0, L)\right)$, $\{u^1, v^1\} \in V$ *and* (3.1) *holds, and the compatibility condition*

$$b u_x^0(L_0) = a(0)\, v_x^0(L_0) \tag{3.2}$$

is verified, then there exists a strong solution $\{u, v\}$ *satisfying*

$$\begin{aligned} \{u, v\} \quad \in \quad & C\left(0, T; H^2(0, L_0) \times H^2(L_0, L)\right) \cap C^1(0, T; V) \cap \\ & C^2\left(0, T; L^2(0, L_0) \times L^2(L_0, L)\right) \end{aligned}$$

Proof. We employ the Galerkin Method to construct a solution. Let

$$\left\{ \{\varphi^i, \psi^i\},\ \ i = 1, 2, \ldots \right\}$$

be a basis to V. We construct approximate solution

$$\{u^m(t), v^m(t)\} = \sum_{i=1}^m h_{im}(t) \{\varphi^i, \psi^i\},$$

which is determinate by the ordinary differential equations

$$\begin{aligned} &\rho_1 \left(u_{tt}^m, \varphi^i\right) + b\left(u_x^m, \varphi_x^i\right) + \left(f_1(u^m), \varphi^i\right) + \rho_2\left(v_{tt}^m, \psi^i\right) \\ &+ a(t)\left(v_x^m, \psi_x^i\right) + \left(g(v_t^m), \psi^i\right) + \left(f_2(v^m), \psi^i\right) = 0, \end{aligned} \tag{3.3}$$

where $i = 1, 2, 3, \ldots$. With the initial conditions

$$\{u^m(0), v^m(0)\} = \{u^0, v^0\},\ \ \{u_t^m(0), v_t^m(0)\} = \{u^1, v^1\}, \tag{3.4}$$

and by standard methods in differential equations we prove the existence of solutions to (3.3)-(3.4) on some interval $[0, T_m[,\ 0 < T_m \leq \infty$.

In order to extend the solution of (3.3)-(3.4) to the whole interval $[0,\infty[$ we need the priori estimate below.

Weak solutions: Multiplying (3.3) by $h'_{im}(t)$, integrating by parts and summing up on i we get

$$\frac{d}{dt}E\left(t,u^m,v^m\right)+\left(g\left(v_t^m\right),v_t^m\right)\leq\frac{|a'(t)|}{a(t)}E\left(t,u^m,v^m\right). \tag{3.5}$$

Integrating (3.5) over $]0,t[$, we find that

$$E\left(t,u^m,v^m\right)+\int_0^t\left(g(v_t^m(s)),v_t^m(s)\right)ds\leq E\left(0,u^0,v^0\right)+\int_0^t\frac{|a'(s)|}{a(s)}E(s,u^m(s),v^m(s)).$$

Employing Gronwall's lemma, from the last inequality, we see that

$$\begin{aligned}&E\left(t,u^m(t),v^m(t)\right)+\int_0^t\left(g\left(v_t^m(s)\right),v_t^m(s)\right)ds\\ \leq&E\left(0,u^0,v^0\right)\exp\left(\int_0^t\frac{|a'(s)|}{a(s)}ds\right),\quad 0\leq t\leq T.\end{aligned} \tag{3.6}$$

Now to obtain a priori estimates, we need the following result.

Lemma 3.4. *Let $\{u^m(t),v^m(t)\}$ be the solution of (3.3)-(3.4) with $\left\{u^0,v^0\right\}\in W$ and $\left\{u^1,v^1\right\}\in L^2(0,L_0)\times L^2(L_0,L)$. If*

$$\alpha=\max\left\{|u_1|\,b^{-1}c_*^{p_1+1}\left[\frac{2}{b}\left(\frac{p_2+1}{p_2-1}\right)E(0)\exp\int_0^T\frac{|a'(s)|}{a(s)}ds\right]^{(p_1-1)/2},\right.$$

$$\left.|\mu_2|\,a_0^{-1}c_*^{p_2+1}\left[\frac{2}{a_0}\left(\frac{p_2+1}{p_2-1}\right)E(0)\exp\int_0^T\frac{|a'(s)|}{a(s)}ds\right]^{(p_2-1)/2}\right\}<1,$$

Then $\{u(t),v(t)\}\in W$ on $[0,T]$, that is, for all $t\in[0,T]$

$$I\left(u^m(t),v^m(t)\right)>0.$$

Proof. Since $I\left(u^0,v^0\right)>0$, it follows from the continuity of $\{u^m(t),v^m(t)\}$ that

$$I\left(u^m(t),v^m(t)\right)\geq 0 \text{ for some interval close to } t=0. \tag{3.7}$$

Let $t_{\max}>0$ be a maximal time (possibly $t_{\max}=T_m$) such that (3.7) holds on $[0,t_{\max}[$. In order to facilitate the notation, we will omit the index m of the solution of the approximate system. Note that

$$J\left(u(t),v(t)\right)\geq\frac{b}{2}\left|u_x\right|^2+\frac{a_0}{2}\left|v_x\right|^2-\frac{\mu_1}{p_1+1}\left|u\right|_{p_1+1}^{p_1+1}-\frac{\mu_2}{p_2+1}\left|v\right|_{p_2+1}^{p_2+1}$$

$$= \frac{1}{p_2+1} I\left(u\left(t\right);v\left(t\right)\right) + \frac{b\left(p_2-1\right)}{2\left(p_2+1\right)} \left|u_x\right|^2 + \frac{a_0\left(p_2-1\right)}{2\left(p_2+1\right)} \left|v_x\right|^2$$

$$+ \frac{\mu_1\left(p_1-p_2\right)}{\left(p_1+1\right)\left(p_2+1\right)} \left|u\right|_{p_1+1}^{p_1+1}$$

$$\geq \frac{p_2-1}{2\left(p_2+1\right)} \left\{b\left|u_x\right|^2 + a_0\left|v_x\right|^2\right\}, \ \forall t \in [0, t_{\max}[\,.$$

Consequently, we get

$$\begin{aligned} b\left|u_x\right|^2 + a_0\left|v_x\right|^2 &\leq \frac{2\left(p_2+1\right)}{p_2-1} J\left(u\left(t\right), v\left(t\right)\right) \\ &\leq \frac{2\left(p_2+1\right)}{p_2-1} E\left(t,u,v\right) \\ &\leq \frac{2\left(p_2+1\right)}{p_2-1}\left[E\left(0,u^0,v^0\right)\exp\int\limits_0^T \frac{\left|a'\left(s\right)\right|}{a\left(s\right)}ds\right] \text{ on } [0,t_{\max}[\,. \end{aligned} \tag{3.8}$$

It follows from the Sobolev-Poincaré inequality and (3.8) that

$$\begin{aligned} \mu_1\left|u\right|_{p_1+1}^{p_1+1} &\leq \left|\mu_1\right| c_*^{p_1+1}\left|u_x\right|^{p_1+1} \\ &= \frac{\left|\mu_1\right|}{b} c_*^{p_1+1}\left|u_x\right|^{p_1-1}\left(b\left|u_x\right|^2\right) \\ &\leq \frac{\left|\mu_1\right|}{b} c_*^{p_1+1}\left[\frac{2}{b}\left(\frac{p_2+1}{p_2-1}\right)E\left(0\right)\exp\int_0^T \frac{\left|a'\left(s\right)\right|}{a\left(s\right)}ds\right]^{\left(p_1-1\right)/2} b\left|u_x\right|^2 \\ &< b\left|u_x\right|^2. \end{aligned} \tag{3.9}$$

Similarly

$$\begin{aligned} \mu_2\left|v\right|_{p_2+1}^{p_2+1} &\leq \frac{\left|\mu_2\right|}{a_0} c_*^{p_2+1}\left[\frac{2}{a_o}\left(\frac{p_2+1}{p_2-1}\right)E\left(0\right)\exp\int\limits_0^T \frac{\left|a'\left(s\right)\right|}{a\left(s\right)}ds\right]^{\left(p_2-1\right)/2} a_0\left|v_x\right|^2 \\ &< a_0\left|v_x\right|^2. \end{aligned} \tag{3.10}$$

Thus from (3.9) and (3.10) we obtain

$$\mu_1\left|u\right|_{p_1+1}^{p_1+1} + \mu_2\left|v\right|_{p_2+1}^{p_2+1} < b\left|u_x\right|^2 + a_0\left|v_x\right|^2. \tag{3.11}$$

Therefore we get $I\left(u\left(t\right);v\left(t\right)\right) > 0$ on $[0,t_{\max}[$, which implies that we can take $t_{\max} = T_m$. This completes the proof of Lemma 3.4. □

Lemma 3.5. *Let* $\{u,v\}$ *be as in Lemma 3.4. Then there is a certain number* η_0*,* $0<\eta_0<1$ *such that*

$$\mu_1\left|u\right|_{p_1+1}^{p_1+1} + \mu_2\left|v\right|_{p_2+1}^{p_2+1} \leq \left(1-\eta_0\right)\left[b\left|u_x\right|^2 + a_0\left|v_x\right|^2\right].$$

Proof. In Lemma 3.4, we have obtained

$$\mu_1 \left|u\right|_{p_1+1}^{p_1+1} \leq \frac{|\mu_1|}{b} c_*^{p_1+1} \left[\frac{2}{b} \frac{p_2+1}{(p_2-1)} E(0) \exp \int_0^T \frac{|a'(s)|}{a(s)} ds \right]^{(p_1-1)/2} b \left|u_x\right|^2,$$

$$\mu_2 \left|v\right|_{p_2+1}^{p_2+1} \leq \frac{|\mu_2|}{a_0} c_*^{p_2+1} \left[\frac{2}{a_0} \frac{p_2+1}{(p_2-1)} E(0) \exp \int_0^T \frac{|a'(s)|}{a(s)} ds \right]^{(p_2-1)/2} a_0 \left|v_x\right|^2.$$

From the above inequalities we get

$$\begin{aligned} \mu_1 \left|u\right|_{p_1+1}^{p_1+1} + \mu_2 \left|v\right|_{p_2+1}^{p_2+1} &\leq \alpha \left(b \left|u_x\right|^2 + a_0 \left|v_x\right|^2 \right) \\ &\leq (1-\eta_0) \left(b \left|u_x\right|^2 + a(t) \left|v_x\right|^2 \right), \end{aligned}$$

where $\eta_0 = 1 - \alpha$. This completes the proof of Lemma 3.5 □

Remark 3.1. From the proof of Lemma 3.4, we get

$$b \left|u_x\right|^2 + a(t) \left|v_x\right|^2 \leq \frac{2(p_2+1)}{p_2-1} J(u(t), v(t)).$$

Using Lemma 3.4, we can deduce a priori estimate for $\{u(t), v(t)\}$. Lemma 3.4 implies that

$$\begin{aligned} E(t, u(t), v(t)) &= \frac{1}{2} \left|u_t(t)\right|^2 + \frac{1}{2} \left|v_t(t)\right|^2 + J(u(t), v(t)) \\ &\geq \frac{1}{2} \left|u_t(t)\right|^2 + \frac{1}{2} \left|v_t(t)\right|^2 + \frac{1}{p_2+1} I(u(t), v(t)) \\ &\quad + \frac{b(p_2-1)}{2(p_2+1)} \left|u_x\right|^2 + \frac{a_0(p_2-1)}{2(p_2+1)} \left|v_x\right|^2 + \frac{\mu_1(p_1-p_2)}{(p_1+1)(p_2+1)} \left|u\right|_{p_1+1}^{p_1+1} \\ &\geq \frac{1}{2} \left|u_t(t)\right|^2 + \frac{1}{2} \left|v_t(t)\right|^2 + \frac{p_2-1}{2(p_2+1)} \left(b \left|u_x\right|^2 + a_0 \left|v_x\right|^2 \right). \end{aligned} \tag{3.12}$$

From (3.6) and (3.12), we get

$$\begin{aligned} &\frac{1}{2} \left|u_t(t)\right|^2 + \frac{1}{2} \left|v_t(t)\right|^2 + \frac{p_2-1}{2(p_2+1)} \left(b \left|u_x\right|^2 + a_0 \left|v_x\right|^2 \right) + \int_0^t \big(g(v_t(s)), v_t(s)\big) ds \\ \leq \quad &E\left(0, u^0, v^0\right) \exp \left(\int_0^T \frac{|a'(s)|}{a(s)} ds \right) \leq L_1, \end{aligned} \tag{3.13}$$

where L_1 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0, T]$.

Thus, we deduce that

$$\{u^m, v^m\} \text{ is bounded in } L^\infty(0, T; V),$$

$$\{u_t^m, v_t^m\} \text{ is bounded in } L^\infty\left(0,T;L^2\left(0,L_0\right)\times L^2\left(L_0,L\right)\right),$$

which imply that

$$\begin{aligned}&\{u^m, v^m\} \rightharpoonup \{u,v\} \quad \text{weakly } * \quad \text{in } L^\infty\left(0,T;V\right),\\ &\{u_t^m, v_t^m\} \rightharpoonup \{u_t,v_t\} \quad \text{weakly } * \quad \text{in } L^\infty\left(0,T;L^2\left(0,L_0\right)\times L^2\left(L_0,L\right)\right).\end{aligned}$$

Using Aubin-Lions compactness lemma, we find that

$$\{u^m, v^m\} \to \{u,v\} \text{ strongly in } L^2\left(0,T;L^2\left(0,L_0\right)\times L^2\left(L_0,L\right)\right),$$

and consequently

$$\begin{aligned}&u^m \to u \text{ a.e in }]0,L_0] \text{ and } f_1\left(u^m\right)\to f_1\left(u\right) \quad \text{a.e. in }]0,L_0],\\ &v^m \to v \text{ a.e in }]L_0,L] \text{ and } f_2\left(v^m\right)\to f_2\left(v\right) \quad \text{a.e. in }]L_0,L].\end{aligned}$$

Besides, from the growth condition in (A1) we have that

$$\begin{aligned}&f_1\left(u^m\right) \text{ is bounded in } L^2\left(0,T;L^2\left(0,L_0\right)\right),\\ &f_2\left(v^m\right) \text{ is bounded in } L^2\left(0,T;L^2\left(L_0,L\right)\right),\end{aligned}$$

and therefore

$$\{f_1\left(u^m\right), f_2\left(v^m\right)\} \rightharpoonup \{f_1\left(u\right), f_2\left(v\right)\} \text{ in } L^2\left(0,T;L^2\left(0,L_0\right)\times L^2\left(L_0,L\right)\right).$$

Now, we note that from (3.13) and the assumption (A2), we get

$$\int_0^t \left|g\left(v_t^m\left(s\right)\right)\right|^2 ds \le L,$$

where L is a positive constant independent of m and t.

So, we can take a subsequence, still denote by (v^m) such that

$$g\left(v_t^m\right) \to \chi \text{ weakly in } L^2\left(\,]L_0,L[\times]0,T[\,\right).$$

Returning to (3.8) and using standard arguments we can show, from the convergence above, that

$$\begin{aligned}\rho_1 u_{tt} - b u_{xx} &= \mu_1 f_1\left(u\right) \text{ in } L^2\left(0,T;H^{-1}\left(0,L_0\right)\right),\\ \rho_2 v_{tt} - a\left(t\right) v_{xx} + \chi &= \mu_2 f_2\left(v\right) \text{ in } L^2\left(0,T;H^{-1}\left(L_0,L\right)\right).\end{aligned}$$

Our goal is to prove that

$$\chi = g\left(v_t\right),$$

but this relation follows from a standard theory of monotone and hemicontinuous operators ($cf.$ [6]), so the proof is omitted. Therefore, $\{u, v\}$ satisfies (1.1)-(1.5).

Regularity of solutions: To get the regularity result, we take a basis

$$B = \left\{ \left\{ \varphi^i, \psi^i \right\},\ i \in \mathbb{N} \right\}$$

such that

$$\left\{ u^0, v^0 \right\}, \left\{ u^1, v^1 \right\} \in \text{Span} \left\{ \left\{ \varphi^0, \psi^0 \right\}, \left\{ \varphi^1, \psi^1 \right\} \right\}.$$

Let us differentiate the approximate equation and multiply by $h''_{im}(t)$. Using a similar argument as before, we obtain that

$$\begin{aligned} &\frac{d}{dt} E_2(t, u, v) + \int_{L_0}^{L} g'\left(v_t^m(x,t)\right)\left(v_{tt}^m(x,t)\right)^2 dx \\ &= \mu_1\left(f_1'(u_m)\, u_t^m, u_{tt}^m\right) + \mu_2\left(f_2'(v_m)\, v_t^m, v_{tt}^m\right) \\ &\quad - a_t(t)\left(v_x^m, v_{xtt}^m\right) + \frac{1}{2} a_t(t) \left|v_{xt}\right|^2, \end{aligned} \tag{3.14}$$

where

$$E_2(t, u, v) = \frac{\rho_1}{2}|u_{tt}|^2 + \frac{b}{2}|u_{xt}|^2 + \frac{\rho_2}{2}|v_{tt}|^2 + \frac{a(t)}{2}|v_{xt}|^2.$$

Note that

$$-a_t\left(v_x^m, v_{xtt}^m\right) = -\left(a_t\left(v_x^m, v_{xt}^m\right)\right)_t + a_{tt}\left(v_x^m, v_{xt}^m\right) + a_t\left|v_{xt}^m\right|^2, \tag{3.15}$$

$E_2(0, u^m, v^m)$ is bounded, because of our choice of the basis.

Now, from the growth condition (A1) and the Sobolev imbedding we have

$$\int_0^{L_0} f_1{}'(u^m)\, u_t^m u_{tt}^m dx \le c\left[\int_0^{L_0} |u_x^m|^2 dx\right]^{(p_1-1)/2} |u_{xt}^m|\,|u_{tt}^m|, \tag{3.16}$$

and similarly

$$\int_{L_0}^{L} f_2{}'(v^m)\, v_t^m v_{tt}^m dx \le c\left[\int_{L_0}^{L} |v_x^m|^2 dx\right]^{(p_2-1)/2} |v_{xt}^m|\,|v_{tt}^m|. \tag{3.17}$$

Taking into account the first estimate (3.13), (3.15)-(3.17), from (3.14) and the Gronwall inequality, we conclude that

$$E_2(t, u^m, v^m) \le c, \tag{3.18}$$

which implies that

$$\{u_t^m, v_t^m\} \rightharpoonup \{u_t, v_t\} \text{ weakly * in } L^\infty\left(0, T; H^1(0, L_0) \times H^1(L_0, L)\right),$$

$$\{u_{tt}^m, v_{tt}^m\} \rightharpoonup \{u_t, v_t\} \text{ weakly * in } L^\infty\left(0, T; L^2(0, L_0) \times L^2(L_0, L)\right).$$

Therefore we have $\{u, v\}$ satisfies (1.1)-(1.5) and we have

$$\left| \begin{array}{ll} -bu_{xx} = -\rho_1 u_{tt} + \mu_1 f_1(u) & \in \quad L^2(0, L_0), \\ -a(t) v_{xx} = -\rho_2 v_{tt} - g(v_t) + \mu_2 f_2(v) & \in \quad L^2(L_0, L), \\ u(L_0, t) = v(L_0, t), \ bu_x(L_0, t) = a(t) v_x(L_0, t), & \\ u(0, t) = 0 = v(L, t). & \end{array} \right.$$

Then using Lemma 3.2 we have the required regularity for $\{u, v\}$. □

4 Exponential Decay

In this section we study the asymptotic behavior of the solution of system (1.1)-(1.7). In the remainder of this paper we denote by c a positive constant which takes different values in different places. We shall suppose that

$$\rho_1 \leq \rho_2 \tag{4.1}$$

and

$$a(t) \leq b, \quad a_t(t) \leq 0, \ \forall t \in]0, \infty[. \tag{4.2}$$

Let f_i, $i = 1, 2$ be such that

$$0 \leq F_1(s) \leq \frac{|\mu_1|}{p_1 + 1} s f_1(s),$$

$$0 \leq F_2(s) \leq \frac{|\mu_2|}{p_2 + 1} s f_2(s),$$

$$F_1(s) \leq F_2(s).$$

Note that odd polynomials satisfy the above inequalities.

Theorem 4.1. *Let $\{u, v\}$ be the weak solution obtained in Theorem 3.3. Suppose that (4.1)-(4.2) and (A2) hold with $p = 1$. If, in addition, the initial data satisfy*

$$v_x^0(L_0) = 0, \tag{4.3}$$

then there exists positive constants γ and c such that

$$E(t) \leq cE(0) e^{-\gamma t}, \ \forall t \geq 0. \tag{4.4}$$

We shall prove this theorem for strong solutions, our conclusion follow by standard density arguments.

The dissipative property of system (1.1)-(1.7) is given by the following lemma.

Lemma 4.2. *The first order energy satisfies*

$$\frac{d}{dt}E_1\left(t,u,v\right)=-\left(g\left(v_t\right),v_t\right)+a_t\left|v_x\right|^2. \tag{4.5}$$

Proof. Multiplying equation (1.1) by u_t, equation (1.2) by v_t and performing an integration by parts we get the result. □

Let $\psi\in C_0^\infty\left(0,L\right)$ be such that $\psi=1$ in $]L_0-\delta,L_0+\delta[$ for some small constant $\delta>0$. Let us introduce the following functional

$$I\left(t\right)=\int_0^{L_0}\rho_1u_tqu_xdx+\int_{L_0}^{L}\rho_2v_t\psi qv_xdx,$$

where $q\left(x\right)=x$.

Lemma 4.3. *There exists* c_1 *such that*

$$\begin{aligned}\frac{d}{dt}I\left(t\right)\le&-\frac{L_0}{2}\left\{\left(\rho_2-\rho_1\right)v_t^2\left(L_0,t\right)+a\left(t\right)\left[1-\frac{a\left(t\right)}{b}\right]\right\}\\&-L_0\left[F_2\left(v\left(L_0,t\right)\right)-F_1\left(u\left(L_0,t\right)\right)\right]-\frac{1}{2}\int_0^{L_0}\left(\rho_1u_t^2+bu_x^2-2F_1\left(u\right)\right)dx\\&+c_1\left\{\int_{L_0+\delta}^{L}\left(v_t^2+v_x^2\right)dx+\int_{L_0}^{L}\left(v^2+g\left(v_t\right)^2\right)dx\right\}+\varepsilon E\left(t,u,v\right).\end{aligned}$$

Proof. Multiplying equation (1.1) by qu_x, equation (1.2) by ψqv_x, integrating by parts and using the corresponding boundary conditions, we have

$$\begin{aligned}\frac{d}{dt}\left(\rho_1u_t,qu_x\right)=&\frac{L_0}{2}\left[\rho_1u_t^2\left(L_0,t\right)+bu_x^2\left(L_0,t\right)\right]\\&+L_0F_1\left(u\left(L_0,t\right)\right)-\frac{1}{2}\int_0^{L_0}\left[\rho_1u_t^2+bu_x^2.-2F_1\left(u\right)\right]dx,\end{aligned}\tag{4.6}$$

$$\begin{aligned}\frac{d}{dx}\left(\rho_2v_t,q\psi v_x\right)\le&-\frac{L_0}{2}\left[\rho_2v_t^2\left(L_0,t\right)+a\left(t\right)v_x^2\left(L_0,t\right)\right]\\&-L_0F_2\left(v\left(L_0,t\right)\right)-\frac{a\left(t\right)}{4}\int_{L_0}^{L_0+\delta}v_x^2dx\\&+c_1\left[\int_{L_0+\delta}^{L}\left(v_t^2+v_x^2\right)dx+\int_{L_0}^{L}\left(g\left(v_t\right)^2+\left|F_2\left(v\right)\right|\right)dx\right].\end{aligned}\tag{4.7}$$

Adding (4.6) and (4.7), we get

$$\frac{d}{dt}I\left(t\right)\le-\frac{L_0}{2}\left[\left(\rho_2-\rho_1\right)v_t^2\left(L_0,t\right)+a\left(t\right)v_x^2\left(L_0,t\right)-bu_x^2\left(L_0,t\right)\right]$$

$$-L_0\left[F_2\left(v\left(L_0,t\right)\right)-F_1\left(u\left(L_0,t\right)\right)\right]-\frac{1}{2}\int\limits_0^{L_0}\left(\rho_1 u_t^2+bu_x^2-2F_1\left(u\right)\right)dx$$

$$-\frac{a\left(t\right)}{4}\int\limits_{L_0}^{L_0+\delta}v_x^2dx+c_1\left[\int\limits_{L_0+\delta}^{L}\left(v_t^2+v_x^2\right)dx+\int\limits_{L_0}^{L}\left(g\left(v_t\right)^2+\left|F_2\left(v\right)\right|\right)dx\right]. \quad (4.8)$$

According to (A.1), we have that

$$\left|F_2\left(s\right)\right|\leq c\left|s\right|^{p_2+1}\leq c\left|s\right|^{2p_2}. \quad (4.9)$$

Now, applying the interpolation inequality

$$\left|z\right|_p\leq\left|z\right|_2^{\alpha}\ \left|z\right|_q^{1-\alpha},\qquad \frac{1}{p}=\frac{\alpha}{2}+\frac{(1-\alpha)}{q},\ \ \alpha\in[0,1],$$

and the inmersion $H^1\left(L_0,L\right)\hookrightarrow L^{2(2p_2-1)}\left(L_0,L\right)$, we get, for all $t\geq 0$,

$$\left|v\left(t\right)\right|_{2p_2}^{2p_2}\leq c_\varepsilon E\left(0\right)^{2(p_2-1)}\left|v\left(t\right)\right|_2^2+\frac{\varepsilon}{E\left(0\right)^{2(p_2-1)}}\left|v_x\left(t\right)\right|_2^{2(2p_2-1)},\ \text{for all }\varepsilon>0.$$

Considering inequality (3.8) we infer that

$$\left|v_x\left(t\right)\right|_2^2\leq cE\left(0\right),$$

then

$$\left|v\left(t\right)\right|_{2p_2}^{2p_2}\leq c_\varepsilon E\left(0\right)^{2(p_2-1)}\left|v\left(t\right)\right|_2^2+\varepsilon E\left(t,u,v\right). \quad (4.10)$$

From $(4.8)-(4.10)$, our conclusion follows. □

Let $\varphi\epsilon C^{\infty}\left(\mathbb{R}\right)$ be a non-negative function such that $\varphi=0$ in $I_{\delta/2}=\left]L_0-\delta/2,L_0+\delta/2\right[$ and $\varphi=1$ in $\mathbb{R}\backslash I_\delta$ and consider the functional

$$J\left(t\right)=\int\limits_{L_0}^{L}\rho_2 v_t\varphi v dx,$$

we have the following estimate.

Lemma 4.4. *Given $\varepsilon>0$, there exists a positive constant c_ε such that*

$$\frac{d}{dt}J\left(t\right)\leq-\frac{a\left(t\right)}{2}\int\limits_{L_0+\delta}^{L}v_x^2dx+\varepsilon\left[a\left(t\right)\int\limits_{L}^{L_0+\delta}v_x^2dx+E\left(t,u,v\right)\right]$$
$$+c_\varepsilon\int\limits_{L_0}^{L}\left(v_t^2+g\left(v_t\right)^2+v^2\right)dx.$$

Proof. Multiplying equation (1.2) by φv and integrating by parts, we get

$$\frac{d}{dt}J(t) = -a(t)(v_x,\varphi v_x) - a(t)(v_x,\varphi_x v) + (v_t,\varphi v_t) - (g(v_t),\varphi v) + \mu_2 (f_2(v),\varphi v).$$

Applying Young's inequality and hypothesis (A.1) and (4.10), our conclusion follows. □

Let us consider the following functional

$$K(t) = I(t) + (2c_1+1) J(t).$$

Using Lemma 4.3 and fixing $\varepsilon = \varepsilon_1$ in Lemma 4.4, where ε_1 is the solution of the equation

$$(2c_1+1)\varepsilon_1 = \frac{1}{8},$$

we have that there exists a positive constant c_2 such that

$$\frac{d}{dt}K(t) \le -E_1(t,u) - \frac{1}{8}a(t)\int_{L_0}^{L} v_x^2 dx + \varepsilon E(t,u,v) + c_2\left(\int_{L_0}^{L}\left(v_t^2 + g(v_t)^2 + v^2\right)dx\right). \tag{4.11}$$

Now in order to estimate the last term of (4.11) we need the following result.

Lemma 4.5. *Let $\{u,v\}$ be a solution in Theorem 4.1. Then there exist $T_0>0$ such that if $T \ge T_0$ we have*

$$\int_S^T |v|^2 ds \le \varepsilon \int_S^T |v_x|^2 ds + +c_\varepsilon \int_S^T \left(|v_t|^2 + |g(v_t)|^2\right) ds \tag{4.12}$$

for any $\varepsilon>0$ and c_ε is a constant depending on T but independent of $\{u,v\}$ and $0<S<T<+\infty$.

Proof. Let suppose that (4.12) does not hold, then there would exist a sequence of solutions $\{u^\nu, v^\nu\}$ such that

$$\int_S^T |v^\nu|^2 ds \ge \gamma \int_S^T \left(|v_t^\nu|^2 + |g(v_t^\nu)|^2 ds\right) + c_0 \int_S^T |v_x^\nu|^2 ds, \tag{4.13}$$

for some $C_0>0$.

We observe that in our work, in view of (3.1), the energy of the initial data $\{\{u^\nu(0), v^\nu(0)\}, \{u_t^\nu(0), v_t^\nu(0)\}\}$, denoted by $E^\nu(0)$, remains uniformly bounded in ν, that is, there exists $M>0$ such that $E^v(0) \le M, \forall v \in \mathbb{N}$.

Consequently, we have $E^{v}(t) \equiv E(t, u^{\nu}, v^{\nu}) \leq M$, $\forall \nu \in \mathbb{N}$, since E^{ν} is a non-increasing function. Then, there exists a subsequence $\{u^{\nu}, v^{\nu}\}$ which we still denote in the same way, such that

$$\begin{aligned} u^{\nu} &\rightharpoonup u \text{ weak star in } L^{\infty}\left(0, T; H^{1}(0, L_0)\right), \\ u_t^{\nu} &\rightharpoonup u_t \text{ weak star in } L^{\infty}\left(0, T; L^{2}(0, L_0)\right), \\ v^{\nu} &\rightharpoonup v \text{ weak star in } L^{\infty}\left(0, T; H^{2}(L_0, L)\right), \\ v_t^{\nu} &\rightharpoonup v_t \text{ weak star in } L^{\infty}\left(0, T; L^{2}(L_0, L)\right). \end{aligned}$$

Applying compactness results we deduce that

$$u^{\nu} \rightarrow u \text{ strongly in } L^{2}\left(0, T; L^{2}(0, L_0)\right), \tag{4.14}$$

$$v^{\nu} \rightarrow v \text{ strongly in } L^{2}\left(0, T; L^{2}(L_0, L)\right). \tag{4.15}$$

According to (4.14)-(4.15), we have that

$$f_1(u^{\nu}) \rightarrow f_1(u) \quad \text{a.e. in }]0, L_0[\times]0, T[,$$

$$f_2(v^{\nu}) \rightarrow f_2(v) \quad \text{a.e. in }]L_0, L[\times]0, T[.$$

From the above convergence and since the sequence $\{f_1(u^{\nu}),\ f_2(v^{\nu})\}$ is bounded in $L^{2}\left(0, T; L^{2}(0, L_0) \times L^{2}(L_0, L)\right)$ we conclude by Lion's Lemma that

$$\{f_1(u^{\nu}), f_2(v^{\nu})\} \rightharpoonup \{f_1(u), f_2(v)\} \text{ weakly in } L^{2}\left(0, T; L^{2}(0, L_0) \times L^{2}(L_0, L)\right). \tag{4.16}$$

The term $\int_S^T |v^{\nu}|^2\, ds$ is bounded since $E^{\nu}(t) \leq M$, $\forall \nu \in \mathbb{N}$, $\forall t \geq 0$ and $|v^{\nu}(t)|^2 \leq cE^{\nu}(t)$, where c is a positive constant independent of ν and t. Then, from (4.13) the term

$$\int_S^T \left(|v_t^{\nu}|^2 + |g(v_t^{\nu})|^2\right) ds \rightarrow 0 \text{ as } \nu \rightarrow +\infty.$$

Particularly, it comes that

$$\int_S^T |g(v_t^{\nu})|^2\, ds \rightarrow 0 \text{ as } \nu \rightarrow +\infty.$$

As S is chosen in the interval $[0, T]$, we can write

$$\lim_{\nu \rightarrow +\infty} \int_S^T |g(v_t^{\nu})|^2\, ds = 0.$$

Therefore

$$g(v_t^{\nu}) \rightarrow 0 \text{ strongly in } L^{2}\left(0, T; L^{2}(L_0, L)\right). \tag{4.17}$$

Using analogous arguments we get from (4.13) that

$$v_t^\nu \to 0 \text{ strongly in } L^2\left(0,T;L^2\left(L_0,L\right)\right). \tag{4.18}$$

Besides, from the uniqueness of the limit we conclude that

$$v_t\left(x,t\right)=0,$$

and therefore

$$v\left(x,t\right)=\varphi\left(x\right).$$

Note that $\{u^\nu, v^\nu\}$ satisfies

$$\left|\begin{array}{ll} \rho_1 u_{tt}^\nu - b u_{xx}^\nu = \mu_1 f_1\left(u^\nu\right) & \text{in }]0,L_0[\times]0,T[, \\ \rho_2 v_{tt}^\nu - a\left(t\right) v_{xx}^\nu + g\left(v_t^\nu\right) = \mu_2 f_2\left(v^\nu\right) & \text{in }]L_0,L[\times]0,T[, \\ u^\nu\left(0,t\right)=0=v^\nu\left(L,t\right), & t>0, \\ u^\nu\left(L_0,t\right)=v^\nu\left(L_0,t\right), & \\ b u_x^\nu\left(L_0,t\right)=a\left(t\right) v_x^\nu\left(L_0,t\right), & \\ u^\nu\left(x,0\right)=u^{\nu,0}\left(x\right), \qquad u_t^\nu\left(x,0\right)=u^{\nu,1}\left(x\right), & \\ v^\nu\left(x,0\right)=v^{\nu,0}\left(x\right), \qquad v_t^\nu\left(x,0\right)=v^{\nu,1}\left(x\right). & \end{array}\right. \tag{4.19}$$

Taking limit in (4.19) as $\nu\to+\infty$, we get, for $\{u,v\}$,

$$\left|\begin{array}{ll} \rho_1 u_{tt} - b u_{xx} = \mu_1 f_1\left(u\right) & \text{in }]0,L_0[\times]0,T[, \\ -a\left(t\right) v_{xx} = \mu_2 f_2\left(v\right) & \text{in }]L_0,L[\times]0,T[, \\ u\left(0,t\right)=0=v\left(L,t\right), & \\ b u_x\left(L_0,t\right)=a\left(t\right) v_x\left(L_0,t\right), & \\ v_t\left(x,t\right)=0 & \text{in }]L_0,L[\times]0,T[, \end{array}\right. \tag{4.20}$$

and for $y=u_t$,

$$\left|\begin{array}{ll} \rho_1 y_{tt} - b y_{xx} = \mu_1 f_1'\left(u\right) y & \text{in }]0,L_0[\times]0,T[, \\ y\left(0,t\right)=0=y\left(L_0,t\right), & \\ b y_x\left(L_0,t\right)=a'\left(t\right) v_x\left(L_0,t\right). & \end{array}\right. \tag{4.21}$$

Here, we observe that

$$\frac{u_{xt}\left(L_0,t\right)}{u_x\left(L_0,t\right)}=\frac{a'\left(t\right)}{a\left(t\right)},$$

and we get after an integration

$$u_x\left(L_0,t\right)=k\,a\left(t\right), \qquad k \text{ is a constant.}$$

But, using the hypothesis we obtain

$$0=\lim_{t\to0^+} u_x\left(L_0,t\right)=k\,a\left(0\right).$$

Consequently $k = 0$ and $u_x(L_0, t) = 0$. Thus, the function y satisfies

$$\left| \begin{array}{ll} \rho_1 y_{tt} - b y_{xx} = \mu_1 f_1'(u)\, y & \text{in} \quad]0, L_0[\times]0, T[\,, \\ y(0,t) = 0 = y(L_0, t) & \text{on} \quad]0, T[\,, \\ y_x(L_0, t) = 0 & \text{on} \quad]0, T[\,. \end{array} \right.$$

Then, using the results of [6] (based on Ruiz arguments [13]) adapted to our case we conclude that $y = 0$, that is $u_t(x,t) = 0$, for T suitable big.

Returning to (4.20) we obtain the following elliptic system

$$\left| \begin{array}{r} -b u_{xx} = \mu_1 f_1(u)\,, \\ -a(t)\, v_{xx} = \mu_2 f_2(v)\,. \end{array} \right.$$

Multiplying by u and v respectively and integrating, then summing up we arrive at

$$b \int_0^{L_0} u_x^2 dx + a(t) \int_{L_0}^{L} v_x^2 dx = \mu_1 \int_0^{L_0} f_1(u)\, u dx + \mu_2 \int_{L_0}^{L} f_2(v)\, v dx.$$

Hence

$$b\, |u_x|^2 + a_0\, |v_x|^2 \leq \mu_1\, |u|_{p_1+1}^{p_1+1} + \mu_2\, |v|_{p_2+1}^{p_2+1}\,.$$

But, this contradicts the Lemma 3.5, if $v \neq 0$. Similarly, if $u \neq 0$ we can obtain a contradiction.

Let us assume that $u = 0$, $v = 0$. Defining

$$\begin{gathered} \lambda_\nu^2 = \textstyle\int_S^T |v_\nu|^2\, ds, \quad w^\nu(x,t) = \dfrac{u^\nu(x,t)}{\lambda_\nu}, \\ z^\nu(x,t) = \dfrac{v^\nu(x,t)}{\lambda_\nu}, \quad 0 \leq t \leq T, \end{gathered} \tag{4.22}$$

we have that $\lambda_\nu \to 0$ and

$$\int_S^T |z_\nu|^2\, ds = 1. \tag{4.23}$$

We also have that

$$\begin{aligned} \tilde{E}^{\nu}(t) = E(t, w^\nu, z^\nu) &\leq \frac{1}{2}\, |w_t^\nu(t)|^2 + \frac{b}{2}\, |w_x^\nu(t)|^2 + \frac{1}{2}\, |z_t^\nu(t)|^2 + \frac{a(t)}{2}\, |z_x^\nu(t)|^2 \\ &= \frac{1}{2\lambda_\nu^2} \left\{ |u_t(t)|^2 + b\, |u_x^\nu(t)|^2 + |v_t^\nu(t)|^2 + a(t)\, |v_x^\nu(t)|^2 \right\}. \end{aligned}$$

From Remark 3.1 we deduce that

$$\tilde{E}^{\nu}(t) \leq \frac{1}{\lambda_\nu^2} \left(\frac{p_2+1}{p_2-1} \right) E^\nu(t)\,. \tag{4.24}$$

Also

$$\widetilde{E}^{\nu}(t) \geq \frac{1}{2}\left\{|w_t^{\nu}(t)|^2 + |z_t^{\nu}(t)|^2 + \frac{p_2-1}{p_2+1}\left(b\,|w_x^{\nu}|^2 + a(t)\,|z_x^{\nu}|^2\right)\right\}$$
$$\geq \frac{1}{\lambda_{\nu}^2}\left(\frac{p_2-1}{p_2+1}\right)E^{\nu}(t). \tag{4.25}$$

On the other hand, applying inequality (4.11) to the solutions $\{u^{\nu}, v^{\nu}\}$ we have

$$\frac{d}{dt}K^{\nu}(t) \leq -\delta_0 E(t, u^{\nu}, v^{\nu}) + c_3 \int_{L_0}^{L}\left(v_t^{\nu\,2} + g(v_x^{\nu})^2 + v^{\nu\,2}\right)dx.$$

Then integrating over $[S, T]$, we obtain

$$K^{\nu}(T) + \delta_0 \int_S^T E(t, u^{\nu}, v^{\nu})\,dt \leq K^{\nu}(S) + c_3 \int_S^T \left(|v_t^{\nu}|^2 + |g(v_t^{\nu})|^2 + |v^{\nu}|^2\right)dt.$$

Since K^{ν} satisfies

$$c_0 E(t, u^{\nu}, v^{\nu}) \leq K^{\nu}(t) \leq c_1 E(t, u^{\nu}, v^{\nu})$$

and that E is a decreasing function, we get

$$E(T, u^{\nu}, v^{\nu}) + \left(\delta_0' - \frac{c_1'}{T}\right)\int_S^T E(t, u^{\nu}, v^{\nu})\,dt$$
$$\leq c_3' \int_S^T \left(|v_t^{\nu}|^2 + |g(v_t^{\nu})|^2 + |v^{\nu}|^2\right)dt. \tag{4.26}$$

Dividing both sides of (4.26) by λ_{ν}^2, applying inequalities (4.24), (4.25), (4.13) and taking T large enough, we conclude that $E(T, w^{\nu}, z^{\nu})$ is bounded.

Integrating (4.5) over $[S, T]$, we obtain

$$E^{\nu}(t) = E^{\nu}(T) + \int_t^T (g(v_t^{\nu}), v_t^{\nu})\,dt - \int_t^T a_t\,|v_x^{\nu}|^2\,dt$$
$$\leq E^{\nu}(T) + c\int_S^T \left(|g(v_t^{\nu})|^2 + |v_t^{\nu}|^2 + |v_x^{\nu}|^2\right)dt.$$

Dividing both sides of this inequality by λ_{ν}^2, we have that, for every $t \in [S, T]$ with $0 \leq S < T < +\infty$,

$$\frac{E^{\nu}(t)}{\lambda_{\nu}^2} \leq \left(\frac{p_2+1}{p_2-1}\right)E(T, w^{\nu}, z^{\nu}) + \frac{c}{\lambda_{\nu}^2}\int_S^T \left(|g(v_t^{\nu})|^2 + |v_t^{\nu}|^2 + |v_x^{\nu}|^2\right)dt.$$

From (4.13) we deduce that

$$\lim_{\nu\to+\infty} \frac{1}{\lambda_{\nu}^2}\int_S^T \left(|g(v_t^{\nu})|^2 + |v_t^{\nu}|^2 + |v_x^{\nu}|^2\right)dt = 0, \tag{4.27}$$

and, consequently, there exists $M > 0$ such that

$$\frac{E^{\nu}(t)}{\lambda_{\nu}^{2}} \leq M$$

for all $t \in [S,T]$ and $\nu \in \mathbb{N}$. From (4.24) we find that

$$\tilde{E}^{\nu}(t) \leq c \tag{4.28}$$

for all $t \in [S,T]$, $0 \leq S < T < +\infty$, and $\nu \in \mathbb{N}$.

Then in particular, for a subsequence $\{w^{\nu}, z^{\nu}\}$, we obtain

$$\begin{aligned} w^{\nu} &\rightharpoonup w \ \text{ weak star in } L^{\infty}\left(0,T;H^{1}(0,L_0)\right), \\ w_t^{\nu} &\rightharpoonup w_t \ \text{ weak star in } L^{\infty}\left(0,T;L^{2}(0,L_0)\right), \\ z^{\nu} &\rightharpoonup z \ \text{ weak star in } L^{\infty}\left(0,T;H^{1}(L_0,L)\right), \\ z_t^{\nu} &\rightharpoonup z_t \ \text{ weak star in } L^{\infty}\left(0,T;L^{2}(L_0,L)\right), \\ w^{\nu} &\to w \ \text{ strongly in } L^{2}\left(0,T;L^{2}(0,L_0)\right), \\ z^{\nu} &\to z \ \text{ strongly in } L^{2}\left(0,T;L^{2}(L_0,L)\right). \end{aligned}$$

In addition, $\{w^{\nu}, z^{\nu}\}$ satisfies

$$\left|\begin{array}{ll} \rho_1 w_{tt}^{\nu} - b w_{xx}^{\nu} = \dfrac{\mu_1}{\lambda_{\nu}} f_1(u^{\nu}) & \text{in } \ ]0,L_0[\times]0,T[, \\ \rho_2 z_{tt}^{\nu} - a(t) z_{xx}^{\nu} + \dfrac{1}{\lambda_{\nu}} g(v_t^{\nu}) = \dfrac{\mu_2}{\lambda_{\nu}} f_2(v^{\nu}) & \text{in } \ ]L_0,L[\times]0,T[, \\ w^{\nu}(0,t) = 0 = z^{\nu}(L,T), & t>0, \\ b w_x^{\nu}(L_0,t) = a(t) z_x^{\nu}(L_0,T), & t>0. \end{array}\right. \tag{4.29}$$

From (4.27) we get that

$$\lim_{\nu\to+\infty} \int_S^T \left|\frac{g(v_t^{\nu})}{\lambda_{\nu}}\right|^2 dt = 0 \qquad \text{and} \qquad \lim_{\nu\to+\infty} \int_S^T \left|z_t^{\nu}\right|^2 dt = 0.$$

Then, in particular, for $S = 0$, we obtain

$$\frac{g(v_t^{\nu})}{\lambda_{\nu}} \to 0 \quad \text{in } \ L^2\left(0,T;L^2(L_0,L)\right) \ \text{ as } \nu \to +\infty. \tag{4.30}$$

In addition

$$z_t^{\nu} \to 0 \qquad \text{in } \ L^2\left(0,T;L^2(L_0,L)\right) \ \text{ as } \nu \to +\infty,$$

$$\begin{aligned} \int_0^T \int_0^{L_0} \left(\frac{f_1(u^{\nu})}{\lambda_{\nu}}\right)^2 dx\, dt &\leq \int_0^T \int_0^{L_0} (w^{\nu})^2 (u^{\nu})^{2(p_1-1)}\, dx\, dt \\ &= \int_0^T \int_{|u^{\gamma}|\leq\varepsilon} (w^{\nu})^2 (u^{\nu})^{2(p_1-1)}\, dx\, dt \end{aligned} \tag{4.31}$$

$$+\int_0^T \int_{|u^\gamma|>\varepsilon} (w^\nu)^2 (u^\nu)^{2(p_1-1)}\, dx\, dt.$$

Since the function $F(s) = |s|^{p_i-1}$ is continuous in $\mathbb{R}$ and $M_\varepsilon = \sup_{|s|\leq\varepsilon} |F(s)|$ is well defined, from (4.31) we get

$$\int_0^T \int_0^{L_0} \left(\frac{f_1(u^\nu)}{\lambda_\nu}\right)^2 dxdt \leq M_\varepsilon^2 |w^\nu|^2_{L^2(Q_0)} + \lambda_\nu^{2(p_1-1)} |w^\nu|^{2p_1}_{L^{2p_1}(Q_0)},$$

where $Q_0 =]0, L_0[\times]0, T[$.

From (4.28), $\{w^\nu\}$ is bounded in

$$L^\infty\left(0, T; H^1(0, L_0)\right) \hookrightarrow L^\infty\left(0, T; L^{2p_1}(0, L_0)\right)$$

and, consequently, there exists $B > 0$ such that

$$\int_0^T \int_0^{L_0} \left(\frac{f_1(u^\nu)}{\lambda_\nu}\right)^2 dxdt \leq B\left[M_\varepsilon^2 + \lambda_\nu^{2(p_1-1)}\right].$$

Then, taking $\varepsilon \to 0$ and $\nu \to +\infty$ we conclude that

$$\frac{f_1(u^\nu)}{\lambda_\nu} \to 0 \text{ in } L^2\left(0, T; L^2(0, L_0)\right) \text{ as } \nu \to +\infty. \tag{4.32}$$

The same argument shows that

$$\frac{f_2(v^\nu)}{\lambda_\nu} \to 0 \text{ in } L^2\left(0, T; L^2(L_0, L)\right) \text{ as } \nu \to +\infty. \tag{4.33}$$

Passing to the limit in (4.29) as $\nu \to +\infty$ and taking (4.30) and (4.32)-(4.33) into account, we obtain

$$\left|\begin{array}{ll} \rho_1 w_{tt} - b w_{xx} = 0 & \text{in } Q_0, \\ z_{xx} = 0 & \text{in } Q_1 =]L_0, L[\times]0, T[, \\ w(0,t) = 0 = z(L,t), & t > 0, \\ b w_x(L_0, t) = a(t) z_x(L_0, t), & t > 0, \\ z_t(x,t) = 0 & \text{in } Q_1. \end{array}\right.$$

Repeating the above procedure in the case $u \neq 0$, we get $w = 0$ and $z = 0$, which contradicts (4.23). So, Lemma 4.5 is proved. □

Proof of Theorem 4.1 It is not difficult to see that $K(t)$ verifies

$$q_0 E(t) \leq K(t) \leq q_1 E(t), \tag{4.34}$$

where q_0 and q_1 are positive constants. Now, from hypothesis on the function g we get

$$|v_t|^2 + |g(v_t)|^2 \leq c \int_{L_0}^L g(v_t)\, v_t dx. \tag{4.35}$$

Applying the inequalities (4.11), (4.34) and (4.35), along with the ones in Lemma 4.5, and integrating from S to T, where $0 \leq S \leq T < +\infty$, and choosing $\varepsilon > 0$ sufficient small, we obtain

$$\int_S^T E(t)\,dt \leq CE(S).$$

In this condition, Lemma 2.1 implies

$$E(t) \leq CE(0)\,e^{-\nu t}.$$

□

5 Polynomial Decay

In this section we study the asymptotic behavior of the solutions of system (1.1)-(1.7) when the function $g(s)$ is non-linear in a neighbourhood to zero like s^p with $p > 1$. In this case we shall prove that the solution decays like $(1+t)^{-2/(p-1)}$.

Theorem 5.1. *With the hypotheses in Theorem 4.1 and $p > 1$ the weak solution decays polynomially, i. e.*

$$E(t) = CE(0)\,(1+t)^{-2/(p-1)}, \quad \forall t \geq 0.$$

Proof. First of all, we shall use some estimates of the previous section which do not depend on the behavior of the function g. From (A.2) and making use of Hölder's inequality, we deduce that

$$\int\limits_{|v_t|>1} \left(v_t^2 + g(v_t)^2\right) dx \leq C \int_{L_0}^L g(v_t)\; v_t dx \tag{5.1}$$

and that

$$\begin{aligned}\int\limits_{|v_t|\leq 1} \left(v_t^2 + g(v_t)^2\right) dx &\leq C \int_{L_0}^L \left(g(v_t)\; v_t\right)^{2/(p+1)} dx \\ &\leq C\left(\int_{L_0}^L g(v_t)\, v_t dx\right)^{2/(p+1)}.\end{aligned} \tag{5.2}$$

Summing inequalities (5.1) and (5.2) we get

$$\int_{L_0}^L \left(v_t^2 + g(v_t)^2\right) dx \leq k_1 \int_{L_0}^L g(v_t)\; v_t dx + k_2\left(\int_{L_0}^L g(v_t)\; v_t dx\right)^{2/(p+1)}. \tag{5.3}$$

Using the inequality (4.11), Lemma 4.3, (5.3), and taking $\varepsilon > 0$ small enough we have that

$$K' \leq -w_1 E - w_2 E' + w_3\left(-E'\right)^{2/(p+1)}$$

for some constants $w_1,\, w_2,\, w_3 > 0$ which are independent of the data. Multiplying this inequality by $E^{(p-1)/2}$ we obtain

$$\begin{aligned} w_1 E^{(p+1)/2} &\leq -w_2 E^{(p-1)/2} E' - K' E^{(p-1)/2} + w_3 E^{(p-1)/2} \left(-E'\right)^{2/(p+1)} \\ &\leq -w_2 E^{(p-1)/2} E' - \left(K E^{(p-1)/2}\right)' + 2\left(p-1\right) K E^{(p-3)/2} \\ &\quad + w_3 E^{(p-1)/2} \left(-E'\right)^{2/(p+1)} . \end{aligned}$$

Integrating the above inequality from S to T, and using (4.34), we get

$$\int_S^T E^{(p+1)/2} dt \leq C E^{(p+1)/2}\left(0\right) E\left(S\right) + \frac{w_3}{w_1} \int_S^T E^{(p-1)/2} \left(-E'\right)^{2/(p+1)} dt. \tag{5.4}$$

Using Hölder and Young's inequalities we can estimate the last term of this inequality by

$$\begin{aligned} &\frac{w_3}{w_1} \int_S^T E^{(p-1)/2} \left(-E'\right)^{2/(p+1)} dt \\ &\leq \frac{1}{2} \int_S^T E^{(p+1)/2} dt + \left(2\frac{w_3}{w_1}\right)^{(p+1)/2} \left(p+1\right)^{-1} \left(\frac{p-1}{p+1}\right)^{(p-1)/2} E\left(S\right). \end{aligned} \tag{5.5}$$

Substituting (5.5) into (5.4) we arrive at

$$\int_S^T E^{(p+1)/2} dt \leq C E\left(S\right).$$

Applying Lemma 2.1, we obtain

$$E\left(t\right) \leq C E\left(0\right) \left(1+t\right)^{-2/(p+1)},$$

which completes the present proof. □

References

[1] M. Aassila, Strong asymptotic stability of isotropic elasticity system with internal damping, *Acta Scientiarium Math* **64** (1998), 103–108.

[2] D. Andrade, L. H. Fatori and J. E. Muñoz Rivera, Non-linear transmission problem with a dissipative boundary condition of memory type, *Elect. J. Diff. Eq.* **2006** (2006), 1–16

[3] V. Komornik, *Exact Controllability and Stabilization, The multiplier method*, John Wiley & Sons, Masson, Paris, 1994.

[4] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quaslinear Elliptic equations,* Academic Press, New York, 1968.

[5] J. Lagnese, Boundary controllability in problem of transmission for a class of second order hyperbolic systems, *ESAIM: Cont. Opt. Cal. Var.* **2** (1997), 343–357.

[6] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Diff. Integ. Eq.* **6** (1993), 507–533.

[7] J. L. Lions, *Quelques Methodes des Resolution des Problemes aux Limites non Lineaires*, Dunod Gauthiers Villars, Paris, 1969.

[8] W. Liu and G. Williams, The exponential stability of the problem of transmission of the wave equation, *Bull. Austral. Math. Soc.* **57** (1998), 305–327.

[9] P. Martinez, A new method to obtain decay rate estimates for dissipative systems with localized damping, *Rev. Mat. Com.* **12** (1999), 251–283.

[10] J. E. Muñoz Rivera and E. Cabanillas, The nonlinear Transmission problem with time dependent coefficients, *Elect. J. Diff. Eq.* **2007** (2007), 1–13

[11] J. E. Muñoz Rivera and Higidio Portillo Oquendo, The transmission problem of viscoelastic waves, *Act. Appl. Math* **62** (2000), 1–21.

[12] M. Nakao, Decays of solutions of the wave equation with a local non-linear dissipation, *Math. Ann.* **305** (1996), 403–417.

[13] A. Ruiz, Unique continuation for weak solutions of the wave equation plus a potential, *J. Math. Pures Appl.* **71** (1992), 445–467.

[14] E. Zuazua, Stability and decays for a class of non-linear hyperbolic problems, *Asymp. Anal.* **1** (1988), 161–185.