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Nonlinear Differential Equations with "Maxima": Parametric Stability in Terms of two Measures

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Abstract: This paper investigates parametric stability for nonlinear differential equations with "maxima". Several sufficient conditions for parametric stability as well as uniform parametric stability are obtained based on the Razumikhin method. Two different types of Lyapunov functions have been applied. A comparison with scalar ordinary differential equations is offered.

Keywords: Parametric stability, Lyapunov functions, Razumikhin method, differential equations, maximum of the unknown function.

1. Introduction

One of the main problems in the qualitative theory of differential equations is stability of the solutions. There are various types of stability. One very useful stability type is connected with parameters. Uncertain parameters in nonlinear systems can cause a shift in the equilibrium state resulting in a loss of stability or destruction of the equilibrium altogether. The concept of parametric stability (see [3,9,15]) has been introduced to consider the joint problem of feasibility and stability of equilibria when parameters belong to a bounded uncertainty set. It is a robust stability concept, which explicitly deals with uncertainty of equilibrium location caused by uncertain values of system parameters. General conditions for parametric stability have been obtained in the context of Lyapunov stability theory and subsequently applied to Lotka–Volterra models in population dynamics (see [9,12]). The study on parametric stability analysis of rotor bearing systems via Lyapunov's direct method is discussed in [2]. Some computer experiments on testing the parametric stability of eukaryotic molecular genetic systems controlling gene expression are given in [16]. Parametric stability is also applied to control theory (see [17,18]).

In the last couple of decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in control theory correspond to maximal deviation of the regulated quantity (see [14]). Such kind of problems could be adequately modeled by differential equations that contain the maxima operator. A. D. Mishkis also points out the necessity to study differential equations with "maxima" in his survey [13]. Various conditions for stability for differential equations with "maxima" are obtained in [6,4,5] (see also [1,7,8]).

In the present paper, the parametric stability of differential equations with "maxima" is studied in the context of Lyapunov functions and the Razumikhin method. Several sufficient conditions for parametric stability and uniform parametric stability in terms of two measures are obtained. Note that stability in terms of two measures is studied in [10,11,7]. In this paper, additionally to the existing results, two different measures for the initial functions and for the solutions are applied. This helps to widen the area of applications of the obtained results to real world problems. Some examples are given to illustrate the concept of the considered type of stability and applications of the obtained results.

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2. Preliminary Notes and Definitions

Let $\mathbb{R}_+ = [0,\infty)$ and $r, t_0 > 0$. Consider the system of nonlinear differential equations with “maxima”

$$x'(t) = f\left(x, \max_{s\in[t-r,t]} x(s), p\right), \quad t \geq t_0 \tag{1}$$

with an initial condition

$$x(t) = \varphi(t-t_0), \quad t \in [t_0 - r, t_0], \tag{2}$$

where $x \in \mathbb{R}^n$, $p \in \mathbb{R}^m$, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $\varphi : [-r,0] \to \mathbb{R}^n$. Denote by $x(\cdot;t_0,\varphi,p)$ the solution of (1), (2). In our further considerations, we will assume that for any initial function $\varphi \in \mathrm{C}([-r,0],\mathbb{R}^n)$, the solution $x(\cdot;t_0,\varphi,p)$ is defined on $[t_0 - r,\infty)$.

We now introduce the class Λ of Lyapunov functions.

Definition 1. *Let $\Delta \subset [-r,\infty)$ and $\Omega \subset \mathbb{R}^n$. We say that the function $V \in \mathrm{C}(\Delta \times \Omega, \mathbb{R}_+)$ belongs to class Λ provided it is Lipschitz with respect to its second argument.*

Let $V \in \Lambda$, $t \in \mathbb{R}_+$, $p \in \mathbb{R}^m$, and $\phi \in \mathrm{C}([t-r,t],\mathbb{R}^n)$. We define a derivative of the function V along the trajectory of system (1) by

$$\mathcal{D}_{(1)}(V,\phi,t,p) = \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\times \left\{V\left(t+\varepsilon, \phi(t) + \varepsilon f\left(\phi(t), \max_{s\in[-r,0]} \phi(t+s), p\right)\right) - V(t,\phi(t))\right\}.$$

Consider the following set:

$$K = \Big\{a \in \mathrm{C}(\mathbb{R}_+,\mathbb{R}_+) : a \text{ is strictly increasing and } a(0) = 0\Big\}.$$

We define the set of measures:

$$\Gamma = \Big\{h \in \mathrm{C}([-r,\infty)\times\mathbb{R}^n,\mathbb{R}_+) : \min_{x\in\mathbb{R}^n} h(t,x) = 0 \text{ for each } t \geq -r\Big\}.$$

Let $\varphi \in \mathrm{C}([-r,0],\mathbb{R}^n)$ and $h_0 \in \Gamma$, $t_0 \in \mathbb{R}_+$. We will use the notation

$$H_0(t_0,\varphi) = \max_{t\in[-r,0]} h_0(t+t_0,\varphi(t+t_0)).$$

The following two definitions are taken from [11].

Definition 2. *Let $h, h_0 \in \Gamma$. Then h_0 is called* uniformly finer *than h if there exist $\delta > 0$ and $a \in K$ such that $h_0(t,x) < \delta$ implies $h(t,x) \leq a(h_0(t,x))$.*

Definition 3. *Let $h \in \Gamma$. The function $V \in \Lambda$ is said to be h-*decrescent *if there exist $\delta > 0$ and $a \in K$ such that $h(t,x) < \delta$ implies $V(t,x) \leq a(h(t,x))$.*

Let $\rho > 0$, $\xi \in \mathbb{R}^n$, and $h \in \Gamma$. We define the following sets:

$$S(h,\xi,\rho) = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : \ h(t,x-\xi) < \rho\},$$
$$S^C(h,\xi,\rho) = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : \ h(t,x-\xi) \geq \rho\}.$$

In our further investigations, we will use the initial value problem for the comparison scalar differential equation

$$u' = g(t,u), \quad t \geq t_0, \quad u(t_0) = u_0, \tag{3}$$

where $u \in \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $u_0 \in \mathbb{R}$. We will also assume that the initial value problem (3) has a solution $u(\cdot;t_0,u_0)$ defined on $[t_0,\infty)$ for any initial point $u_0 \in \mathbb{R}$.

Definition 4. *A constant point $\xi^* \in \mathbb{R}^n$ is called an* equilibrium *of the system of differential equations with “maxima” (1) for $p = p^* \in \mathbb{R}^m$ if ξ^* is a solution (1), i.e., $f(\xi^*,\xi^*,p^*) = 0$.*

Note that the equilibrium of (1) depends on p and we denote it by ξ_p.

We now introduce the concept of parametric stability: Let us assume that for some value p^*, there exists an equilibrium ξ_{p^*} which is stable. Let the parameter p change its value from p^* to another value $\tilde{p}$. Then is there a new equilibrium state $\xi_{\tilde{p}}$, i.e., $f(\xi_{\tilde{p}},\xi_{\tilde{p}},\tilde{p}) = 0$, and if there is, then how far is it from the old one ξ_{p^*} and is it stable as the old one? The answer of this question is given by the concept of parametric stability as studied for ordinary differential equations in [9,15] and for differential equations with delay in [3]. In order to generalize the concept of parametric stability, we will use two different measures in the definition of stability of equilibrium.

Definition 5. *The equilibrium ξ_p of (1) for $p \in \mathbb{R}^m$ is called* stable in terms of both measures *(h_0,h), in short, (h_0,h)-stable, if for any $t_0 \in \mathbb{R}_+$ and $\varepsilon > 0$, there exists $\delta = \delta(t_0,\varepsilon,p) > 0$ such that for any $\varphi \in \mathrm{C}([-r,0],\mathbb{R}^n)$, $H_0(t_0,\varphi-\xi_p) < \delta$ implies $h(t,x(t;t_0,\varphi,p)-\xi_p) < \varepsilon$ for $t \geq t_0$.*

Let us introduce the following definitions of parametric stability in terms of two measures from the set Λ.

Definition 6. *Let $h, h_0 \in \Gamma$. The system of differential equations with “maxima” (1) is said to be*

(S1) parametrically stable *at $p^* \in \mathbb{R}^m$ in terms of both measures (h_0,h) if there exist an (h_0,h)-stable equilibrium ξ_{p^*} of (1) for $p = p^*$ and a neighborhood $N(p^*)$ such that for any $p \in N(p^*)$,*
  (i) *there exists an equilibrium ξ_p of the system of differential equations with “maxima” (1);*
  (ii) *the equilibrium ξ_p is (h_0,h)-stable.*

(S2) uniformly parametric stable *at $p^* \in \mathbb{R}^m$ in terms of both measures (h_0,h) if the number δ in (S1) is independent of $t_0 \in \mathbb{R}_+$.*

Remark. Note that if $h(t, x) = h_0(t, x) \equiv \|x\|$, then the defined types of stabilities in Definitions 5 and 6 are known in the literature as *stability* and *parametric stability*, respectively.

We will give some examples to illustrate the concepts of parametric stability and to compare this type of stability with stability of an equilibrium. To simplify the considerations, we consider the case of $h(t, x) = h_0(t, x) \equiv \|x\|$ and $r = 0$, i.e., ordinary differential equations.

Example 1. Consider the scalar linear differential equation $x' = -x + p$, where p is a parameter. The equation has a solution $x(t; t_0, x_0, p) = p+(x_0-p)e^{-(t-t_0)}$, where $x(t_0) = x_0$. For any p^*, the equation has an equilibrium $\xi_{p^*} = p^*$, which is stable. At the same time, the equation is parametrically stable, where $N(p^*)$ is an arbitrary neighborhood.

Example 2. Consider the scalar linear differential equation $x' = -x + px$, where p is a parameter. The solution of this equation is $x(t; t_0, x_0, p) = x_0 e^{(p-1)(t-t_0)}$, where $x(t_0) = x_0$.

Let $p^* = 0$. The differential equation has an equilibrium $\xi^* = 0$ which is stable. Let $N(p^*) = \{p : |p| < 1\}$. Then the equation is parametrically stable at $p^* = 0$ since $|x(t; t_0, x_0, p)| \le x_0$ for $t \ge t_0$ and $p \in N(p^*)$.

Let $p^* = 1$. The differential equation has an equilibrium $\xi^* = 0$ which is stable. At the same time, for any $p = 1 + b$, where $b > 0$ is an arbitrary small number, there is an equilibrium $\xi_p = 0$ which is not stable, since the corresponding solution is $x(t; t_0, x_0, p) = x_0 e^{b(t-t_0)}$. Therefore, the differential equation is not parametrically stable at $p^* = 1$.

Example 3. Consider the scalar differential equation [15]: $x' = p - \cos(x)$, where p is a parameter. For $p^* = 1$, the equation has the form $x' = 1 - \cos(x) = 2\sin^2(\frac{x}{2})$, and it has an equilibrium $x^* = 0$ which is unstable. At the same time, the instability of the equilibrium $x^* = 0$ does not persist for small variations of the parameter since for any small variation of the parameter $p > 1$, there is no equilibrium.

In the proofs of our main results, we will use the following comparison result.

Lemma 1. *Let following conditions be fulfilled:*

1. $f \in C(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ *and* $\varphi \in C([-r, 0], \mathbb{R}^n)$.
2. *There exists an equilibrium* $\xi_p \in \mathbb{R}^n$ *of the differential equation with "maxima"* (1) *for a fixed value* p *of the parameter.*
3. *The initial value problem* (1), (2) *possesses a solution* $x(\cdot; t_0, \varphi, p)$ *defined on* $[t_0 - r, T)$, *where* $T \le \infty$.
4. $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$.
5. $V : [t_0, T) \times \mathbb{R}^n \to \mathbb{R}_+$, $V \in \Lambda$, *and for any number* $t \in [t_0, T)$ *and any function* $\psi \in C([t-r, t], \mathbb{R}^n)$ *such that*
$$V(t, \psi(t) - \xi_p) > V(t + s, \psi(t + s) - \xi_p)$$
for $s \in [-r, 0)$, *the inequality*
$$\mathcal{D}_{(1)}(V, \psi - \xi_p, t, p) \le g(t, V(t, \psi(t) - \xi_p))$$
holds.

Then the inequality $\max_{s\in[-r,0]} V(t_0 + s, \varphi(s) - \xi_p) \le u_0$ *implies*
$$V(t, x(t; t_0, \varphi, p) - \xi_p) \le u^*(t; t_0, u_0) \ \text{for} \ t \in [t_0, T),$$
where $u^*(\cdot; t_0, u_0)$ *is the maximal solution of the initial value problem* (3).

Proof. Let $u_0 \in \mathbb{R}_+$ be such that
$$\max_{s\in[-r,0]} V(t_0 + s, \varphi(s) - \xi_p) \le u_0.$$
Let $n \in \mathbb{N}$ and v_n be the maximal solution of the problem
$$u' = g(t, u) + \frac{1}{n}, \quad u(t_0) = u_0 + \frac{1}{n}.$$
From $g(t, u) + \frac{1}{n} > 0$ on $[t_0, T)$, it follows that v_n is increasing on $[t_0, T)$.

Define a function $m \in C([t_0, T), \mathbb{R}_+)$ by
$$m(t) = V(t, x(t; t_0, \varphi, p) - \xi_p).$$
Because of the fact that $u^*(t; t_0, u_0) = \lim_{n\to\infty} v_n(t)$, it is enough to prove that for any $n \in \mathbb{N}$, we have
$$m(t) \le v_n(t) \quad \text{for} \ t \in [t_0, T). \tag{4}$$
Note that for any $n \in \mathbb{N}$, the inequality $m(t_0) < v_n(t_0)$ holds. Assume inequality (4) is not true. Let $n \in \mathbb{N}$ be such that there exists a point $\eta \in (t_0, T)$ with $m(\eta) > v_n(\eta)$. Define
$$t^* = \sup \{t \in [t_0, T) : \ m(s) < v_n(s) \text{ for } s \in [t_0, t)\}.$$
Note $t^* \in (t_0, T)$. Therefore
$$m(t^*) = v_n(t^*), \quad m(t) < v_n(t) \ \text{ for } t \in [t_0, t^*). \tag{5}$$
From inequality (5), it follows that
$$\begin{aligned} D_- m(t^*) &= \lim_{h\to 0-} \frac{m(t^* + h) - m(t^*)}{h} \\ &\ge \lim_{h\to 0-} \frac{v(t^* + h) - v(t^*)}{h} = v_n'(t^*) \\ &= g(t^*, v_n(t^*)) + \frac{1}{n} \\ &= g(t^*, m(t^*)) + \frac{1}{n}. \end{aligned} \tag{6}$$
As Case 1, suppose $t^* - r \ge t_0$. From the inequality $m(t^*) = v_n(t^*) \ge v_n(s) > m(s)$ for $s \in [t^* - r, t^*)$, according to condition 5, we get
$$D_- m(t^*) \le g(t^*, m(t^*)) < g(t^*, m(t^*)) + \frac{1}{n},$$

contradicting (6). Therefore, (4) holds. As Case 2, suppose $t^* - r < t_0$. Hence for $s \in [t^* - r, t_0]$, we obtain the relations

$$\begin{aligned} m(s) &= V(s, x(s; t_0, \varphi, p) - \xi_p) \\ &\le \max_{s\in[t_0-r,t_0]} V(s, x(s; t_0, \varphi, p) - \xi_p) \\ &= \max_{s\in[-r,0]} V(t_0 + s, \varphi(s) - \xi_p) \le u_0 < u_0 + \frac{1}{n} \\ &= v_n(t_0) \le v_n(t^*) = m(t^*). \end{aligned}$$

For $s \in [t_0, t^*]$, as in Case 1, we obtain $m(s) < m(t^*)$. Therefore applying condition 5, we get

$$D_-m(t^*) \le g(t^*, m(t^*)) < g(t^*, m(t^*)) + \frac{1}{n},$$

contradicting (6).

3. Main Results

Initially, we will give some sufficient conditions for parametric stability applying only one Lyapunov function and the Razumikhin method.

Theorem 1. *Let the following conditions be fulfilled:*

1. $f \in C(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$.
2. $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *and* $g(t, 0) \equiv 0$ *for* $t \in \mathbb{R}_+$.
3. $h_0, h \in \Gamma$, *and* h_0 *is uniformly finer than* h.
4. *There exist a point* $p^* \in \mathbb{R}^m$, *an* (h_0, h)*-stable equilibrium* ξ_{p^*} *of* (1), *and a neighborhood* $N(p^*)$ *of* p^* *such that for any* $p \in N(p^*)$, *there exists an equilibrium* ξ_p *of the system of differential equations with "maxima"* (1) *for* p.
5. *There exist a function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $V \in \Lambda$, *and a constant* $\rho > 0$ *such that:*
   (i) *for any* $p \in N(p^*)$, *any number* $t \in \mathbb{R}_+$, *and any* $\psi \in C([t - r, t], \mathbb{R}^n)$ *with* $(t, \psi(t)) \in S(h, \xi_p, \rho)$ *and* $V(t, \psi(t) - \xi_p) > V(t + s, \psi(t + s) - \xi_p)$ *for* $s \in [-r, 0)$, *the inequality*
   $$\mathcal{D}_{(1)}(V, \psi - \xi_p, t, p) \le g(t, V(t, \psi(t) - \xi_p))$$
   holds;
   (ii) *for any* $p \in N(p^*)$ *and any* $(t, x) \in S(h, \xi_p, \rho)$, *the inequality*
   $$a(h(t, x - \xi_p)) \le V(t, x - \xi_p) \le b(h_0(t, x - \xi_p))$$
   holds, where $a, b \in K$.

Then the system of differential equations with "maxima" (1) *is parametrically stable (uniformly parametrically stable) at* p^* *in terms of both measures* (h_0, h) *if the zero solution of the scalar differential equation* (3) *is stable (uniformly stable).*

Proof. Let $t_0 \in \mathbb{R}_+$ be an arbitrary fixed point, $p \in N(p^*)$ be a fixed parameter, and $\varepsilon \in (0, \rho)$. There exists an equilibrium ξ_p of (1). Let the zero solution of the scalar differential equation (3) be stable, i.e., there exists $\delta_1 \in (0, \rho)$, $\delta_1 = \delta_1(t_0, \varepsilon)$ such that $|u_0| < \delta_1$ implies

$$|u(t)| < a(\varepsilon), \quad t \ge t_0, \tag{7}$$

where u is a solution of (3). According to condition 3, there exist $\delta_2 > 0$ and a function $\psi \in K$ such that

$$h_0(t, x - \xi_p) < \delta_2 \tag{8}$$

implies

$$h(t, x - \xi_p) \le \psi(h_0(t, x - \xi_p)). \tag{9}$$

Since $b \in K$ and $\psi \in K$, we can find $\delta_3 = \delta_3(\varepsilon) > 0$, $\delta_3 < \min(\delta_2, \rho)$, such that

$$\psi(\delta_3) < \varepsilon, \qquad b(\delta_3) < \delta_1. \tag{10}$$

Choose an initial function $\varphi \in C([-r, 0], \mathbb{R}^n)$ such that

$$H_0(t_0, \varphi - \xi_p) < \delta_3 < \delta_2. \tag{11}$$

From inequalities (8), (10), and (11), we get the inequality

$$\begin{aligned} h(t_0, \varphi - \xi_p) &\le \psi(h_0(t_0, \varphi - \xi_p)) \\ &\le \psi(H_0(t_0, \varphi - \xi_p)) \\ &< \psi(\delta_3) < \varepsilon. \end{aligned}$$

We will prove that if (11) is satisfied, then

$$h(t, x(t; t_0, \varphi, p) - \xi_p) < \varepsilon, \ \ t \ge t_0. \tag{12}$$

Assume inequality (12) is not true. Then there exists a point $t^* > t_0$ such that

$$\begin{aligned} h(t^*, x(t^*; t_0, \varphi_0, p) - \xi_p) &= \varepsilon, \\ h(t, x(t; t_0, \varphi_0, p) - \xi_p) &< \varepsilon, \ \ t \in [t_0, t^*). \end{aligned} \tag{13}$$

Denote $x(s) = x(s; t_0, \varphi, p)$ for $s \in [t_0 - r, t^*]$. Since $\varepsilon < \rho$, we have $(t, x(t)) \in S(h, \xi_p, \rho)$ for $t \in [t_0, t^*]$. If we assume that $h_0(t^*, x(t^*) - \xi_p) \le \delta_3$, then from the choice of δ_3 and inequalities (8), (9), and (10), we obtain

$$h(t^*, x(t^*) - \xi_p) \le \psi(h_0(t^*, x(t^*) - \xi_p)) \le \psi(\delta_3) < \varepsilon,$$

contradicting (13). Hence

$$h_0(t^*, x(t^*) - \xi_p) > \delta_3, \ \ H_0(t_0, \varphi - \xi_p) < \delta_3. \tag{14}$$

Therefore there exists a point $t_0^* \in (t_0, t^*)$ such that

$$\begin{aligned} h_0(t_0^*, x(t_0^*) - \xi_p) &= \delta_3, \\ h_0(t, x(t) - \xi_p) &< \delta_3, \ \ t \in [t_0 - r, t_0^*). \end{aligned}$$

Let $u^*(\cdot; t_0^*, u_0^*)$ be the maximal solution of (3) with initial value

$$u_0^* = \max_{t\in[t_0^*-r,t_0^*]} V(t, x(t) - \xi_p).$$

According to Lemma 1, the inequality

$$V(t, x(t) - \xi_p) \leq u^*(t; t_0^*, u_0^*), \quad t \in [t_0^*, t^*] \tag{15}$$

holds. From (10) and condition 5(ii), we get

$$V(t, x(t) - \xi_p) \leq b(h_0(t, x(t) - \xi_p)) < \delta_1$$

for all $t \in [t_0^* - r, t_0^*]$. Hence $|u_0^*| < \delta_1$, and according to (7),

$$|u^*(t; t_0^*, u_0^*)| < a(\varepsilon), \quad t \geq t_0^*. \tag{16}$$

From (15), (16), and condition 4(ii), we obtain

$$a(\varepsilon) = a(h(t^*, x(t^*) - \xi_p)) \leq V(t^*, x(t^*) - \xi_p) \\ \leq u^*(t^*; t_0^*, u_0^*) < a(\varepsilon).$$

This contradiction proves that the inequality (12) holds.

Corollary 1. *Let the conditions of Theorem 1 be satisfied for $g(t,x) \equiv 0$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then the system* (1) *is uniformly parametric stable at p^* in terms of both measures (h_0, h).*

In the case when the Lyapunov function does not satisfy all the desired conditions, it is useful to perturb the function rather than discard it. In this case, we will use two different comparison scalar differential equations:

$$u' = g_1(t, u), \quad t \geq t_0, \tag{17}$$

and

$$v' = g_2(t, v), \quad t \geq t_0, \tag{18}$$

where $u, v \in \mathbb{R}$, $t_0 \in \mathbb{R}_+$.

Theorem 2. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 4 of Theorem 1 are satisfied.*
2. *There exist a constant $\rho > 0$ and an h_0-decrescent function $V_1 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $V_1 \in \Lambda$, such that:*
   (i) *for any $p \in N(p^*)$, $t > 0$, and $\psi \in \mathrm{C}([t-r, t], \mathbb{R}^n)$ such that $(t, \psi(t)) \in S(h, \xi_p, \rho)$ and*

   $$V_1(t, \psi(t) - \xi_p) > V_1(t+s, \psi(t+s) - \xi_p)$$

   for $s \in [-r, 0)$, the inequality

   $$\mathcal{D}_{(1)}(V_1, \psi - \xi_p, t, p) \leq g_1(t, V_1(t, \psi(t) - \xi_p))$$

   holds, where $g_1 \in \mathrm{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $g_1(t, 0) \equiv 0$ for $t \in \mathbb{R}_+$.
3. *For any $\mu > 0$, there exists $V_2^{(\mu)} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $V_2^{(\mu)} \in \Lambda$, such that:*
   (ii) *for any $p \in N(p^*)$ and any point*

   $$(t, x) \in S(h, \xi_p, \rho) \cap S^C(h_0, \xi_p, \mu),$$

   the inequality

   $$a(h(t, x - \xi_p)) \leq V_2^{(\mu)}(t, x - \xi_p) \\ \leq b(h_0(t, x - \xi_p))$$

   holds, where $a, b \in K$;

   (iii) *for any $p \in N(p^*)$, $t > 0$, and $\psi \in \mathrm{C}([t-r, t], \mathbb{R}^n)$ such that $(t, \psi(t)) \in S(h, \xi_p, \rho) \cap S^C(h_0, \xi_p, \mu)$ and*

   $$V_1(t, \psi(t) - \xi_p) + V_2^{(\mu)}(t, \psi(t) - \xi_p) \\ > V_1(t+s, \psi(t+s) - \xi_p) \\ + V_2^{(\mu)}(t+s, \psi(t+s) - \xi_p)$$

   for $s \in [-r, 0)$, the inequality

   $$\mathcal{D}_{(1)}(V_1, \psi - \xi_p, t, p) \\ + \mathcal{D}_{(1)}(V_2^{(\mu)}, \psi - \xi_p, t, p) \\ \leq g_2\Big(t, V_1(t, \psi(t) - \xi_p) + V_2^{(\mu)}(t, \psi(t) - \xi_p)\Big)$$

   holds, where $g_2 \in \mathrm{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$, $g_2(t, 0) \equiv 0$ for $t \in \mathbb{R}_+$.
4. *The zero solution of the scalar differential equation* (18) *is uniformly stable.*

Then the system of differential equations with "maxima" (1) *is parametrically stable (uniformly parametrically stable) at p^* in terms of both measures (h_0, h) if the zero solution of the scalar differential equation* (17) *is stable (uniformly stable).*

Proof. Let $t_0 \in \mathbb{R}_+$ be an arbitrary fixed point, $p \in N(p^*)$ be a fixed parameter, and $\varepsilon \in (0, \rho)$. There exists an equilibrium ξ_p of (1). Since V_1 is h_0-decrescent, there exist $\rho_1 \in (0, \rho)$ and a function $\psi_1 \in K$ such that

$$h_0(t, x) < \rho_1, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{19}$$

implies

$$V_1(t, x) \leq \psi_1(h_0(t, x)). \tag{20}$$

From the properties of the measures $h_0(t, x)$ and $h(t, x)$, it follows that there exist $\rho_0 > 0$ and a function $\psi_2 \in K$ such that

$$h_0(t, x) < \rho_0 \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{21}$$

implies

$$h(t, x) \leq \psi_2(h_0(t, x)). \tag{22}$$

From condition 4, there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that inequality $|v_0| < \delta_1$ implies

$$|v(t; t_0, v_0)| < a(\varepsilon) \ \text{ for } t \geq t_0, \tag{23}$$

where $v(\cdot; t_0, v_0)$ is a solution of the scalar differential equation (18) with the initial condition $v(t_0) = v_0$. Since $b, \psi_2 \in K$, we can find $\delta_2 = \delta_2(\varepsilon) > 0$, $\delta_2 < \varepsilon$, such that the inequalities

$$b(\delta_2) < \frac{\delta_1}{2} \tag{24}$$

and

$$\psi_2(\delta_2) < \varepsilon \tag{25}$$

hold. Let the zero solution of the scalar differential equation (17) be stable, i.e., there exists $\delta_3 = \delta_3(t_0, \varepsilon) > 0$ such that the inequality $|u_0| < \delta_3$ implies

$$|u(t; t_0, u_0)| < \frac{\delta_1}{2}, \quad t \geq t_0, \tag{26}$$

where $u(\cdot; t_0, u_0)$ is a solution of the scalar equation (17) with the initial condition $u(t_0) = u_0$. Since $\psi_1 \in K$, there exists $\delta_4 = \delta_4(\delta_3) > 0$ such that the inequality $|u| < \delta_4$ implies

$$\psi_1(u) < \delta_3. \tag{27}$$

From inequalities (19) and (27), it follows that there exists $\delta_5 = \delta_5(\delta_4) > 0$, $\delta_5 < \min\{\delta_4, \rho_1\}$, such that the inequality $h_0(t, x) < \delta_5$ implies

$$V_1(t, x) \leq \psi_1(h_0(t, x)) < \delta_3. \tag{28}$$

Choose the initial function $\varphi \in \mathrm{C}([-r, 0], \mathbb{R}^n)$ such that

$$H_0(t_0, \varphi - \xi_p) < \delta_6, \tag{29}$$

where $\delta_6 = \min\{\delta_2, \delta_5\}$, $\delta_6 = \delta_6(t_0, \varepsilon) > 0$. From inequalities (21) and (22), it follows that

$$\begin{aligned} h(t_0, \varphi - \xi_p) &\leq \psi_2(h_0(t_0, \varphi - \xi_p)) \\ &\leq \psi_2(H_0(t_0, \varphi - \xi_p)) \\ &< \psi_2(\delta_6) \leq \psi_2(\delta_2) < \varepsilon. \end{aligned}$$

We will prove that if inequality (29) is satisfied, then

$$h(t, x(t; t_0, \varphi, p) - \xi_p) < \varepsilon \ \text{ for } \ t \geq t_0 - r. \tag{30}$$

From inequality (29), it follows that

$$h_0(t, x(t; t_0, \varphi, p) - \xi_p) < \varepsilon, \quad t \in [t_0 - r, t_0].$$

Assume that inequality (30) is not true for $t \geq t_0$. According to the assumption, there exists a point $t^* > t_0$ such that

$$h(t, x(t^*; t_0, \varphi, p) - \xi_p) = \varepsilon \tag{31}$$

and

$$h(t, x(t; t_0, \varphi, p) - \xi_p) < \varepsilon, \quad t \in [t_0 - r, t^*). \tag{32}$$

Denote $x(t) = x(t; t_0, \varphi, p)$, $t \in [t_0 - r, t^*]$. From the choice of φ and δ_6, we have $h_0(t_0, \varphi - \xi_p) < \delta_6 \leq \delta_2$. If we assume that $h_0(t^*, x(t^*) - \xi_p) \leq \delta_2$, then from the choice of δ_2 and inequalities (22) and (25), we obtain

$$h(t^*, x(t^*; t_0, \varphi, p) - \xi_p) \leq \psi_2(h_0(t^*, x(t^*) - \xi_p)) < \varepsilon,$$

contradicting (31). Hence

$$h_0(t^*, x(t^*) - \xi_p) > \delta_2, \quad H_0(t_0, \varphi - \xi_p) < \delta_2. \tag{33}$$

Then there exists a point $t_0^* \in (t_0, t^*)$ such that

$$\begin{gathered} h_0(t, x(t_0^*) - \xi_p) = \delta_2, \\ h_0(s, x(t) - \xi_p) < \delta_2 \ \text{ for } \ s \in [t_0 - r, t_0^*) \end{gathered}$$

and $(t, x(t)) \in S(h, \xi_p, \varepsilon) \cap S^C(h_0, \xi_p, \delta_2)$, $t \in [t_0^*, t^*]$. From the choice of ε, it follows that

$$(t, x(t)) \in S(h, \xi_p, \rho) \cap S^C(h_0, \xi_p, \delta_2), \quad t \in [t_0^*, t^*].$$

Let $r_1(\cdot; t_0, u_0)$ be the maximal solution of the differential equation (17), where

$$u_0 = \max_{s \in [-r, 0]} V_1(t_0 + s, \varphi(s) - \xi_p).$$

From condition 3(i) and Lemma 1, it follows that

$$V_1(s, x(s) - \xi_p) \leq r_1(s; t_0, u_0), \quad s \in [t_0, t^*]. \tag{34}$$

From inequality (29), we obtain

$$h_0(t_0 + s, \varphi(s) - \xi_p) < \delta_5, \quad s \in [-r, 0]. \tag{35}$$

From (35) and (28), we get

$$V_1(t_0 + s, \varphi(s) - \xi_p) \leq \psi_1(h_0(s, \varphi(s) - \xi_p)) < \delta_3,$$

$s \in [-r, 0]$, from which, together with (26), it follows that

$$r_1(t; t_0, u_0) < \frac{\delta_1}{2}. \tag{36}$$

From (34) and (36), we obtain the inequality

$$V_1(t_0^* + s, x(t_0^* + s) - \xi_p) < \frac{\delta_1}{2}, \quad s \in [-r, 0]. \tag{37}$$

Consider the function $V_2^{(\delta_2)}(t, x)$ defined in condition 3, where $\mu = \delta_2$. Let us define $V : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$, $V \in \Lambda$, such that

$$V(t, x) = V_1(t, x) + V_2^{(\delta_2)}(t, x).$$

From inequality (24), condition 3, and the choice of t_0^*, we get

$$\begin{aligned} &V_2^{(\delta_2)}(t_0^* + s, x(t_0^* + s) - \xi_p) \\ &\quad \leq b(h_0(t_0^* + s, x(t_0^* + s) - \xi_p)) \\ &\quad \leq b(\delta_2) < \frac{\delta_1}{2}, \quad s \in [-r, 0]. \end{aligned}$$

Hence

$$\begin{aligned} &V(t_0^* + s, x(t_0^* + s) - \xi_p) \\ &\quad = V_1(t_0^* + s, x(t_0^* + s) - \xi_p) \\ &\quad + V_2^{(\delta_2)}(t_0^* + s, x(t_0^* + s) - \xi_p) \\ &\quad < \frac{\delta_1}{2} + \frac{\delta_1}{2}, \quad s \in [-r, 0]. \end{aligned} \tag{38}$$

The function V satisfies the conditions of Lemma 1 on $[t_0^*, t^*]$. Hence, from Lemma 1, it follows that

$$V(t, x(t) - \xi_p) \leq v^*(t; t_0^*, v_0^*), \quad t \in [t_0^*, t^*], \tag{39}$$

where v^* is the maximal solution of the scalar differential equation (18) with initial condition

$$v_0^* = \max_{s\in[-r,0]} V(t_0^* + s, x(t_0^* + s) - \xi_p).$$

From (38), it follows that $v_0^* < \delta_1$, and according to (23), the inequality

$$v^*(t; t_0^*, v_0^*) < a(\varepsilon), \quad t \geq t_0^* \tag{40}$$

holds. From (39), (40) and condition 4(ii), we obtain

$$\begin{aligned} a(\varepsilon) = a(h(t^*, x(t^*) - \xi_p)) &\leq V(t^*, x(t^*) - \xi_p) \\ &\leq v^*(t^*; t_0^*, v_0^*) < a(\varepsilon). \end{aligned}$$

This contradiction proves the validity of the inequality (30) and the claim.

Now we will give an example in order to illustrate some of the obtained sufficient conditions on a system of differential equations with "maxima".

Example 4. Consider the system of differential equations with "maxima"

$$\begin{aligned} x'(t) &= -p^2 e^{2y} + \max_{s\in[t-r,t]} y(s), \\ y'(t) &= -y + 2p\sqrt{x}e^y - \max_{s\in[t-r,t]} x(s), \end{aligned} \tag{41}$$

where p is a parameter. There exists no explicit solution of the considered system (41), but we will apply the above results in order to investigate the parametric stability of the system.

For $p^* = 0$, the system (41) reduces to

$$\begin{aligned} x'(t) &= \max_{s\in[t-r,t]} y(s), \\ y'(t) &= -y - \max_{s\in[t-r,t]} x(s). \end{aligned} \tag{42}$$

The system (42) has an equilibrium $\xi^* = 0, \xi^* = (\xi_1^*, \xi_2^*)$, which is stable.

For any $p \in N(p^*) = \{p : \ |p| < \sqrt{0.5}\}$, the corresponding algebraic system to the system of differential equations with "maxima" (41) is

$$\begin{aligned} &-p^2 e^{2y} + y = 0, \\ &-y + 2p\sqrt{x}e^y - x = 0. \end{aligned} \tag{43}$$

The algebraic system (43) has a solution $\xi_p = (\xi_{1,p}, \xi_{2,p})$, where $\xi_{2,p} > 0$ is a solution of the equation $y = p^2 e^{2y}$, and $\xi_{1,p} = \xi_{2,p}$. Let $V(t,x,y) = \frac{1}{2}(x^2 + y^2) \in \Lambda$. Let $t \in \mathbb{R}_+$ be an arbitrary point, the parameter $p \in N(p^*)$, and the function $\psi \in C([t-r,t], \mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that $|\psi_k(t) - \xi_{k,p}| > |\psi_k(t+s) - \xi_{k,p}|$ for $s \in [-r, 0)$, $k = 1, 2$. Then we have

$$\begin{aligned} &\mathcal{D}_{(42)}(V, \psi - \xi_p, t, p) = (\psi_1(t) - \xi_{1,p})\times \\ &\Big(-p^2 e^{2(\psi_2(t)-\xi_{2,p})} + \max_{s\in[-r,0]} (\psi_2(t+s) - \xi_{2,p})\Big) \\ &+ (\psi_2(t) - \xi_{2,p})\times \\ &\Big(-(\psi_2(t) - \xi_{2,p}) + 2p\sqrt{\psi_1(t) - \xi_{1,p}}e^{\psi_2(t)-\xi_{2,p}} \\ &- \max_{s\in[-r,0]} (\psi_1(t+s) - \xi_{1,p})\Big) \\ \leq &(\psi_1(t) - \xi_{1,p}) \Big(-p^2 e^{2(\psi_2(t)-\xi_{2,p})} + (\psi_2(t) - \xi_{2,p})\Big) \\ &+ (\psi_2(t) - \xi_{2,p})\Big(- (\psi_2(t) - \xi_{2,p}) \\ &+ 2p\sqrt{\psi_1(t) - \xi_{1,p}}e^{\psi_2(t)-\xi_{2,p}} - (\psi_1(t) - \xi_{1,p})\Big) \\ = &- \Big(p\sqrt{\psi_1(t) - \xi_{1,p}}e^{\psi_2(t)-\xi_{2,p}} - (\psi_2(t) - \xi_{2,p})\Big)^2 \\ \leq &0. \end{aligned}$$

According to Corollary 1, the considered system of differential equations with "maxima" is uniformly parametric stable at $p^* = 0$.

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