Applied Mathematics & Information Sciences
An International Journal

Semiring Orders in a Semiring

Jeong Soon Han[1], *Hee Sik Kim*[2] *and J. Neggers*[3]

[1] Department of Applied Mathematics, Hanyang University, Ahnsan, 426-791, Korea
[2] Department of Mathematics, Research Institute for Natural Research, Hanyang University, Seoal, Korea
[3] Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U.S.A

Received: Received May 03, 2011; Accepted August 23, 2011
Published online: 1 January 2012

Abstract: Given a semiring it is possible to associate a variety of partial orders with it in quite natural ways, connected with both its additive and its multiplicative structures. These partial orders are related among themselves in an interesting manner is no surprise therefore. Given particular types of semirings, e.g., commutative semirings, these relationships become even more strict. Finally, in terms of the arithmetic of semirings in general or of some special type the fact that certain pairs of elements are comparable in one of these orders may have computable and interesting consequences also. It is the purpose of this paper to consider all these aspects in some detail and to obtain several results as a consequence.

Keywords: semiring, semiring order, partial order, commutative.

The notion of a semiring was first introduced by H. S. Vandiver in 1934, but implicitly semirings had appeared earlier in studies on the theory of ideals of rings ([2]). Semirings occur in different mathematical fields, i.e., as ideals of a ring, as positive cones of partially ordered rings and fields, in the context of topological considerations, and in the foundations of arithmetic, including questions raised by school education. Semirings have become of great interest as a tool in different branches of computer science ([4]).

By a *semiring* ([1]) we shall mean a set R endowed with two associative binary operations called an *addition* and a *multiplication* (denoted by $+$ and $\cdot$, respectively) satisfying the following conditions:

(i). addition is a commutative operation,
(ii). there exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$, and
(iii). multiplication distributes over addition both from the left and from the right.

1. Preliminaries

There are two ways to define a partially ordered set on a set: (i) weak inclusion: reflexive, anti-symmetric and transitive; (ii) strong inclusion: irreflexive and transitive, and they are equivalent (see [7]). J. Neggers et al. ([5, 6]) discussed the notion of semiring order in semirings, and obtained some results related with the notion of fuzzy left ideals of the semirings.

Suppose that R is a semiring. Define a relation $<_R$ on R as follows:

$$x <_R y \quad \text{provided} \quad x + y = y \quad \text{and} \quad xy = x, x \neq y.$$

Thus, since $0 + y = y$ and $0y = 0$, it follows that if $y \neq 0$, then $0 <_R y$ always, i.e., 0 is a unique miminal element.

L1. *$x <_R y$ and $y <_R x$ is impossible.*

L2. *$x <_R y$ and $y <_R z$ implies $x <_R z$.*

The set $(R, <_R)$ is a poset with unique miminal element 0. We shall refer to it as the *semiring order* of R. A non-empty subset I of a semiring order $(R, <_R)$ is called an *order ideal* if $x \in I, y <_R x$ imply $y \in I$.

Example 2.1. Let R^+ be the collection of non-negative real numbers with the usual operations "$+$" and "$\cdot$". Then $(R^+, +, \cdot)$ is a semiring. Also, if $x <_{R^+} y$ then $x + y = y$ means $x = 0$ and $y \neq 0$. In particular, if $x \neq y$, and $x \neq 0, y \neq 0$, then $x \circ y$, i.e., x and y are incomparable. Hence, $(R^+ - \{0\}, <_{R^+})$ is an antichain. We shall consider R^+ to be an *antichain semiring*.

Example 2.2. Let R^+ be the collection of non-negative real numbers. Define operations "$\oplus$" and "$\odot$" by $x \oplus y :=$

* Corresponding author: e-mail: heekim@hanyang.ac.kr

$\max\{x,y\}, x\odot y := \min\{x,y\}$. Then we are dealing with a semiring. Indeed, suppose that $x <_{R^+} y$. Then $x \oplus y = y$ and $x \odot y = x$. If $r \in R^+$, then $x <_{R^+} y$ implies $r \odot x <_{R^+} r \odot y$ as well. Hence $(r \odot x) \oplus (r \odot y) = r \odot y = r \odot (x \oplus y)$. Thus $(R^+, \oplus, \odot)$ is a semiring.

In this case, if $x < y$ in R^+, then also $x <_R y$ in the semiring order, whence the two orders are the same since an order extension of a chain is precisely the chain itself. Thus, we shall consider $(R^+, \oplus, \odot)$ to be a *chain semiring*.

J. Neggers *et. al.* [6] obtained that if $\mu : R \to L$ is an L-fuzzy left ideal of the semiring $(R, +, \cdot)$, then μ_t is an order ideal of $(R, <_R)$ and any finite order ideal I is a level subset of μ. Moreover, they proved that if $\mu : R \to L$ is an L-fuzzy left ideal of a finite chain semiring $(R, +, \cdot)$ then the collection of order ideals of $(R, <_R)$ is the collection of level subsets of μ. Furthermore, this collection is linearly ordered by set inclusion. This paper is a continuation of [6] on the study of semiring orders in a semiring.

2. Some semiring orders

In this section we introduce several semiring orders in semirings, and investigate some relations between them.

Proposition 3.1. *Let $(R, +, \cdot)$ be a semiring and $x, y \in R$. If we define a relation $<_+$ on R by $x <_+ y$ if and only if $x + y = y, x \neq y$, then it is a partial order.*

Proof. Clearly, $<_+$ is irreflexive. If $x <_+ y, y <_+ z$, then $x + y = y, y + z = z, x \neq y$, and $y \neq z$. Hence $x + z = x + (y + z) = (x + y) + z = y + z = z$. We claim that $x \neq z$. Assume that $x = z$. Then $z = y + z = y + x = x + y = y$, a contradiction. Hence $x <_+ z$, proving the proposition. □

Since $0 + x = x$ for any $x \in R$, $0 <_+ x$ for any $x \in R - \{0\}$. Hence $(R, <_+)$ is a poset with unique minimal element 0.

Proposition 3.2. *Let $(R, +, \cdot)$ be a commutative semiring and $x, y \in R$. If we define a relation $<_\cdot$ on R by $x <_\cdot y$ if and only if $xy = y, x \neq y$, then it is a partial order.*

Proof. Clearly, $<_\cdot$ is irreflexive. If $x <_\cdot y, y <_\cdot z$, then $xy = x, yz = y$, and hence $xz = (xy)z = x(yz) = xy = x$. We claim that $x \neq z$. Assume that $x = z$. Then $x = xy = zy = yz = y$, a contradiction, since $(R, \cdot)$ is commutative.

Since $0x = 0$ for any $x \in R$, $0 <_\cdot x$ for any $x \in R - \{0\}$. Hence $(R, <_\cdot)$ is a poset with unique minimal element 0.

Example 3.3. Let $R := \{0, 1, 2, 3\}$ be a set with the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	3	3

Then $(R, +, \cdot)$ is a non-commutative semiring. We can see that $2 <_\cdot 3, 3 <_\cdot 2$, but not $3 <_\cdot 3$, i.e., $<_\cdot$ is not a partial order on R.

Even though we obtained the posets as in Propositions 3.1 and 3.2, they were made by just one binary operation in semirings, while semirings were defined by two binary operations. This means the partial orders discussed in Proposition 3.1 and 3.2 have some defects.

The semiring order $<_R$ discussed in section 2 is an intersection of $<_+$ and $<_\cdot$ in a (not necessarily commutative) semiring, i.e., $<_R = <_+ \cap <_\cdot$.

Proposition 3.4. *Let $(R, +, \cdot)$ be a semiring with $xy = x \Rightarrow x + y = y, \forall x, y \in R$. Then $<_\cdot$ is a partial order on R.*

Proof. Let $x <_\cdot y, y <_\cdot z$. Then $xy = x, yz = y, x \neq y, y \neq z$. By assumption $x + y = y, y + z = z$. This means that $x <_R y, y <_R z$. Since $<_R$ is a partial order, $x <_R z$, proving $x \neq z$. Hence $x <_\cdot z$.

We define another semiring order on commutative semirings as follows.

Theorem 3.5. *Let $(R, +, \cdot)$ be a commutative semiring and $x, y \in R$. Define a binary relation $<_B$ on R by $x <_B y$ if and only if $x + y + xy = y, xy = x, x \neq y$. Then $(R, <_B)$ is a poset.*

Proof. Let $x <_B y$ and $y <_B z$. Then $x + y + xy = y, xy = x, x \neq y$ and $y + z + yz = z, yz = y, y \neq z$, and hence $xz = (xy)z = x(yz) = xy = x$. We claim that $x \neq z$. Assume that $x = z$. Then $z = y + z + yz = y + x + yx = x + y + xy = y$, a contradiction. Moreover,

$$\begin{aligned} x + z + xz &= x + (y + z + yz) + xz \\ &= (x + y + xz) + z + yz \\ &= (x + y + x) + z + yz \\ &= (x + y + xy) + z + yz \\ &= y + z + yz \\ &= z. \end{aligned}$$

Hence $x <_B z$.

Proposition 3.6. *If $(R, +, \cdot)$ is a commutative semiring, then $<_R \subseteq <_B$.*

Proof. If $(x, y) \in <_R$, then $x + y = y, xy = x, x \neq y$. Hence $x + y + xy = x + y + x = y + x = x + y = y$, proving $(x, y) \in <_B$.

Note that $<_R = <_B$ does not hold in non-commutative rings in general. See the following example.

Example 3.7. Let $R := \{0, 1, 2, 3\}$ be a set with the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	2
3	0	1	3	3

Then $(R, +, \cdot)$ is a non-commutative semiring. Since $3 \cdot 2 = 3, 3 + 2 + 3 \cdot 2 = 2$, $3 <_B 2$. But $3 + 2 = 3 \neq 2$ implies that $3 \not<_R 2$.

Proposition 3.8. *If $(R, +, \cdot)$ is a commutative semiring with $x + x = x$ for all $x \in R$, then $<_R = <_B$.*

Proof. If $x <_B y$, then $xy = x, x + y + xy = y, x \neq y$ and hence $x+y = x = x+y = x+y+x = x+y+xy = y$, proving that $x <_R y$. From Proposition 3.6, we obtain the proposition.

Proposition 3.9. *Let $(R, +, \cdot)$ be a semiring and $x, y \in R$. If $x <_B y, y <_B x$, then $x + x = y + y$.*

Proof. Let $x <_B y, y <_B x$. Then $x + y + xy = y, xy = x, x \neq y$ and $y + x + yx = x, yx = y, y \neq x$. It follows that $y = x + y + xy = x + y + x = 2x + y$. Similarly, $x = 2y+x$. Hence $2y = y+y = (2x+y)+y = 2x + 2y = 2y + 2x = 2x$.

For Boolean algebras it is unfortunately the case that $2x = 0$ for all x, so that the particular argument above will not work. It is also true however that $xy = yx$ so that $x + y + xy = y, y + x + yx = x$ immediately yields $y = x$. Thus the general condition is a sort of "weak commutativity rule": if $x + y + xy = y, y + x + yx = x$ and $xy = x, yx = y$, then $x = y$. This suggests that the condition "$x + y + xy = y, y + x + yx = x$ implies $x = y$" may also be interesting. Thus, e.g., if R contains a multiplicative identity 1_R, then the condition above becomes: $(1_R + x)(1_R + y) = 1_R + y, (1_R + y)(1_R + x) = 1_R + x$ implies $1_R + x = 1_R + y$.

Let $(R, +, \cdot)$ be a semiring and $x, y \in R$. Define a binary relation ρ_2 on R by

$$x \ \rho_2 \ y \Leftrightarrow 2x + y = y, x \neq y$$

A semiring $(R, +, \cdot)$ is called a ρ_2*-semiring* if ρ_2 is a partial order on R.

Example 3.10. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	3	1
3	3	3	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	3
3	0	1	3	2

Then it is easy to show that $(X, +, \cdot)$ is a ρ_2-semiring.

Example 3.11. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	2
3	0	1	3	2

Then the semiring $(X, +, \cdot)$ is not a ρ_2-semiring, since $2\rho_2 3$ and $3\rho_2 2$, but $2 \neq 3$.

Proposition 3.12. *If $(R, +, \cdot)$ is a commutative semiring, then $<_R \subseteq <_B \ \cap \rho_2$.*

Proof. If $x <_R y$, then $xy = x$ and hence $x+y+xy = x + y + x = x + y = y$. Hence $<_R \subseteq \rho_2$. By Proposition 3.6, the conclusion follows.

Proposition 3.13. *Let $(R, +, \cdot)$ be a semiring with $x + x = y + y \Rightarrow x = y, \forall x, y \in R$. Then $(R, +, \cdot)$ is a ρ_2-semiring.*

By Proposition 3.13, we see that (R, ρ_2) is a poset.

3. Order computations

Proposition 4.1. *Let $(R, +, \cdot)$ be a semiring. If $<_R$ is a semiring order, then $x \not<_R 2x$ for any $x \in R$.*

Proof. Assume that there exists $x \in R$ such that $x <_R 2x$. Then $x + 2x = 2x, x(2x) = x$, but $x \neq 2x$. It follows that $x = x(2x) = x(x + x) = x^2 + x^2 = 2x^2$, i.e., $x = 2x^2$. Hence $x = 2x^2 = x(2x) = x(x+2x) = x^2+2x^2 = x^2 + x$. Since $2x^2 = x$, we obtain $2x^3 = x(2x^2) = x^2$ and hence $4x^3 = 2x^3 + 2x^3 = x^2 + x^2 = 2x^2 = x$, i.e., $4x^3 = x$. Since $x + 2x = 2x$, i.e., $3x = 2x$, we have $3x^3 = 2x \cdot x^2 = 2x^3 = x^2$. Thus $3x^3 = x^2$. Moreover, if we add x to each side of $3x = 2x$, then $4x = 3x + x = 2x + x = 3x$. If we multiply $4x = 3x^3$ by x^2, then $4x^3 = 3x^3$. Hence we obtain $x^2 = 3x^3 = 4x^3 = x$. It follows that $x = 2x^2 = 2x$, a contradiction.

Theorem 4.2. *Let $(R, +, \cdot)$ be a semiring and let $<_R$ be a semiring order. If $x \in R$ such that $2x <_R x$, then $(2n + 1)x = x, 2nx = 2x$ for any natural number n.*

Proof. Assume that $2x <_R x$. Then $2x + x = x, (2x)x = 2x$, but $2x \neq x$. It follows that $3x = x, 5x = 3x + 2x = x + 2x = 3x = x$. By induction, we obtain $(2n+1)x = x$ for any natural number n. Since $2x+x = x$, we have $x + (2x + x) = x + x$, i.e., $4x = 2x$. Hence $4x + 2x = 2x + 2x = 4x = 2x$. By induction, we have $2nx = 2x$ for any natural number n.

Proposition 4.3. *Let $(R, +, \cdot)$ be a semiring. If $<_R$ is a semiring order, then $x \not<_R x^2$ for any $x \in R$.*

Proof. Assume that there exists $x \in R$ such that $x <_R x^2$. Then $x + x^2 = x^2, x \cdot x^2 = x$, but $x \neq x^2$. It follows that $x = x^3 = (x + x^2)x = x^2 + x^3 = x^2 + x = x^2$, a contradiction.

Theorem 4.4. *Let $(R, +, \cdot)$ be a semiring and let $<_R$ be a semiring order. If $x \in R$ such that $x^2 <_R x$, then $nx^2 = x^2 = x^n$ for any natural number $n \geq 2$.*

Proof. Assume that $x^2 <_R x$. Then $x^2 + x = x, x^2 \cdot x = x^2$, but $x^2 \neq x$. It follows that $x^2 = x(x^2 + x) = x^3 + x^2 = x^2 + x^2 = 2x^2$ and hence $3x^2 = 2x^2 + x^2 = x^2 + x^2 = 2x^2 = x^2$. By induction, we obtain $nx^2 = x$ for any $n \geq 1$.

Now, $x^4 = x \cdot x^3 = x \cdot x^2 = x^3 = x^2$ and $x^5 = x \cdot x^4 = x \cdot x^2 = x^3 = x^2$. By induction, we obtain $x^n = x^2$ for any natural number $n \geq 2$.

Proposition 4.5. *Let $(R, +, \cdot)$ be a commutative semiring and let $<_R$ be a semiring order. If $x \in R$ such that $xy <_R x$, then $xy + x^n = x^n$ for any natural number n.*

Proof. Assume that $xy <_R x$. Then $xy + x = x, xy \cdot x = xy$, but $xy \neq x$. Since R is commutative, $x^2y = xy$. It follows that $x^2y + x^2 = x(xy + x) = x^2$, i.e., $xy + x^2 = x^2$, and hence $x^2y + x^3 = x^3$. Similarly, we obtain $xy + x^4 = x^2y + x^4 = x(xy + x^3) = x^4$. By induction, we have $xy + x^n = x^n$.

Next, we perform some computations as examples involving the semiring order $<_B$ on a commutative semiring R.

Proposition 4.6. *Let* $(R,+,\cdot)$ *be a commutative semiring. If* $<_B$ *is a semiring order, then* $x \not<_B 2x$ *for any* $x \in R$.

Proof. Assume that there exists $x \in R$ such that $x <_B 2x$. Then $x+2x+x(2x) = 2x, x(2x) = x$, but $x \neq 2x$. It follows that $3x+2x^2 = 2x, 2x^2 = x$ and hence $4x = 2x$. Thus $2x = 2x^2+2x^2 = 4x^2 = 2x^2 = x$, a contradiction.

Proposition 4.7. *Let* $(R,+,\cdot)$ *be a commutative semiring and let* $<_B$ *be a semiring order. If* $x \in R$ *such that* $2x <_B x$*, then* $5nx = nx$ *for any natural number* n.

Proof. Assume that $2x <_B x$. Then $2x+x+2x\cdot x = x, (2x)x = 2x$, but $2x \neq x$. It follows that $3x+2x^2 = x, 2x^2 = 2x$, proving that $5x = x$. By induction, we obtain $5nx = x$ for any natural number n.

Proposition 4.8. *Let* $(R,+,\cdot)$ *be a commutative semiring and let* $<_B$ *be a semiring order. If* $x \in R$ *such that* $x <_B x^2$*, then* $x^{2n} = x^2, x^{2n+1} = x$ *for any natural number* n.

Proof. Assume that $x <_B x^2$. Then $x+x^2+x\cdot x^2 = x^2, x\cdot x^2 = x$, but $x \neq x^2$. It follows that $x^2 = x+x^2+x^3 = x+x^2+x = 2x+x^2$ and $x^3 = x(2x+x^2) = 2x^2+x^3 = 2x^2+x$. Hence $x^4 = x(2x^2+x) = 2x^3+x^2 = 2x+x^2 = x^2$ and $x^5 = x\cdot x^4 = x\cdot x^2 = x^3 = x, x^6 = x^2, x^7 = x^3 = x$ consequently, proving the proposition.

Proposition 4.9. *Let* $(R,+,\cdot)$ *be a commutative semiring and let* $<_B$ *be a semiring order. If* $x \in R$ *such that* $x^2 <_B x$*, then* $(2n+1)x^2 = x^2, (2n)x^2 = 2x^2$ *for any natural number* n.

Proof. Assume that $x^2 <_B x$. Then $x^2+x+x^2\cdot x = x, x^2\cdot x = x^2$, but $x \neq x^2$. It follows that $x = x^2+x+x^3 = 2x^2+x$ and hence $x^2 = 2x^3+x^2 = 3x^2$. Thus $5x^2 = 2x^2+3x^2 = 2x^2+x^2 = x^2$. By induction, we have $(2n+1)x^2 = x^2$ for any natural number n.

Since $4x^2 = 3x^2+x^2 = x^2+x^2 = 2x^2$, $6x^2 = 4x^2+2x^2 = 4x^2 = 2x^2$ and $8x^2 = 6x^2+2x^2 = 2x^2$. By induction, we obtain $(2n)x^2 = 2x^2$ for any natural number n.

References

[1] P. J. Allen, Proc. Amer. Math. Soc. **21** (1969), 412.

[2] K. Głazek, A guide to the literature on semirings and their applications in mathematics and information sciences with complete bibliograph (Kluwer Academic, Dordrecht, 2002).

[3] J. S. Golan, Semirings and their applications (Kluwer Academic, Dordrecht, 1999).

[4] U. Hebisch and H. J. Weinert, Semirings, algebraic theory and applications in computer science, (World Scientific, Singapore, 1993).

[5] Y. B. Jun, J. Neggers and H. S. Kim, Czech. Math. J. **48** (1998), 669.

[6] J. Neggers, Y. B. Jun and H. S. Kim, Czech. Math. J. **49** (1999), 127.

[7] J. Neggers and H. S. Kim, Basic posets (World Scientific Pub., Singapore, 1998).

Professor Jeong Soon Han teaches applied mathematics at the College of Science and Technology of Hanyang University since 1988. She received a B.S.(1979), an M.S.(1981) and a Ph.D.(1986) in Mathematics from Hanyang University. She has taught various courses such as Calculus, Topology, Topological Geometry, History of Mathematics, Manifold. She translated “Using History to Teach Mathematics” into Korean. She has also authored 39 articles in domestic and abroad academic journals. She has been active as a member of National Council of Teachers of Math, Korean Mathematical Society, Korea Society of Educational Studies in Mathematics. She won Best Teacher Award from Hanyang University several times.

Dr. Hee Sik Kim is working at Dept. of Mathematics, Hanyang University as a professor. He has received his Ph.D. at Yonsei University. He has publised a book, Basic Posets with professor J. Neggers, and published 143 papers in several journals. He is working as an (managing) editor of 3 journals. His mathematical research areas are BCK-algebras, fuzzy algebras, poset theory and theory of semirings, and he is reviewing many papers in this areas. He has concerned on martial arts, photography and poetry also.

Dr. Joseph Neggers received a Ph.D. from the Florida State University in 1963. After positions at the Florida State University, the University of Amsterdam, King‘s College (London, UK), and the University of Puerto Rico, he joined the University of Alabama in 1967, where he is still engaged in teaching, research and writing poetry often in a calligraphic manner as well as enjoying friends and family through a variety of media both at home and abroad. He has reviewed over 500 papers in Zentralblatt Math., and published 76 research papers, and published a book, Basic Posets, with Professor Kim. He has studied several areas: poset theory, algebraic graph theory and combinatorics and include topics which are of an applied as well as a pure nature.