Applied Mathematics & Information Sciences
An International Journal

A New Ranking Model Using the Power Method

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Received: Received March 01, 2011, Revised June 15, 2011, Accepted Sep. 13 2011.
Published online: 1 January 2012

Abstract: In a previous study, we presented two rankings, Ranking(I) and Ranking(II), and their mathematical structures. In the present paper, a new ranking, denoted as Ranking(III), is presented, and a solution that avoids the rank inversion problem that may occur when applying Ranking(I) or Ranking(II) is given.

Keywords: Ranking, Perron-Frobenius theorem, Irreducible matrix.

1. Review

In general, a ranking is obtained through either competition or trial for a certain set of elements. Such sets, which are referred to herein as *constructed sets* and are denoted by C, include baseball teams and students in a class. The process of determining the ranking usually takes the results of data for either competition or trial into account. For example, if the set C is a class of students and the competition or trial is an examination in some subject, the ranking will be based on the scores achieved by the students. The simplest method is to generate the ranking according to the score. This ranking scheme has the advantage of simplicity and provides an understanding of overall trends of a class. However, in a large-scale examination, say more than 100 students, it is not unusual for several students to achieve the same score, because such examinations designed to be easy to grade, e.g., multiple-choice examinations. As a result, several students will achieve the same score. In this situation, the grader would like to be able to provide precise grades for students who achieve the same scores. As a solution to this problem, we have presented two new ranking methods[**?**][**?**]. These two ranking methods are outlined below:

Ranking (I)
This ranking is constructed by comparing examinees. Each of rankings in C has the characteristic whereby the examinees who achieved high scores for each question have uniformly high rank.

Ranking (II)
This ranking is constructed by comparing examinees and questions. Each of the rankings in C has the characteristic whereby the examinees who achieved high scores for difficult questions receive a higher rank than any examinees who achieved low scores for difficult questions.

In the present paper, we present a method of generating Ranking(III), which is formed as the unified ranking of Ranking(I) and Ranking(II). Using Ranking(III), a new ranking that can reflect a more complicated situation than using either Ranking(I) or Ranking(II) is obtained.

As mentioned above, Ranking(I) and Ranking(II) can determine a clear ranking even among examinees who achieved identical scores. However, irregular results may simultaneously occur in both Ranking(I) and Ranking(II), such that the generated rankings do not depend on the actual data (rank inversion phenomenon). In the present study, we also present a technique by which to address this problem.

In Section 2, Ranking(I) and Ranking(II) are described in outline. In Section 3, we present the method of applying Ranking(I) and Ranking(II). In Section 4, the model of Ranking(III) is shown, and the characteristics of Ranking(III) are presented. In Section 5, we present the method of controlling the rank inversion phenomenon using actual data.

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2. Ranking(I) and Ranking(II)

This section presents simple explanations of Ranking(I) and Ranking(II) (detailed explanations are provided in [**?**]). A constructed set of elements, which are the objects of ranking through either competition or trial, is defined as

$C = \{c(1), c(2), \cdots, c(n)\}.$

The basis behind generating either Ranking(I) and Ranking(II) is the construction of an irreducible matrix $\mathbf{M} = \{m[i,j]\}$ comparing two values of $c(i)$ and $c(j)$ through either competition or trial and to apply the power method to $\mathbf{M}$.

2.1. Ranking(I)

In this subsection, a simple review of Ranking(I) is presented.

Let $\mathbf{M}_{(\mathrm{I})} = \{m_{(\mathrm{I})}(i,j)\}_{1\le i,j\le n}$ be a matrix generated by comparing two elements in C through either competition or trial. Each element in $\mathbf{M}_{(\mathrm{I})}$ is determined in accordance with the following conditions:

Matrix $\mathbf{M}_{(I)}$ is irreducible and primitive.
The value of $m_{(I)}(i,j)$ represents the non-negative ratio of superiority of $c(i)$ over $c(j)$.
The ratio of superiority is determined depending on a common rule through either competition or trial among elements in C

From (2.1b) of Condition 2.1, no element of matrix $\mathbf{M}_{(\mathrm{I})}$ is negative, and so $\mathbf{M}_{(\mathrm{I})}$ is non-negative. A matrix $\mathbf{M}_{(\mathrm{I})}$ that satisfies Condition 2.1 is referred to as *evaluation matrix(I) corresponding to C*. Then, we have the following remark and definition.

Remark.[**?**][**?**] From the Perron-Frobenius Theorem and Conditions 2.1-(2.1a) and 2.1-(2.1b), there exists an eigenvector $r_{\mathrm{M}_{(\mathrm{I})}} = {}^{\mathrm{T}}(x_1, x_2, \cdots, x_n)$, the elements of which are all positive, that corresponds to the largest positive eigenvalue $\lambda_{\mathrm{M}_{(\mathrm{I})}}$ of $\mathbf{M}_{(\mathrm{I})}$.

Definition 1.*The eigenvector $r_{\mathrm{M}_{(\mathrm{I})}}$, denoted in* Remark 2.1*, is referred to as the ranking vector corresponding to matrix $\mathbf{M}_{(\mathrm{I})}$ and is normalized with respect to l_2-norm.*

In the present study, each element in the initial vector is equal to 1 in the application of the power method. Next, the properties of each element in the ranking vector are given.

2.1.1. Process of generating the ranking vector for $\mathbf{M}_{(\mathrm{I})}$

In this subsection, the mathematical meaning of each element in the ranking vector during the process of generating the ranking vector is described. From (2.1a) of Condition 2.1, we can generate the ranking vector for $\mathbf{M}_{(\mathrm{I})}$ using the power method. Then, the initial vector is given as $r_0 = {}^{\mathrm{T}}(1,1,\cdots,1)$ and

$$\mathbf{M}_{(\mathrm{I})} r_0 \equiv r_1 = {}^{\mathrm{T}}\big(r_1(1),\ r_1(2),\ \cdots,\ r_1(n)\big). \tag{1}$$

In Eq. (1), the vector $p_{[1]\mathrm{M}_{(\mathrm{I})}}$ is calculated as follows:

$$\begin{aligned} p_{[1]\mathrm{M}_{(\mathrm{I})}} &= \frac{r_1}{||r_1||_2} \\ &= {}^{\mathrm{T}}\big(p_{[1]\mathrm{M}_{(\mathrm{I})}}(1),\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(2), \cdots,\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(n)\big). \end{aligned}$$

An entry $p_{[1]\mathrm{M}_{(\mathrm{I})}}(i)$ in $p_{[1]\mathrm{M}_{(\mathrm{I})}}$ is referred to as the *first potential for $c(i)$ in C*, and $p_{[1]\mathrm{M}_{(\mathrm{I})}}$ is referred to as the *first potential vector for $\mathbf{M}_{(\mathrm{I})}$*. Elements $p_{[1]\mathrm{M}_{(\mathrm{I})}}(i)$ $(i = 1, \cdots n)$ in $p_{[1]\mathrm{M}_{(\mathrm{I})}}$ represent the total degree of superiority of $c(i)$ to other elements $c(j)$ (including the superiority of $c(i)$ to $c(i)$). Then, calculating $\mathbf{M}_{(\mathrm{I})} p_{[1]\mathrm{M}_{(\mathrm{I})}}$, we obtain

$$\begin{aligned} \mathbf{M}_{(\mathrm{I})} p_{[1]\mathrm{M}_{(\mathrm{I})}} &= {}^{\mathrm{T}}\begin{pmatrix} \sum_{k=1}^{n} m_{(\mathrm{I})}(1,k)\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(k), \cdots, \\ \sum_{k=1}^{n} m_{(\mathrm{I})}(n,k)\ p_{[1]\mathrm{M}_{(\mathrm{I})}}(k) \end{pmatrix} \\ &\equiv r_2 = {}^{\mathrm{T}}\big(r_2(1),\ r_2(2),\ \cdots,\ r_2(n)\big). \end{aligned} \tag{2}$$

Thus, the value of $r_2(i)$ in r_2, where the corresponding element $c_2(i)$ has a high rate of superiority to other elements having high first potentials, becomes characteristically larger than that of $r_2(j)$, where the corresponding element $c_2(j)$ has a high rate of superiority to other elements having low first potentials. As in the case of generating $p_{[1]\mathrm{M}_{(\mathrm{I})}}$, $p_{[2]\mathrm{M}_{(\mathrm{I})}}$ is obtained by normalizing r_2, referred to as *the second potential vector*. Thus, the characteristic of $\{r_2(i)\}$ $(i = 1, \cdots, n)$ mentioned above is retained by $\{p_{[2]\mathrm{M}_{(\mathrm{I})}}(i)\}$, $(i = 1, \cdots, n)$. This characteristic is usually satisfied in each stage of $p_{[3]\mathrm{M}_{(\mathrm{I})}}$, $p_{[4]\mathrm{M}_{(\mathrm{I})}}$,$\cdots$. Therefore, for the vector

$p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}} = {}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(1),\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(2),\ \cdots,\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(n)\big),$

we have the following property:

Property 1. The value of element $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(i)$ in $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}$, where the element corresponding to $c(i)$ has a high rate of superiority to the elements having high $(k-1)$ potentials $\{p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{I})}}(t)\}$, becomes larger than that of element $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}(j)$, where the element corresponding to $c(j)$ has a high rate of superiority to elements having low $(k-1)$ potentials $\{p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{I})}}(s)\}$.

The matrix $\mathbf{M}_{(\mathrm{I})}$ is assumed to be irreducible and primitive. From the iteration of these manners above, we can generate the ranking vector $r_{\mathrm{M}_{(\mathrm{I})}}$, defined in Definition 1, corresponding to the largest positive eigenvalue $\lambda_{\mathrm{M}_{(\mathrm{I})}}$. This iteration process is identical to generating the process of $r_{\mathrm{M}_{(\mathrm{I})}}$ by the power method. Therefore, we have

$\lim_{k\to\infty} p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}} = r_{\mathrm{M}_{(\mathrm{I})}}.$

We refer to

$p_{[\infty]\mathrm{M}_{(\mathrm{I})}} \equiv \lim_{k\to\infty} p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{I})}}$

as the *final potential* for $\mathbf{M}_{(\mathrm{I})}$. A vector $p_{[\infty]\mathrm{M}_{(\mathrm{I})}}$ is generated through the successive transition of each step's potentials for all elements in C. Thus, we obtain another property for $r_{\mathrm{M}_{(\mathrm{I})}}$ as follows:

Property 2. The value of $c(i)$ in $r_{\mathrm{M}_{(\mathrm{I})}}$ is determined based on its superiority to elements $\{c(j)\}$, which have relatively high potentials.

In the present paper, a ranking that is ordered according to the highest-value element in $r_{\mathrm{M}_{(\mathrm{I})}}$ is referred to as Ranking(I) for $\mathbf{M}_{(\mathrm{I})}$ in C.

2.2. Ranking(II)

In this section, Ranking(II) is described. Unlike Ranking(I), two constructed sets, $C = \{c(1), c(2), \cdots, c(n)\}$ and $D = \{d(1), d(2), \cdots, d(m)\}$, are needed in order to determine Ranking(II). A matrix $\mathbf{M}_{(\mathrm{II})}$ is generated by evaluating the superiority of $c(i)$, $(i = 1, \cdots, n)$ to $d(j)$, $(j = 1, \cdots, m)$ and then applying the Perron-Frobenius theorem. The conditions for generating $\mathbf{M}_{(\mathrm{II})}$ are as follows:

(2.2a) Matrix $\mathbf{M}_{(\mathrm{II})}$ is irreducible.
(2.2b) The value of $m_{(\mathrm{II})}(i, n+j)$ is the non-negative ratio of superiority of $c(i)$ to $d(j)$, and the value of $m_{(\mathrm{II})}(n+j, i)$ is the non-negative ratio of superiority of $d(j)$ to $c(i)$.
(2.2c) The superiority is determined by maintaining the conditions such that

$$m_{(\mathrm{II})}(i, n+j) + m_{(\mathrm{II})}(n+j, i) = h(const) > 0.$$

(2.2d) The ratio of superiority is determined based on a common rule for either competition or trial among all 1-paired elements that do not belong to the same set.
(2.2e) The ratio of superiority is assumed to be zero among all 1-paired elements that belong to the same set.

From Conditions 2.2-(2.2b) and 2.2-(2.2e), no element of $\mathbf{M}_{(\mathrm{II})}$ is negative, and so $\mathbf{M}_{(\mathrm{II})}$ is non-negative. A matrix $\mathbf{M}_{(\mathrm{II})} = \{m_{(\mathrm{II})}(i,j)\}_{1 \le i,j \le n+m}$, which satisfies Condition 2.2 is referred to as *evaluation matrix (II) for constructed sets C and D*. As for the case of evaluation matrix (I), we have the following remark and definition:

Remark. From the Perron-Frobenius Theorem and (2.2a), (2.2b), and (2.2e) of Condition 2.2, there exists an eigenvector $r_{\mathrm{M}_{(\mathrm{II})}} = {}^{\mathrm{T}}(x_1, x_2, \cdots, x_{n+m})$, the elements of which are all positive, that corresponds to the largest positive eigenvalue $\lambda_{\mathrm{M}_{(\mathrm{II})}}$ of $\mathbf{M}_{(\mathrm{II})}$.

Definition 2. *The vector* $r_{\mathrm{M}_{(\mathrm{II})}}$*, denoted in* Remark 2.2*, is referred to as the ranking vector corresponding to the matrix* $\mathbf{M}_{(\mathrm{II})}$.

Evaluation matrix $\mathbf{M}_{(\mathrm{II})}$ has the following form:

$$\mathbf{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}, \tag{3}$$

where

$$\mathbf{V} = \begin{pmatrix} m_{(\mathrm{II})}(1, n+1) & \cdots & m_{(\mathrm{II})}(1, n+m) \\ \vdots & \ddots & \vdots \\ m_{(\mathrm{II})}(n, n+1) & \cdots & m_{(\mathrm{II})}(n, n+m) \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} m_{(\mathrm{II})}(n+1, 1) & \cdots & m_{(\mathrm{II})}(n+1, n) \\ \vdots & \ddots & \vdots \\ m_{(\mathrm{II})}(n+m, 1) & \cdots & m_{(\mathrm{II})}(n+m, n) \end{pmatrix}.$$

The sizes of $\mathbf{V}$ and $\mathbf{W}$ are $[n \times m]$ and $[m \times n]$, respectively. Here, a property for $\mathbf{M}_{(\mathrm{II})}$ is given.

Property 3. Matrix $\mathbf{M}_{(\mathrm{II})}$ is not primitive and has a period of 2.

2.3. Characteristics of the ranking vector for $\mathbf{M}_{(\mathrm{II})}$

In this subsection, the mathematical properties of each element in ranking vector $r_{\mathrm{M}_{(\mathrm{II})}}$ are given. As is mentioned in Property 3, the matrix $\mathbf{M}_{(\mathrm{II})}$ has a period of 2. Therefore, we cannot obtain the characteristics of each element in $r_{\mathrm{M}_{(\mathrm{II})}}$ by the transition of successive potential in applying the power method for the case of $\mathbf{M}_{(\mathrm{I})}$. Therefore, we present the following theorems concerning $\mathbf{M}_{(\mathrm{II})}$.

Theorem 1. *For a matrix* $\mathbf{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}$*, matrices* $\mathbf{VW}$ *and* $\mathbf{WV}$ *are irreducible in* $\mathbf{M}^2_{(\mathrm{II})} = \begin{pmatrix} \mathbf{VW} & \mathbf{0} \\ \mathbf{0} & \mathbf{WV} \end{pmatrix}$.

Theorem 2. *If* $s_1 = {}^{\mathrm{T}}(x_1, \cdots, x_n, y_1, \cdots, y_m)$, $||s_1||_2 = 1$ *is the ranking vector for a matrix* $\mathbf{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}$ *corresponding to the largest positive eigenvalue* α*, then*

$$s_2 = {}^{\mathrm{T}}(x_1, \cdots, x_n, -y_1, \cdots, -y_m)$$

is also an eigenvector for $\mathbf{M}_{(\mathrm{II})}$ *corresponding to an eigenvalue* $-\alpha$.

The matrix $\mathbf{M}_{(\mathrm{II})}$ has a period of 2 and does not converge to the eigenvector corresponding to the largest positive eigenvalue α in the application of the power method. However, the form of the eigenvector s_2 corresponding to the eigenvalue -α was determined by Theorem 2. Then, the following corollary for $\mathbf{M}_{(\mathrm{II})}$ is taken in the process of applying the power method.

Corollary 1. *Let* $r_{\mathrm{M}_{(\mathrm{II})}} = {}^{\mathrm{T}}(x_1, x_2, \cdots, x_n, y_1, \cdots, y_m)$, $||r_{\mathrm{M}_{(\mathrm{II})}}||_2 = 1$ *be the ranking vector corresponding to the largest positive eigenvalue* α *for a matrix* $\mathbf{M}_{(\mathrm{II})} = \begin{pmatrix} \mathbf{0} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}$*. Then, if the power method is applied to* $\mathbf{M}_{(\mathrm{II})}$*, the following two vectors are obtained:*

$$\begin{cases} w_1 = \frac{1}{w_{f1}} {}^{\mathrm{T}}\big((c_1 + c_2)x_1, \cdots, \\ (c_1 + c_2)x_n, (c_1 - c_2)y_1, \cdots, (c_1 - c_2)y_m\big), \\ w_2 = \frac{1}{w_{f2}} {}^{\mathrm{T}}\big((c_1 - c_2)x_1, \cdots, \\ (c_1 - c_2)x_n, (c_1 + c_2)y_1, \cdots, (c_1 + c_2)y_m\big). \end{cases} \tag{4}$$

where w_{f1} *and* w_{f2} *are constants to normalize the vectors* w_1 *and* w_2*, respectively, with respect to* $l_2 - norm$.

If the following iteration, (5), is applied to evaluation matrix $\mathbf{M}_{(\mathrm{II})}$ with initial vector $u_0 = {}^{\mathrm{T}}(1,1,\cdots,1)$,

$$\begin{cases} v_{\mathrm{M}_{(\mathrm{II})}[\mathrm{k}+1]} \equiv \mathbf{M}_{(\mathrm{II})}\, u_{\mathrm{M}_{(\mathrm{II})}[\mathrm{k}]}, a \\ u_{\mathrm{M}_{(\mathrm{II})}[\mathrm{k}+1]} \equiv \dfrac{v_{\mathrm{M}_{(\mathrm{II})}[\mathrm{k}+1]}}{\|v_{\mathrm{M}_{(\mathrm{II})}[\mathrm{k}+1]}\|}, \end{cases} \tag{5}$$

$k = 0,1,\cdots n-1$

then the n-th potential for $\mathbf{M}_{(\mathrm{II})}$ is defined in a similar manner by applying the power method to $\mathbf{M}_{(\mathrm{I})}$ as follows:

$$p_{[\mathrm{n}]\mathrm{M}_{(\mathrm{II})}} \equiv u_{\mathrm{M}_{(\mathrm{II})}[\mathrm{n}]},$$

but $\lim_{n\to\infty} p_{[\mathrm{n}]\mathrm{M}_{(\mathrm{II})}} = p_{[\infty]\mathrm{M}_{(\mathrm{II})}}$ oscillates between w_1 and w_2 in Corollary 1. Then, from the process of generating w_1 and w_2 for $\mathbf{M}_{(\mathrm{II})}$, we can assume that, in Eq. (4), $c_1 > c_2$. Next, we have the following property:

Property 4. In Eq. (4) of Corollary 1, denoting

$$\begin{cases} w_1 = {}^{\mathrm{T}}\big(w_1(1), w_1(2), \cdots, w_1(n+m)\big), \\ w_2 = {}^{\mathrm{T}}\big(w_2(1), w_2(2), \cdots, w_2(n+m)\big), \end{cases}$$

where all $w_1(i),\ w_2(i),\ (i = 1,\cdots,n+m)$ are positive, the following equations are satisfied:

$$\begin{aligned} &\frac{1}{\sqrt{\sum_{v=1}^{n} w_1(v)^2}}\,{}^{\mathrm{T}}\big(w_1(1),\cdots,w_1(n)\big) \\ &= \frac{1}{\sqrt{\sum_{v=1}^{n} w_2(v)^2}}\,{}^{\mathrm{T}}\big(w_2(1),\cdots,w_2(n)\big) \\ &= \frac{1}{\sqrt{\sum_{v=1}^{n} x_v^2}}\,{}^{\mathrm{T}}(x_1,\cdots,x_n), \end{aligned}$$

$$\begin{aligned} &\frac{1}{\sqrt{\sum_{v=n+1}^{m} w_1(v)^2}}\,{}^{\mathrm{T}}\big(w_1(n+1),\cdots,w_1(n+m)\big) \\ &= \frac{1}{\sqrt{\sum_{v=n+1}^{m} w_2(v)^2}}\,{}^{\mathrm{T}}\big(w_2(n+1),\cdots,w_2(n+m)\big) \\ &= \frac{1}{\sqrt{\sum_{v=1}^{m} y_v^2}}\,{}^{\mathrm{T}}(y_1,\cdots,y_m). \end{aligned}$$

Since the k-th potential $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}$ for $\mathbf{M}_{(\mathrm{II})}$ in the process of applying the power method is denoted as follows:

$$\begin{aligned} &p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}} \\ &= {}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(1),\cdots,p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n), \\ &\quad p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+1),\cdots,p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+m)\big), \end{aligned}$$

each element of $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(i)$ $(1 \le i \le n)$ represents the k-th potential for $c(i)$ and is calculated as follows:

$$p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(i) = \sum_{v=1}^{m} m_{(\mathrm{II})}(i, n+v)\, p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{II})}}(n+v). \tag{6}$$

In Eq. (6), the value of $p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(i)$ indicates that the element $c(i)$, which has a high degree of superiority to $\{d(j)\}$ with high potentials $\{p_{(\mathrm{II})k-1}(n+v)\}_{1\le v\le m}$, is becoming larger. Similarly, the k-th potential of $d(j)$ $(n+1 \le j \le n+m)$ is

$$p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+j) = \sum_{v=1}^{n} m_{(\mathrm{II})}(n+j, v)\, p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{II})}}(v). \tag{7}$$

This means that the value of element $d(j)$, which has relatively high superiority to $\{c(i)\}$ having high potentials $\{p_{[\mathrm{k}-1]\mathrm{M}_{(\mathrm{II})}}(i)\}_{1\le i\le n}$, is becoming larger. From Corollary 1, the final potential $p_{[\infty]\mathrm{M}_{(\mathrm{II})}}$ oscillates between w_1 and w_2, and the ratios of the relation among the first through n-th elements in w_1 and w_2 are identical and the relation among the $(n+1)$-th through $(n+m)$-th elements in w_1 and w_2 are identical (see Property 4). Therefore, we can redefine the k-th potential for C, denoted by $p_{[k](C)}$, and k-th potential for D, denoted by $p_{[k](D)}$, as follows:

$$\begin{aligned} p_{[k](C)} &= \frac{1}{\sqrt{\sum_{v=1}^{n} p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(v)^2}}\,{}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(1), \\ &\qquad \cdots,\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n)\big), \\ p_{[k](D)} &= \frac{1}{\sqrt{\sum_{v=n+1}^{n+m} p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(v)^2}}\,{}^{\mathrm{T}}\big(p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+1), \\ &\qquad \cdots,\ p_{[\mathrm{k}]\mathrm{M}_{(\mathrm{II})}}(n+m)\big). \end{aligned} \tag{8}$$

Therefore, we have the final potential for C as $x_{(C)\mathrm{M}_{(\mathrm{II})}}$ and that for D as $y_{(D)\mathrm{M}_{(\mathrm{II})}}$, as follows:

$$x_{(C)\mathrm{M}_{(\mathrm{II})}} = \frac{1}{\sqrt{\sum_{v=1}^{n} x_v^2}}\,{}^{\mathrm{T}}(x_1,\cdots,x_n) = \lim_{k\to\infty} p_{[k](C)} \tag{9}$$

$$y_{(D)\mathrm{M}_{(\mathrm{II})}} = \frac{1}{\sqrt{\sum_{v=1}^{m} y_v^2}}\,{}^{\mathrm{T}}(y_1,\cdots,y_m) = \lim_{k\to\infty} p_{[k](D)} \tag{10}$$

Finally, from Eqs. (6), (7), (9), and (10), the following property of Ranking (II) for $\mathbf{M}_{(\mathrm{II})}$ is obtained:

Property 5. Among the elements belonging to C, the rank of element $c(i)$, which has a high superiority to $\{d(j)\}$ having high potential, is increasing, and among the elements belonging to D, the rank of element $d(j)$, which has a high superiority to $\{c(i)\}$ having high potential, is increasing.

3. Application of Ranking(I) and Ranking(II)

In this section, examples of the application of Ranking(I) and Ranking(II) are given. Table 1 lists actual data representing the distribution of scores achieved by 20 examinees on a certain examination. The examination was constructed with four questions and 10 points are assigned to each question. Let $C = \{c(1), c(2),\cdots,c(20)\}$ be the set of examinees that is ordered according to the total score. Let $D = \{d(1), d(2), d(3), d(4)\}$ be the set of questions, and let $s(i,j)$ be the points obtained by examinee $c(i)$ on question $d(j)$. The common problem in applying Ranking(I) and Ranking(II) is how the evaluation matrices ought to be created in accordance with the characteristics of the given data. Thus, the methods of generating $\mathbf{M}_{(\mathrm{I})}$ and $\mathbf{M}_{(\mathrm{II})}$ are presented in the next subsection.

3.1. Application of Ranking(I)

In this subsection, a method by which to create an evaluation matrix for Ranking(I) is presented. Based on the value $s(i,j)$, an evaluation matrix $\mathbf{M}_{(\mathrm{I})}$ is created using the following equation:

$$m_{(\mathrm{I})}(i,j) = \frac{1}{m}\frac{\sum_{v=1}^{m} s(i,v)}{\sum_{k=v}^{m}\{s(i,v)+s(j,v)\}}. \tag{11}$$

The value $m_{(\mathrm{I})}(i,j)$ indicates the superiority of $c(i)$ with respect to $c(j)$ considering the distribution of scores for each question between two examinees. The superiority of $c(i)$ with respect to $c(i)$ is 0.5. However, in calculating Eq. (11), if $c(i)$ obtained no points for question $d(k)$, then the superiority of $c(j)$, $(j \neq i)$ with respect to $c(i)$ is not dependent on the points obtained by $c(j)$, and the superiority of $c(j)$ with respect to $c(i)$ is equal to 1 for guestion $d(k)$. In order to avoid this situation, $s(i,j)$ is modified as follows:

$$s'(i,j) = s(i,j) + \mu, \quad \mu > 0 \tag{12}$$

and apply $s'(i,j)$ to Eq. (11). Determination of the value μ in Eq. (12) depends on the grader, and, in the present study, $\mu = 1$, which indicates 10% of the full score for each question, is assigned. From Conditions (11) and (12), all entries in $\mathbf{M}_{(\mathrm{I})}$ are positive, and consequently, matrix $\mathbf{M}_{(\mathrm{I})}$ satisfies Condition 2.1.

The column $r_{(\mathrm{I})}$ in Table 2 represents the ranking vector for $\mathbf{M}_{(\mathrm{I})}$ in applying Ranking(I) to Table 1, and the column Rank.(I) is the ranking according to the highest value among entries in $r_{(\mathrm{I})}$.

3.2. Application of Ranking(II)

Unlike the method of applying Ranking(I), an evaluation matrix $\mathbf{M}_{(\mathrm{II})}$ is created based on the superiority of examinee $c(i)$ with respect to question $d(j)$. The actual data used to create the $\mathbf{M}_{(\mathrm{II})}$ is based on the value $s'(i,j)$ instead of $s(i,j)$. This is because if $c(i)$ obtained no points for $d(j)$, the potential for $c(i)$ with respect to $d(j)$ does not increase regardless of the existence of $d(j)$'s potential in the power method. And then, $s'(i,j)$ is modified as follows:

$$s''(i,j) = \frac{s'(i,j)}{k+2\mu}, \tag{13}$$

where k is the full score each for question. Based on the value of $s''(i,j)$, the superiority of $c(i)$ with respect to $d(j)$ is represented as $s''(i,j)$ and the superiority of $d(j)$ with respect to $c(i)$ is represented as $1 - s''(i,j)$. Thus, an evaluation matrix $\mathbf{M}_{(\mathrm{II})}$ can be created and a ranking vector is generated. The column $r_{\mathrm{M}_{(\mathrm{II})}}$ in Table 2 lists the ranking vector and the column Rank.(II) lists the order which is ranked by the highest value among entries in $r_{\mathrm{M}_{(\mathrm{II})}}$. In columns Rank.(I) and Rank.(II), $*$ indicates that the rank inversion phenomenon corresponds to each ranking.

Table 1 Scores for each student.

	Ques.1	Ques.2	Ques.3	Ques.4	Total	Rank
c(1)	5	10	8	9	32	1
c(2)	8	9	6	9	32	1
c(3)	4	8	10	7	29	3
c(4)	5	8	7	9	29	3
c(5)	6	6	7	9	28	5
c(6)	6	8	5	8	27	6
c(7)	8	7	4	7	26	7
c(8)	8	5	4	9	26	7
c(9)	4	4	7	10	25	9
c(10)	5	4	8	8	25	9
c(11)	8	6	3	8	25	9
c(12)	3	8	7	6	24	12
c(13)	5	4	5	9	23	13
c(14)	5	4	8	5	22	14
c(15)	3	8	7	4	22	14
c(16)	0	7	7	7	21	16
c(17)	3	4	6	7	20	17
c(18)	5	0	5	9	19	18
c(19)	0	4	6	6	16	19
c(20)	4	3	4	5	16	19
Mean	4.75	5.85	6.2	7.55	24.35	

Table 2 Rankings for each evaluation matrix.

	$r_{\mathrm{M}_{(\mathrm{I})}}$	$r_{\mathrm{M}_{(\mathrm{II})}}$	Rank.(I)	Rank.(II)
c(1)	0.252706	0.279733	2	2
c(2)	0.253880	0.286006	1	1
c(3)	0.241885	0.257901	4	3
c(4)	0.244273	0.255280	3	4
c(5)	0.241634	0.249009	5	5
c(6)	0.237474	0.243814	6	6
c(7)	0.232595	0.242308	7	7
c(8)	0.231076	0.237102	8	8
c(9)	0.226400	0.217842	11	*12
c(10)	0.228909	0.224786	9	10
c(11)	0.226422	0.231695	10	9
c(12)	0.223749	0.218082	12	*11
c(13)	0.220657	0.206376	13	*15
c(14)	0.217735	0.207931	14	*13
c(15)	0.214706	0.206846	15	14
c(16)	0.195399	0.184965	*17	16
c(17)	0.208082	0.182806	*16	17
c(18)	0.187857	0.173491	*19	18
c(19)	0.175465	0.146674	20	20
c(20)	0.191024	0.157500	*18	19

4. Ranking(III)

In this section, the matrix $\mathbf{M}_{(\mathrm{III})}$, which is obtained by unifying $\mathbf{M}_{(\mathrm{I})}$ and $\mathbf{M}_{(\mathrm{II})}$, is defined. The size of matrix

$\mathbf{M}_{(\mathrm{III})}$ is the same as that of $\mathbf{M}_{(\mathrm{II})}$, and its shape is constructed from four block matrices as follows:

$$\mathbf{M}_{(\mathrm{III})} = \begin{pmatrix} \mathbf{C} & \mathbf{V} \\ \mathbf{W} & \mathbf{D} \end{pmatrix} \tag{14}$$

The matrix $\mathbf{M}_{(\mathrm{III})}$ in Eq.(14) satisfies the following conditions:

(3a) Each entry of two matrices $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{M}_{(\mathbf{III})}$ satisfies Conditions 2.1(a) through 2.1(d).
(3b) Each entry of two matrices $\mathbf{V}$ and $\mathbf{W}$ in $\mathbf{M}_{(\mathbf{III})}$ satisfies Conditions 2.2(b) and 2.2(c) and has a positive value.

The matrix $\mathbf{M}_{(\mathrm{III})}$ is referred to as *evaluation matrix(III) corresponding to C and D*. From Condition 4, since $\mathbf{M}_{(\mathrm{III})}$ is primitive, we have the following remark:

Remark. From the Perron-Frobenius Theorem and Condition 4, there exists an eigenvector $r_{\mathrm{M}_{(\mathrm{III})}} = {}^{\mathrm{T}}(z_1, z_2, \cdots, z_{n+m})$, the elements of which are all positive, corresponding to the largest positive eigenvalue $\lambda_{\mathrm{M}_{(\mathrm{III})}}$ of $\mathbf{M}_{(\mathrm{III})}$.

4.1. Processes of generating Ranking(III)

Since $\mathbf{M}_{(\mathrm{III})}$ is primitive, $r_{\mathrm{M}_{(\mathrm{III})}}$ is easily obtained using the power method. However, the characteristics of the processes of generating $r_{\mathrm{M}_{(\mathrm{III})}}$ cannot be expressed as the transition of the k-th potential, for example, in the manner that $\mathbf{M}_{(\mathrm{I})}$ and $\mathbf{M}_{(\mathrm{II})}$ are performed. Then, for an initial vector $u_{\mathrm{M}_{(\mathrm{III})}[0]} = {}^{\mathrm{T}}(1, \cdots, 1)$, we divide the vector $v_{\mathrm{M}_{(\mathrm{III})}[1]}$ as follows:

$$\begin{aligned} v_{\mathrm{M}_{(\mathrm{III})}[1]} &\equiv \mathbf{M}_{(\mathrm{III})} \mathbf{u}_{\mathrm{M}_{(\mathrm{III})}[0]} \\ &= \begin{pmatrix} \sum_{v=1}^{n} m_{(\mathrm{III})}(1,v) + \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(1,v) \\ \vdots \qquad\qquad \vdots \\ \sum_{v=1}^{n} m_{(\mathrm{III})}(n,v) + \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(n,v) \\ \sum_{v=1}^{n} m_{(\mathrm{III})}(n+1,v) + \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(n+1,v) \\ \vdots \qquad\qquad \vdots \\ \sum_{v=1}^{n} m_{(\mathrm{III})}(n+m,v) + \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(n+m,v) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{v=1}^{n} m_{(\mathrm{III})}(1,v) \\ \vdots \\ \sum_{v=1}^{n} m_{(\mathrm{III})}(n,v) \\ \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(n+1,v) \\ \vdots \\ \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(n+m,v) \end{pmatrix} + \begin{pmatrix} \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(1,v) \\ \vdots \\ \sum_{v=n+1}^{n+m} m_{(\mathrm{III})}(n,v) \\ \sum_{v=1}^{n} m_{(\mathrm{III})}(n+1,v) \\ \vdots \\ \sum_{v=1}^{n} m_{(\mathrm{III})}(n+m,v) \end{pmatrix} \end{aligned}$$

(16)

The first through n-th entries in the vector that is the first term in Eq. (15) represent the sum of superiority for $c(i)$, $(i = 1, \cdots, n)$ to elements $c(k)$, $(k = 1, \cdots, n)$, depending on the comparison of superiority among only elements in C. Furthermore, the $(n+1)$-th through $(n+m)$-th entries in the vector that is the first term in Eq. (15) represent the sum of superiority for $d(j)$, $(j = n+1, \cdots, n+m)$ to elements $d(s)$, $(s = n+1, \cdots, n+m)$. On the other hand, the vector that is the second term in Eq. (15) is the same as the vector $v_{\mathrm{M}_{(\mathrm{II})}[1]}$ in Eq. (5). Then, generating a vector $u_{\mathrm{M}_{(\mathrm{III})}[1]}$ by multiplying $v_{\mathrm{M}_{(\mathrm{III})}[1]}$ in Eq. (15) by $1/||v_{\mathrm{M}_{(\mathrm{III})}[1]}||$, the first term of the right-hand side is denoted as $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1]}$ and the second term of the right-hand side is denoted as $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1]}$. We then define $p_{\mathrm{M}_{(\mathrm{III})}[1]}$, which is referred to as the *first potential for* $\mathbf{M}_{(\mathrm{III})}$, as follows:

$$p_{\mathrm{M}_{(\mathrm{III})}[1]} \equiv u_{\mathrm{M}_{(\mathrm{III})}[1]} = p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1]} + p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1]} \tag{17}$$

Furthermore, each of the two vectors in Eq. (17), $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1]}$ and $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1]}$, is divided into two vectors as follows:

$$p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1]} = \begin{pmatrix} p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} \\ p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} \end{pmatrix}, \quad p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1]} = \begin{pmatrix} p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} \\ p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} \end{pmatrix} \tag{18}$$

The i-th entry of $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})}$ in Eq. (18) represents the total degree of superiority for $c(i)$ to any other elements belonging to D, and, conversely, the j-th entry of $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})}$ represents the total degree of superiority for $d(j)$ to any other elements belonging to C. Thus, each entry in $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1]}$ represents the total degree of superiority based on the comparison between elements of different sets C and D. Then, $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1]}$ is referred to as the *first external potential for* $\mathbf{M}_{(\mathrm{III})}$. Each entry in $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1]}$ represents the total degree of superiority based on the comparison between elements of the same set. Then, $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1]}$ is referred to as the *first internal potential for* $\mathrm{M}_{(\mathrm{III})}$.

Next, $v_{\mathrm{M}_{(\mathrm{III})}[2]}$ is defined as follows:

$$\begin{aligned} v_{\mathrm{M}_{(\mathrm{III})}[2]} &= \mathbf{M}_{(\mathbf{III})}\, \mathbf{u}_{\mathrm{M}_{(\mathrm{III})}[1]} \\ &= \begin{pmatrix} \mathbf{C} & \mathbf{V} \\ \mathbf{W} & \mathbf{D} \end{pmatrix} \left(p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1]} + p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1]}\right) \end{aligned}$$

$$= \begin{pmatrix} \mathbf{C} & \mathbf{V} \\ \mathbf{W} & \mathbf{D} \end{pmatrix} \left\{ \begin{pmatrix} p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} \\ p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} \end{pmatrix} + \begin{pmatrix} p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} \\ p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \mathbf{C}p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} + \mathbf{V}p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} \\ \mathbf{D}p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} + \mathbf{W}p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} \end{pmatrix} + \begin{pmatrix} \mathbf{C}p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} + \mathbf{V}p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} \\ \mathbf{D}p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{D})} + \mathbf{W}p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[1](\mathrm{C})} \end{pmatrix}. \tag{19}$$

Then, $u_{\mathrm{M}_{(\mathrm{III})}[2]}$ is defined by multiplying $v_{\mathrm{M}_{(\mathrm{III})}[2]}$ by $1/||v_{\mathrm{M}_{(\mathrm{III})}[2]}||$, the vector corresponding to the first term in Eq. (19) is defined as $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[2]}$ and the vector corresponding to the second term in Eq. (19) is defined as $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[2]}$. The vector $p_{\mathrm{M}_{(\mathrm{III})}[2]}$, referred to as the *second potential for* $\mathbf{M}_{(\mathbf{III})}$, is defined as follows:

$$p_{\mathrm{M}_{(\mathrm{III})}[2]} \equiv u_{\mathrm{M}_{(\mathrm{III})}[2]} = p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[2]} + p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[2]} \tag{20}$$

As in Eq. (18), the elements in $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[2]}$ and $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[2]}$ are divided between 1 through n elements and $(n+1)$ through $(n+m)$, respectively, as follows:

$$p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[2]} = \begin{pmatrix} p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[2](\mathrm{C})} \\ p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[2](\mathrm{D})} \end{pmatrix}, \quad p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[2]} = \begin{pmatrix} p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[2](\mathrm{C})} \\ p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[2](\mathrm{D})} \end{pmatrix} \tag{21}$$

From Eq. (21), the entries in $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[2]}$ and $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[2]}$, which corresponds to the element that has a high rate of superiority to elements having high first internal or external potentials, becomes larger. From the iteration with respect to $\mathbf{M}_{(\mathrm{III})}$ by Eq. (22), $u_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}]}$ and $p_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}]}$ are defined as follows:

$$\begin{cases} v_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]} \equiv \mathbf{M}_{(\mathrm{III})}\, u_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}]}, \\ u_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]} \equiv \dfrac{v_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]}}{||v_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]}||}, \\ p_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]} \equiv u_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]} \\ = p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]} + p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}+1]} \end{cases} \tag{22}$$

$k = 2, 3, \cdots$

Since the matrix $\mathbf{M}_{(\mathrm{III})}$ is primitive, an eigenvector $r_{\mathrm{M}_{(\mathrm{III})}}$ of $\mathbf{M}_{(\mathrm{III})}$, which is referred to as the ranking vector for $\mathbf{M}_{(\mathrm{III})}$, is taken from Eq. (22) as follows:

$$r_{\mathrm{M}_{(\mathrm{III})}} = \lim_{k\to\infty} p_{\mathrm{M}_{(\mathrm{III})}[\mathrm{k}]}$$

$$= p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]} + p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]} \tag{23}$$

Here, $r_{\mathrm{M}_{(\mathrm{III})}} = p_{\mathrm{M}_{(\mathrm{III})}[\infty]}$ is referred to as the ranking vector or the final potential vector for $\mathbf{M}_{(\mathrm{III})}$, and $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}$ is referred to as the final internal potential vector. In addition, $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}$ is referred to as the final external potential vector. From Eqs. (22) and (23), we have the following properties for $r_{\mathrm{M}_{(\mathrm{III})}}$.

Property 6. From the processes of generating $r_{\mathrm{M}_{(\mathrm{III})}}$, each value in $r_{\mathrm{M}_{(\mathrm{III})}}$ has the following properties:

(a) An element that has a high rate of superiority to other elements with high internal potentials becomes larger than an element that has a high rate of superiority to other elements with low internal potentials.
(b) An element that has high rate of superiority to other elements with high external potentials becomes larger than an element that has high rate of superiority to other elements with low external potentials.

4.2. Dependence of the final potential vector

In this subsection, the method for generating the indices that represent the rate of dependence of $p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}$ and $p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}$ over $r_{\mathrm{M}_{(\mathrm{III})}}$ is presented. Using these indices, the degree to which each value in $r_{\mathrm{M}_{(\mathrm{III})}}$ is influenced by the difference in superiority among elements belonging to the same set or that among elements belonging to different sets can be determined.

4.2.1. Method for generating indices of dependence

Based on the assumption, the following equation is satisfied:

$$||r_{\mathrm{M}_{(\mathrm{III})}}|| = ||p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]} + p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}|| = 1.$$

Then, we have the following property.

Property 7. Form Eqs.(22) and (23), the following equation is satisfied:

$$r_{\mathrm{M}_{(\mathrm{III})}} = \gamma_1\, p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]} = \gamma_2\, p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}, \quad \gamma_1,\ \gamma_2 > 0 \tag{24}$$

Thus, from Eq. (24), we obtain the following equation:

$$||r_{\mathrm{M}_{(\mathrm{III})}}|| = ||p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]} + p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}||$$

$$= ||p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}|| + ||p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}|| = 1.$$

Then, two indices of dependence, $\kappa^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}}$ and $\kappa^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}}$, for the final internal potential vector and the final external potential vector over $r_{\mathrm{M}_{(\mathrm{III})}}$, respectively, are defined as follows:

$$\kappa^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}} = ||p^{\mathrm{E}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}||, \quad \kappa^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}} = ||p^{\mathrm{I}}_{\mathrm{M}_{(\mathrm{III})}[\infty]}||$$

Example 1 is presented to show the κ^{E} and κ^{I} for given matrix $\mathbf{M}_{1(\mathrm{III})}$.

Example 1. The matrix $\mathbf{M}_{1(\mathrm{III})}$ is given as follows:

$$\mathbf{M}_{1(\mathrm{III})} = \begin{pmatrix} 0 & 0.1 & 0.1 & \frac{1}{6} & \frac{5}{6} \\ 0.1 & 0 & 0.1 & \frac{1}{3} & \frac{2}{3} \\ 0.1 & 0.1 & 0 & \frac{1}{2} & \frac{5}{12} \\ \frac{5}{6} & \frac{2}{3} & \frac{1}{2} & 0 & 0.1 \\ \frac{1}{6} & \frac{1}{3} & \frac{7}{12} & 0.1 & 0 \end{pmatrix}.$$

From the distributions of the superiority of $\mathbf{M}_{1(\mathrm{III})}$, the superiorities of different elements belonging to C are even valued and are equal to 0.1, and the superiority of an element with respect tot itself belonging to C is zero. Using the method of Ranking(III), ranking vectors $r_{\mathrm{M}_{1(\mathrm{III})}}$, $p^{\mathrm{E}}_{\mathrm{M}_{1(\mathrm{III})}[\infty]}$, and $p^{\mathrm{I}}_{\mathrm{M}_{1(\mathrm{III})}[\infty]}$ are generated as follows:

$$r_{\mathrm{M}_{1(\mathrm{III})}}$$

$$= {}^{\mathrm{T}}(0.375268, 0.403941, 0.410677, 0.621392, 0.375796)$$

$$p^{\mathrm{E}}_{\mathrm{M}_{1(\mathrm{III})}[\infty]}$$

$$= {}^{\mathrm{T}}(0.332046, 0.357416, 0.363377, 0.549821, 0.332513)$$

$$p^{\mathrm{I}}_{\mathrm{M}_{1(\mathrm{III})}[\infty]}$$

$$= {}^{\mathrm{T}}(0.0432222, 0.0465247,$$

$$0.0473006, 0.0715704, 0.0432833)$$

and, the indices of dependence, κ^{E} and κ^{I} are generated as follows:

$$\kappa^{\mathrm{E}} = ||p^{\mathrm{E}}_{\mathrm{M}_{1(\mathrm{III})}[\infty]}|| = 0.884823$$

$$\kappa^{\mathrm{I}} = ||p^{\mathrm{I}}_{\mathrm{M}_{1(\mathrm{III})}[\infty]}|| = 0.115177$$

5. Controlling the rank inversion

As mentioned in Section 3, the advantage of Ranking(I) and Ranking(II) is the ability to determine a clear ranking, even if examinees belonging in C obtain the same score. However, irregular results in rankings may also occur, such that generated rankings are not ordered to the actual scores achieved by examinees (see Table 2). Therefore, in this section, we will present methods for controlling the change of ranking with respect to both Ranking(I) and Ranking(II). In order to simplify the explanation of this method, we generate modified rankings, which completely solve the rank inversion problem, along with the data given in Tables 1 and 2.

Before presenting the method used to solve the rank inversion problem, we group examinees according to score. Let

$$G = \{G_1, G_2, \cdots, G_k\}$$

be the set of scores, and each element G_i is composed of examinees who achieved identical scores G_i. The order of elements in G is assumed to be arranged according to score. As such, set G is an ordered set having following properties:

$$G_i \cap G_j = \phi, \qquad \sum_{i=1}^{k} |G_i| = n, \tag{25}$$

where $|G_i|$ is the number of elements.

5.1. Controlled ranking method for Ranking(I)

In this section, the method of controlling the change in ranking with respect to Ranking(I), referred to as *Modified Ranking(I) method*, is presented. Dummy elements $c(n+1)$, $c(n+2), \cdots$ belonging in C are added and the rankings among $c(1), c(2), \cdots, c(n)$ are changed by controlling the ratio of superiority for $c(n+1)$, $c(n+2), \cdots$ to other elements $c(1), c(2), \cdots, c(n)$. Then, the ratio of superiority of dummy elements to other elements $c(1), c(2), \cdots, c(n)$ is assumed to be same, and the ratio of superiority between dummy elements is assumed to be even.

First, in order to incorporate the above assumption, the evaluation matrix $\mathbf{M}_{(\mathrm{I})}$ must be modified as follows:

$$\mathbf{M}_{(\mathrm{I})\mathrm{R}} = \begin{pmatrix} & & & f + w_{(\mathrm{I})}(1) & \cdots \\ & \mathbf{M}_{(\mathrm{I})} & & \vdots & \cdots \\ & & & f + w_{(\mathrm{I})}(n) & \cdots \\ f - w_{(\mathrm{I})}(1) & \cdots & f - w_{(\mathrm{I})}(n) & f & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{26}$$

where $\mathbf{M}_{(\mathrm{I})\mathrm{R}}$ is the modified evaluation matrix, and the value of f is the standard value obtained using the rule to create an evaluation matrix $\mathbf{M}_{(\mathrm{I})}$. In this case, the value of f is determined to be 0.5 from the characteristics of creating $\mathbf{M}_{(\mathrm{I})}$. The value $f + w_{(\mathrm{I})}(i)$ in matrix $\mathbf{M}_{(\mathrm{I})\mathrm{R}}$ represents the weight of examinee $c(i)$ with respect to dummy element $c(n+1)$. The steps of generating the matrix of $\mathbf{M}_{(\mathrm{I})\mathrm{R}}$ are $\mathrm{Step}_{(\mathrm{I})}1$ through $\mathrm{Step}_{(\mathrm{I})}4$, as follows:

$\mathbf{Step}_{(\mathrm{I})}\mathbf{1}$

Take one dummy element $c(n+1)$ and search the entry from the low value in $r_{\mathrm{M}_{(\mathrm{I})}}$ and select two entries, $c(x)$ and $c(y)$, $(x < y)$, for which the first rank inversion occurred.

$\mathbf{Step}_{(\mathrm{I})}\mathbf{2}$

Let $G(i)$ and $G(j)$, $(i < j)$, be elements in G such that $G(i)$ belongs $c(x)$ and $G(j)$ belongs $c(y)$. This means that rank inversion does not occur among examinees who belongs to $\cup_{z>j} G_z$ or among examinees who belong to $\cup_{i<z<j} G_z$. In this case, in order to control the rank inversion between $c(x)$ and $c(y)$, we add a uniform weight $d_{(\mathrm{I})[1]}$ to examinees $\{c(p)\}$ who belong to $\cup_{z=1}^{i} G_z$, which means that $c(p)$ is superior to $c(n+1)$, and add a uniform weight $-d_{(\mathrm{I})[1]}$ to examinees $\{c(q)\}$ who belong to $\cup_{z \geq j} G_z$, which means that $c(q)$ is inferior to $c(n+1)$, until the rank inversion between $c(x)$ and $c(y)$ is corrected within 3 significant digits. Similarly, we then select examinees, $c(u)$ and $c(v)$, $(u < v)$, for which the second rank inversion occurred. Let $G(k)$ and $G(l)$, $(k < l)$, be elements in G such that $G(k)$ belongs $c(u)$ and $G(l)$ belongs $c(v)$. This means that rank inversion does not occur among examinees who belong to $\cup_{z>l}$ or among examinees who belong to $\cup_{k<z<l} G_z$. In this case, in order to control the rank inversion between $c(u)$ and $c(v)$, we add a uniform weight $d_{(\mathrm{I})[2]}$ to examinees belonging to $\cup_{z=1}^{k} G_z$

and a uniform weight $-d_{(\mathrm{I})[2]}$ to examinees belonging to $\cup_{z \geq l} G_z$ until the rank inversion between $c(u)$ and $c(v)$ is corrected.

$\mathrm{Step}_{(\mathrm{I})}3$

If the rank inversion phenomenon among elements belonging C is not completed within the value of $f - w_{(\mathrm{I})}(n)$, $(i = 1, \cdots n)$ is nonnegative, add more dummy elements $c(n+2)$ as per Eq. (26).

$\mathrm{Step}_{(\mathrm{I})}4$

Repeat $\mathrm{Step}_{(\mathrm{I})}1$ and $\mathrm{Step}_{(\mathrm{I})}3$ until all examinees' rankings are modified in accordance with the order of actual scores.

Perform $\mathrm{Step}_{(\mathrm{I})}1$ through $\mathrm{Step}_{(\mathrm{I})}4$, two dummy elements were needed to solve the rank inversion problem in $r_{\mathrm{M}_{(\mathrm{I})}}$. Each weight $w_{(\mathrm{I})}(i)$ corresponding to $c(i)$ in Eq. (26) is generated as follows:

$$w_{(\mathrm{I})}(i) = \begin{cases} d_{(\mathrm{I})[1]} + d_{(\mathrm{I})[2]} & i = 1, \cdots 16 \\ d_{(\mathrm{I})[1]} - d_{(\mathrm{I})[2]} & i = 17,\ 18 \\ -d_{(\mathrm{I})[1]} - d_{(\mathrm{I})[2]} & i = 19,\ 20 \end{cases} \tag{27}$$

$$\text{where } d_{(\mathrm{I})[1]} = 0.047,\ d_{(\mathrm{I})[2]} = 0.24$$

From Eq. (27), the modified evaluation matrix $\mathbf{M}_{(\mathrm{I})\mathrm{R}}$ and the modified ranking vector $r'_{\mathrm{M}_{(\mathrm{I})}} = \{r'_{\mathrm{M}_{(\mathrm{I})}}(i)\}$ for $\mathbf{M}_{(\mathrm{I})\mathrm{R}}$ are determined. The column $r'_{\mathrm{M}_{(\mathrm{I})}}$ in Table 3 represents the value of $\{r'_{\mathrm{M}_{(\mathrm{I})}}(i)\}$ and MRank.(I) is the modified ranking according to the order of values among $\{r'_{\mathrm{M}_{(\mathrm{I})}}(i)\}$. Comparing Rank and MRank.(I), MRank.(I) is ordered depending on the actual scores and is determined even among examinees who achieved the same scores.

5.2. Controlled ranking method for Ranking(II)

In this subsection, the method of controlling the change of ranking with respect to Ranking(II), referred to as the *modified Ranking (II) method*, is presented. While Ranking(I) is determined among elements belonging to C, Ranking(II) is determined considering the relation between two elements that belong to different sets C and D. Therefore, in order to control the change in ranking, a method by which to determine the weight of each element is needed. As the target to order of elements is the examinee, we add a weight to each element belonging to C. Then, we rewrite the evaluation matrix $\mathbf{M}_{(\mathrm{II})}$ in Eq. (3) as follows:

$$\mathbf{M}_{(\mathrm{II})\mathrm{R}} = \begin{pmatrix} \mathbf{C} & \mathbf{V} \\ \mathbf{W} & \mathbf{0} \end{pmatrix}, \tag{28}$$

where $\mathbf{0}$ is the zero matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & w_{(\mathrm{II})}(1) & \cdots & \cdots & w_{(\mathrm{II})}(1) \\ w_{(\mathrm{II})}(2) & 0 & w_{(\mathrm{II})}(2) & \cdots & w_{(\mathrm{II})}(2) \\ w_{(\mathrm{II})}(3) & w_{(\mathrm{II})}(3) & 0 & \cdots & w_{(\mathrm{II})}(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(\mathrm{II})}(n) & w_{(\mathrm{II})}(n) & \cdots & w_{(\mathrm{II})}(n) & 0 \end{pmatrix} \tag{29}$$

Table 3 Modified Rankings of Ranking(I) and Ranking(II).

	$r'_{\mathrm{M}_{(\mathrm{I})}}$	$r'_{\mathrm{M}_{(\mathrm{II})}}$	Rank	MRank.(I)	MRank.(II)
c(1)	0.252768	0.280103	1	2	2
c(2)	0.253852	0.286299	1	1	1
c(3)	0.242784	0.258497	3	4	3
c(4)	0.244986	0.255870	3	3	4
c(5)	0.242548	0.249655	5	5	5
c(6)	0.238721	0.244496	6	6	6
c(7)	0.234223	0.242996	7	7	7
c(8)	0.232816	0.237832	7	8	8
c(9)	0.228495	0.218776	9	11	11
c(10)	0.230809	0.225668	9	9	10
c(11)	0.22853	0.232471	9	10	9
c(12)	0.226065	0.218499	12	12	12
c(13)	0.223202	0.206871	13	13	13
c(14)	0.220508	0.206185	14	14	14
c(15)	0.217729	0.205108	14	15	15
c(16)	0.199943	0.183410	16	16	16
c(17)	0.199398	0.181257	17	17	17
c(18)	0.180793	0.172004	18	18	18
c(19)	0.166957	0.145442	19	20	20
c(20)	0.181284	0.156158	19	19	19

From Eq. (28), the matrix $\mathbf{M}_{(\mathrm{II})\mathrm{R}}$ is one of the type of $\mathbf{M}_{(\mathrm{III})}$ in case of $\mathbf{D}$ is zero matrix in Eq.14. So the manner to add the weight to each element in C can be explained along with the characteristics of $\mathbf{M}_{(\mathrm{III})}$. In order to control Ranking(III), the rank inversion problem is addressed by adding various weights to the first internal potentials corresponding to the elements belonging to C. The method of controlling Ranking(II), denoted by MRank.(II), is given by $\mathrm{Step}_{(\mathrm{II})}1$ through $\mathrm{Step}_{(\mathrm{II})}3$.

$\mathrm{Step}_{(\mathrm{II})}1$

As in $\mathrm{Step}_{(\mathrm{I})}1$, the entry is searched from the lowest value in $r_{\mathrm{M}_{(\mathrm{II})}}$ and two entries, $c(x')$ and $c(y')$, $(x' < y')$, for which the first rank inversion occurred, are selected.

$\mathrm{Step}_{(\mathrm{II})}2$

Let $G(i')$ and $G(j')$, $(i' < j')$, be elements in G such that $G(i')$ belongs $c(x')$ and $G(j')$ belongs to $c(y')$. In this case, in order to control the rank inversion between $c(x')$ and $c(y')$, we add a uniform weight $d_{(\mathrm{II})[1]}$ to examinees belonging to $\cup_{z=1}^{i} G_z$ until the rank inversion between $c(x')$ and $c(y')$ is corrected within 3 significant digits. Similarly, we select examinees, $c(u')$ and $c(v')$, $(u' < v')$, for which the second rank inversion occurred. Let $G(k')$ and $G(l')$, $(k' < l')$ be elements in G such that $G(')$ belongs to $c(u')$ and $G(l')$ belongs to $c(v')$. This indicates that rank inversion did not occur among examinees who belong to $\cup_{z>l'}$ or among examinees who belong to $\cup_{k'<z<l'} G_z$. In this case, in order to control the rank inversion between $c(u')$ and $c(v')$, we add a uniform weight $d_{(\mathrm{II})[2]}$ to examinees belonging to $\cup_{z=1}^{k'} G_z$ until the

rank inversion between $c(u')$ and $c(v')$ is corrected. From $\mathrm{Step}_{(\mathrm{II})}1$ through $\mathrm{Step}_{(\mathrm{II})}2$, the values of $w_{(\mathrm{II})}(i)$ are obtained as follows:

$G(1)$	$w_{(\mathrm{II})}(1) = d_{(\mathrm{I})[1]} + d_{(\mathrm{I})[2]}$
$\vdots$	$\vdots$
$G(k')$	$w_{(\mathrm{II})}(u') = d_{(\mathrm{I})[1]} + d_{(\mathrm{I})[2]}$
$\vdots$	$\vdots$
$G(l')$	$w_{(\mathrm{II})}(v') = d_{(\mathrm{I})[1]}$
$\vdots$	$\vdots$
$G(i')$	$w_{(\mathrm{II})}(y') = d_{(\mathrm{I})[1]}$
$\vdots$	$\vdots$
$G(j')$	$w_{(\mathrm{II})}(x') = 0$
$\vdots$	$\vdots$

$\mathrm{Step}_{(\mathrm{II})}3$

Repeat $\mathrm{Step}_{(\mathrm{II})}1$ and $\mathrm{Step}_{(\mathrm{II})}2$ until the rankings of all examinees are modified in accordance with the order of actual scores.

Perform $\mathrm{Step}_{(\mathrm{II})}1$ through $\mathrm{Step}_{(\mathrm{II})}3$ and generate each weight $w_{(\mathrm{II})}(i)$ corresponding to $c(i)$ in Eq. (29) as follows:

$$w_{(\mathrm{II})}(i) = \begin{cases} d_{(\mathrm{II})[1]} + d_{(\mathrm{II})[2]} & i = 1, \cdots 11 \\ d_{(\mathrm{I})[1]} & i = 12,\ 13 \\ 0 & i = 14, \cdots, 20 \end{cases} \tag{30}$$

where $d_{(\mathrm{II})[1]} = 0.0023$, $d_{(\mathrm{I})[2]} = 0.00054$. From Eq. (**??**), the modified evaluation matrix $\mathbf{M}_{(\mathrm{II})\mathrm{R}}$ and the modified ranking vector $r'_{\mathrm{M}_{(\mathrm{II})}} = \{r'_{\mathrm{M}_{(\mathrm{II})}}(i)\}$ for $\mathbf{M}_{(\mathrm{II})\mathrm{R}}$ are determined. The column $r'_{\mathrm{M}_{(\mathrm{II})}}$ in Table 3 represents the value of $r'_{\mathrm{M}_{(\mathrm{II})}}$, and MRank.(II) is the modified ranking according to the order of the values among $\{r'_{\mathrm{M}_{(\mathrm{II})}}(i)\}$. Comparing Rank and MRank.(II), MRank.(II) depends on the actual scores and is determined even among examinees who achieved the same score.

6. Conclusions

In a previous paper, we presented Ranking(I) and Ranking(II) and their mathematical structures in [**?**]. We discussed the disadvantage whereby irregular results may occur in both Ranking(I) and Ranking(II), such that the generated rankings are not ordered according to the actual data for each element. In order to address this problem, we herein presented a method of generating Ranking(III) which is formed as the unified ranking of Ranking(I) and Ranking(II) and presented a solution by which to avoid rank inversion in applying Ranking(I) and Ranking(II) to given data. However, since Ranking(III) may be applied to the rank inversion problem as well as problems in various other fields, in a future study, we intend to examine various applications using Ranking(III).

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