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Homomorphisms of C^*-Algebras

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Abstract: In this note we give a straightforward proof of the fact that every continuous homomorphism from a C^*-algebra into a weakly sequential complete Banach algebra is a finite rank operator. We also study Dieudonne type homomorphisms of the unital C^*-algebras.

Keywords: C^*-algebra, homomorphism, V-algebra, Dieudonne operator.

1. Introduction

Recently, many authors have been interested in the structure of compact and weakly compact homomorphisms of Banach algebras [4,5,7]. In particular, homomorphisms of C^*-algebras have been studied extensively in the literature. In [5], Ghahramani proved that every compact homomorphism from a C^*-algebra is a finite rank operator. Extending this result, Galé-Ransford-White [4] proved that every weakly compact homomorphism from a C^*-algebra is a finite rank operator. Mathieu [7] give more elementary proof of the Galé-Ransford-White result.

Let K be a compact Hausdorff space and let $C(K)$ be the space of all continuous functions on K. It is well known [6] that an arbitrary bounded linear operator from $C(K)$ into a weakly sequentially complete Banach space is weakly compact. Generalizing this result, Akemann-Dodds-Gamlen [1] proved that an arbitrary bounded linear operator from a C^*-algebra into a weakly sequentially complete Banach space is weakly compact. Combining the Akemann-Dodds-Gamlen result with the Galé-Ransford-White result, we can assert that every continuous homomorphism from a C^*-algebra into a weakly sequential complete Banach algebra is a finite rank operator. In this note, we give more elementary proof of the last result without using of Akemann-Dodds-Gamlen Theorem. We also study Dieudonne type homomorphisms of the unital C^*-algebras.

2. C*-Algebras

Let X be a complex Banach space and let X^* be its dual. A sequence $(x_n)_{n\in N}$ in X such that $(\varphi(x_n))_{n\in N}$ is a Cauchy sequence of scalars for each $\varphi \in X^*$ is called a *weak Cauchy sequence*. Recall that the space X is said to be *weakly sequentially complete* if every weak Cauchy sequence has a weak limit. In this section, we prove the following

Theorem 1. *Every continuous homomorphism from a C^*-algebra into a weakly sequentially complete Banach algebra is of finite rank.*

For the proof we need some preliminary results.

Let A be an arbitrary complex unital Banach algebra with the unit element 1_A. We will denote by $S(A)$ the set of all normalized states on A, namely,

$$S(A) = \{\Phi \in A^* : \|\Phi\| = \Phi(1_A) = 1\}.$$

An element $h \in A$ is said to be *Hermitian* if $\Phi(h) \in R$ for all $\Phi \in S(A)$. It is well known [2, Corollary 10.13] that $h \in A$ is Hermitian if and only if $\|\exp(ith)\| = 1$ for all $t \in R$. For example, if A is a unital C^*-algebra, then $h \in A$ is Hermitian if and only if h is self-adjoint. Furthermore, each $a \in A$ can be written as $a = h + ik$, where h and k are self-adjoint elements of A.

By $Her(A)$ we will denote the set of all Hermitian elements of A. It can be seen that $Her(A)$ is a closed real subspace of A. The algebra A is said to be a *V-algebra* if each $a \in A$ is of the form $a = h + ik$, where $h, k \in Her(A)$. The Vidav-Palmer Theorem [2, Theorem

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38.14] states that a V-algebra with involution defined by $(h+ik)^* = h - ik$ is a C^*-algebra. Recall also that for an arbitrary $h \in Her(A)$,

$$\|h\| = \sup\{|\Phi(h)| : \Phi \in S(A)\} \quad (1)$$

(see [2, Theorem 10.17 and Lemma 38.3]).

Let A be an arbitrary complex Banach algebra. It is well known [3] that the second dual A^{**} of A can be equipped with two Banach algebra multiplications $\circ$ and $*$ (the first and the second Arens multiplication) which extend the original multiplication in A (canonically embedded into A^{**}). Namely, for $a \in A$, $\varphi \in A^*$, and $F, G \in A^{**}$ we set $< F \circ G, \varphi > = < F, G \cdot \varphi >$ and $< F * G, \varphi > = < G, \varphi \cdot F >$, where $G \cdot \varphi$ and $\varphi \cdot F$ are functionals on A defined by $< G \cdot \varphi, a > = < G, \varphi \cdot a >$ and $< \varphi \cdot F, a > = < F, a \cdot \varphi >$. If $F \circ G = F * G$ for every $F, G \in A^{**}$, then A is said to be *Arens regular*. For example, C^*-algebras are Arens regular [3].

Lemma 1. *Let A be a unital C^*-algebra and let B be an arbitrary complex Banach algebra. If there exists a contractive homomorphism $\omega : A \mapsto B$ with dense range, then B is also a C^*-algebra and B is $*$-isomorphic to a quotient C^*-algebra of A.*

Proof. Since $ker(\omega)$ is a closed two-sided ideal of A, $ker(\omega)$ is self-adjoint. Hence the quotient algebra $Aker(\omega)$ is a unital C^*-algebra. We can see that the induced mapping $\bar{\omega} : A\ker(\omega) \mapsto B$, defined by $\bar{\omega}(a + ker\omega) = \omega(a)$ is an contractive and injective homomorphism with dense range. Hence, we can suppose without loss of generality that ω is injective. Next, we will prove that B is a C^*-algebra and B is $*$-isomorphic to the algebra A. We can easily see that $\omega(1_A)$ is the unit element of B and $\|\omega(1_A)\| = 1$. Hence, B is unital. Now let $h \in Her(A)$. Since $\omega(\exp(ith)) = \exp(it\omega(h)), t \in R$, we have $\|\exp(it\omega(h))\| \leq \|\omega\| \|\exp(ith)\| \leq 1$, for all $t \in R$. It follows that $\omega(h) \in Her(B)$. Let us show that B is a V-algebra. To see this, let $b \in B$ be given. Then, there exists a sequence $(a_n)_{n\in N}$ in A such that $\omega(a_n) \to b$. Let $a_n = h_n + ik_n$ $(n = 1, 2, ...)$, where $(h_n)_{n\in N}$ and $(k_n)_{n\in N}$ are the sequences in Her(A). Then, we have that $\omega(h_n) + i\omega(k_n) \to b$. Hence, for an arbitrary $\varepsilon > 0$, there exists an integer N such that $\|\omega(h_n) - \omega(h_m) + i(\omega(k_n) - \omega(k_m))\| \leq \varepsilon$, for all $n, m > N$. Since $\{\omega(h_n) - \omega(h_m)\}$ and $\{\omega(k_n) - \omega(k_m)\}$ are in Her(B), it follows that for all $\Phi \in S(B)$, $|\Phi(\omega(h_n) - \omega(h_m))| \leq \varepsilon$, $|\Phi(\omega(k_n) - \omega(k_m))| \leq \varepsilon$. Taking into account (2.1), we obtain $\|\omega(h_n) - \omega(h_m)\| \leq \varepsilon$, $\|\omega(k_n) - \omega(k_m)\| \leq \varepsilon$. Since $Her(B)$ is a real Banach space, there exist Hermitian elements l and m in B such that $\omega(h_n) \to l$ and $\omega(k_n) \to m$. Consequently, we have $b = l + im$, where $l, m \in Her(B)$. Thus B is a V-algebra. By the Vidav-Palmer Theorem [2, Theorem 38.14], B is a C^*-algebra with the involution defined by $b^* = l - im$. Furthermore, for an arbitrary $a = h + ik \in A$, we have

$$\omega(a^*) = \omega(h - ik) = \omega(h) - i\omega(k)$$
$$= (\omega(h) + i\omega(k))^* = (\omega(h + ik))^* = \omega(a)^*.$$

Therefore, ω is a $*$-homomorphism. By [12, Corollary 1.2.6], ω is an isometry. Since ω has dense range, ω is a surjective isometry. Hence ω is a $*$-isomorphism. This completes the proof.

Proof(Proof of Theorem 1). Let A be C^*-algebra and let B be a weakly sequentially complete Banach algebra. Let $\omega : A \mapsto B$ be a continuous homomorphism. Since the space $\overline{\omega(A)}$ is weakly sequentially complete, we lose no generality if we assume that $\overline{\omega(A)} = B$. Furthermore, since A is Arens regular and B is a weakly sequentially complete, by [13, Theorem 4.1], B has the unit element 1_B. Let $(e_i)_{i\in I}$ be an approximate identity for A such that $\sup_i \|e_i\| \leq 1$. Then, $(\omega(e_i))_{i\in I}$ is a bounded approximate identity for B. It follows that $\omega(e_i) \to 1_B$. Let $A \oplus C$ be the C^*- unitization of A with the norm $|||a + \lambda||| = \sup_{\|b\|\leq 1} \|ab + \lambda b\|$ .Then the mapping $\tilde{\omega} : A \oplus C \mapsto B$, defined by $\tilde{\omega}(a + \lambda) = \omega(a) + \lambda 1_B$ is a homomorphism with dense range. Moreover, since $\tilde{\omega}(a + \lambda) = \lim_i \omega(ae_i + \lambda e_i)$, $\|\tilde{\omega}(a + \lambda)\| \leq \|\omega\| \sup_i \|ae_i + \lambda e_i\| \leq \|\omega\| \|a + \lambda\|$ .Hence, $\tilde{\omega}$ is bounded. This shows that ω can be extended to $A \oplus C$ as a continuous homomorphism. Therefore, we may assume that A has a unit element. Renorming B if necessary, we can assume that ω is contractive. By the preceding lemma, B is a C^*-algebra. But we know that weakly sequentially complete C^*-algebras are finite-dimensional [11, Proposition 2]. Hence, ω is a finite rank operator. The proof is complete.

3. Dieudonne Type Homomorphisms

Let X and Y be two Banach spaces and let $T : X \mapsto Y$ be a bounded linear operator. The operator T is said to be a *Dieudonne operator* if T sends weakly Cauchy sequences in X into weakly convergent ones (see [6]). For example, if either X or Y is a weakly sequentially complete, then every bounded linear operator $T : X \mapsto Y$ is a Dieudonne operator. Assume that the operator $\overline{T} : X\ker T \mapsto Y$ is defined by $\overline{T}(x + \ker T) = Tx$. One can easily see that if $\overline{T}$ is a Dieudonne operator, then so is T. The following example shows that the converse is not true in general.

Example Let G be a non-discrete locally compact abelian group and let $A(G)$ be the Fourier algebra of G. For a compact subset K of G, we denote by $A(K)$, the algebra of all functions on K which are the restrictions to K of the functions in $A(G)$ with the norm

$$\|f\|_{A(K)} = \inf\left\{ \|h\|_{A(K)} : h|_K = f \right\}.$$

Clearly, the algebra $A(K)$ can be identified with the quotient algebra $A(G)I_K$, where I_K is the largest closed ideal in $A(G)$ whose hull is K;$I_K = \{f \in A(G) : f(K) = \{0\}\}$ .Recall that K is said to be a *Helson set* if every $f \in C(K)$ is the restriction to K of a member of $A(G)$. It can be seen that if K is a Helson set, then $A(K)$ is isomorphic to $C(K)$. As is known [10, Chapter 5], there exists a Helson set in any non-discrete locally compact abelian group. Since $A(G)$ is a weakly sequentially complete, the canonical quotient map $\pi : A(G) \mapsto A(K)$ is a Dieudonne operator. Now, assume on the contrary that $\overline{\pi}$ is also a Dieudonne operator. Since $\overline{\pi}$ is the identity operator on $A(K)$, it follows that $A(K)$ is a weakly sequentially complete. But this is not possible if K is an infinite Helson set. We say that $T : X \mapsto Y$ is a *Dieudonne type operator* if $\overline{T}$ is a Dieudonne operator.

Proposition 1.*If $T : X \mapsto Y$ is a weakly compact linear operator, then T is a Dieudonne type operator.*

*Proof.*Assume that T is weakly compact. Let $\pi : X \mapsto X \ker T$ be the canonical quotient map. Then, $T = \overline{T} \circ \pi$. Since π is open, every bounded subset of $X/\ker T$ is the image of some bounded set in X. It follows that $\overline{T}$ is also a weakly compact operator. Thus we can assume that T is injective. Let us show that T is a Dieudonne operator. Let $(x_n)_{n\in N}$ be a weakly Cauchy sequence in X. Since the sequence $(x_n)_{n\in N}$ is bounded, there exists a subsequence $(x_{n_k})_{k\in N}$ such that $(Tx_{n_k})_{k\in N}$ is weakly convergent. Since $(Tx_n)_{n\in N}$ is a weakly Cauchy sequence, it follows that the sequence $(Tx_n)_{n\in N}$ converges weakly.

Let X be an infinite dimensional non-reflexive and weakly sequentially complete Banach space (for instance, such as $L^1(\mu)$). Then the identity operator on X is a Dieudonne operator but not weakly compact. Now assume that $T : X \mapsto Y$ is a Dieudonne operator. We remark that if X does not contain an isomorphic copy of ℓ^1, then T is weakly compact. To see this, let $(x_n)_{n\in N}$ be a bounded sequence in X. By Rosenthal's ℓ^1-theorem [9], the sequence $(x_n)_{n\in N}$ has a weakly Cauchy subsequence $(x_{n_k})_{k\in N}$. Since T is a Dieudonne operator, the sequence $(Tx_{n_k})_{k\in N}$ is weakly convergent.

Let A and B be two Banach algebras and let $\theta : A \mapsto B$ be a continuous homomorphism. We say that θ is a *Dieudonne type homomorphism* if $\overline{\theta}$ is a Dieudonne type operator.

Corollary 1.*Every Dieudonne type homomorphism from a unital C^*-algebra into a Banach algebra is of finite rank.*

*Proof.*Let $\theta : A \mapsto B$ be such a homomorphism. We can suppose without loss of generality that θ has dense range. Renorming B if necessary, we can assume that θ is contractive. By Theorem 2.1, B is a C^*-algebra and $\overline{\theta}$ is a $*$-isomorphism between $A \ker \theta$ and B. Let $(b_n)_{n\in N}$ be a weakly Cauchy sequence in B. Then, $\left(\overline{\theta}^{-1}(b_n)\right)_{n\in N}$ is a weakly Cauchy sequence in $A \ker \theta$. Since $\overline{\theta}$ is a Dieudonne operator, it follows that the sequence $(b_n)_{n\in N}$ converges weakly. Hence, B is a weakly sequential complete C^*-algebra. However, weakly sequential complete C^*-algebras are finite-dimensional [11, Proposition 2].

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