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Probabilistic Interpretation of Kober Fractional Integral of Non-Integer Order

Vasily E. Tarasov[1,*] *and Svetlana S. Tarasova*[2]

[1] Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia
[2] Faculty of Information Technologies and Applied Mathematics, Moscow Aviation Institute (National Research University), Moscow 125993, Russia

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Abstract: In this paper, probabilistic interpretation of the Kober fractional integration of non-integer order is proposed. We prove that the fractional integral, which is proposed by Kober, can be interpreted as an expected value of a random variable up to a constant factor. In this interpretation, the random variable describes dilation (scaling), which has the gamma distribution. The Erdelyi-Kober fractional integration also has a probabilistic interpretation. Fractional differential operators of Kober and Erdelyi-Kober type have analogous probabilistic interpretation. The proposed interpretation leads to a possibility of generalization of the fractional integration and differentiation by using continuous probability distributions.

Keywords: Kober fractional integral, fractional calculus, dilation operator, scale effect, distributed dilation, probability distribution, beta distribution.

1 Introduction

There are different types of fractional integrals and derivatives of non-integer orders, the most famous of which are the operators that are proposed by Liouville and Riemann, Grnwald and Letnikov, Riesz, Erdelyi and Kober, Caputo [1,2,3,4]. The history of the theory of fractional integrals and derivatives includes more than three hundred years of development [5,6,7,8]. These operators of non-integer order have a wide application in different sciences (for example, see the encyclopedia on fractional calculus and its applications, which will be published in 2019 in eight volumes [9]). These applications include fractional relaxation and oscillation, fractional diffusion and waves, spatial non-locality and fading memory, the openness of systems and dissipation, long-range interactions and spatial and frequency dispersion of power type and many others.

Various interpretations of fractional differentiation and integration such as physical interpretations [10,11,12,13,14], geometric interpretations [13,14,15,16,17], economic interpretation [18,19] and informatic interpretation [20] have been proposed. An important thing is the probabilistic interpretation of fractional derivatives [21,22] and fractional integrals [23].

In this article, we discuss probabilistic interpretation of the Kober fractional integration of non-integer order that has been proposed by Kober [24] in 1940 as a generalization of the well-known Riemann-Liouville fractional integration.

2 Kober fractional integration and its interpretation

The Riemann-Liouville fractional integral of the order $\alpha > 0$ [4] is defined by the equation

$$(I^{\alpha}_{RL,0+}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\,\varphi(\tau)d\tau. \tag{1}$$

* Corresponding author e-mail: tarasov@theory.sinp.msu.ru

Here it is assumed that the function $\varphi(t)$ is measurable on the interval $(0,t)$ and that the condition

$$\int_0^t |\varphi(\tau)|\, d\tau < \infty \tag{2}$$

holds. For positive integer $\alpha = n \in \mathbb{N}$, the fractional integral (1) coincides (see equation 2.1.3 of [4], p. 70) with the nth integral of the form

$$(I^{\alpha}_{RL,0+}\varphi)(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\ \varphi(t_n) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}\varphi(\tau)\, d\tau \tag{3}$$

As a generalization of the Riemann-Liouville fractional integral (1), new fractional integration of non-integer order has been suggested by Kober [24] in 1940. The Kober fractional integral [4], p. 106] is defined by the expression

$$\left(I^{\alpha}_{K;0+;\eta}\varphi\right)(t) = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t \tau^{\eta}\ (t-\tau)^{\alpha-1}\varphi(\tau)d\tau, \tag{4}$$

where $\alpha > 0$ is the order of integration and $\eta \in \mathbb{R}$. For the functions $\varphi(t) \in L_p(\mathbb{R}_+)$, where $1 \leq p < \infty$, and $\eta > (1-p)/p$, operator (4) is bounded [1], p.323. For $\eta = 0$, operator (4) can be expressed through the Riemann-Liouville integration (1) by the expression

$$\left(I^{\alpha}_{K;0+;1}\varphi\right)(t) = t^{-\alpha}\ (I^{\alpha}_{RL,0+}\varphi)(t). \tag{5}$$

Using expression [25], p. 296, in the form

$$\int_0^t \tau^{\eta}(t-\tau)^{\alpha-1}d\tau = t^{\alpha+\eta}\ B(\eta+1,\alpha), \tag{6}$$

where $B(\alpha,\beta)$ is the beta function, we get

$$\left(I^{\alpha}_{K;0+;\eta}1\right)(t) = \frac{1}{\Gamma(\alpha)}B(\eta+1,\alpha) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)}. \tag{7}$$

Equality (7) means that the kernel of the operator (4) can be considered as a density function (pdf) of beta-distribution up to a constant factor (7). In this case, to have a probabilistic interpretation of (4) we should interpret the variable τ as a random variable.

Let us give a probabilistic interpretation of the fractional integration (4). Making the change of variable by $x = \tau/t$, the Kober operator (4) is represented as

$$\left(I^{\alpha}_{K;0+;\eta}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 x^{\eta}\ (1-x)^{\alpha-1}\ \varphi(x\,t)dx. \tag{8}$$

Expression (8) gives a possibility to use the probability density function (pdf) of the beta-distribution [26,27,28,29] up to a constant factor (7), in the form

$$f_{\alpha;\beta}(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1} \tag{9}$$

for $x \in [0,1]$ and $f_{\alpha;\beta}(x) = 0$ if $x \notin [0,1]$, where $B(\alpha,\beta)$ is the beta function. Note that the beta distribution describes the fraction of the sum of two terms that fall on each of them, if the terms are random variables that have a gamma distribution. Using (9), the Kober fractional integral is represented by the equation

$$\left(I^{\alpha}_{K;0+;\eta}\varphi\right)(t) = \frac{\Gamma(\eta+\alpha+1)}{\Gamma(\eta+1)}\int_0^1 f_{\eta+1;\alpha}(x)\ \ \varphi(x\,t)dx. \tag{10}$$

It should be emphasized that expression (10) contains $\varphi(x\,t)$ instead of $\varphi(x)$. Therefore we can consider the variable $x > 0$ as a random variable that describes scaling (dilation), which has the gamma distribution. To describe the change of scale (dilation) we can use the operator S_x (see [1], p. 95-96, [4], p. 11) such that

$$(S_x\varphi)(t) = \varphi(x\,t), \tag{11}$$

where $x>0$. It is known that the dilation of Euclidean geometric figures changes the size, when the shape is not changed. In physics and economics, the dilation is the change of scale of objects and processes. Using the scaling operator (11), the Kober fractional integral is represent by the equation

$$\left(I^{\alpha}_{K;0+;\eta}\varphi\right)(t)=\frac{\Gamma(\eta+\alpha+1)}{\Gamma(\eta+1)}\int_0^1 f_{\eta+1;\alpha}(x)\,(S_x\varphi)(t)dx. \quad (12)$$

We see from equation (12) that this operator can be interpreted as an expected value of a random variable $x>0$ that described the scaling and has the beta distribution up to numerical factor (7). Therefore we have the case that the interpretation of the Kober integrals is related with probability theory.

Since the Riemann-Liouville fractional integral (1) can be expressed through the Kober fractional integral with $\eta=0$ multiplied by a power function t^{α}, then we can use the probabilistic interpretation for the Riemann-Liouville integration of non-integer orders.

As a result, expression (12) gives a possibility to state that the Kober operator (4) can be considered as a continuously-distributed dilation operator, in which the scaling variable has the beta distribution up to a constant factor (7). Analogously, we can give the probabilistic interpretation for the Kober fractional differentiation [4], p. 108.

3 Probabilistic interpretation and generalization of fractional integrals and derivatives

The proposed interpretation of the fractional-order integration allows us to generalize the fractional integrations of non-integer and integer orders. We can define general form of differential and integral operators with continuously-distributed scaling (see Section 9 of [31]). For example, the generalized integral operator of continuously-distributed scaling (dilation) is defined [31] by the expression

$$(I_S\varphi)(t)=\int_0^{\infty} f_S(x)\,(S_x\varphi)(t)\,dx=\int_0^{\infty} f_S(x)\ \varphi(x\,t)\,dx, \quad (13)$$

where $f_S(x)\geq 0$ is the probability density function such that

$$\int_0^{\infty} f_S(x)dx=1. \quad (14)$$

In equation (13) it is assumed that the integral $\int_0^{\infty} f_S(x)\ |\varphi(x)|\ dx$ converges, where $\varphi(x)$ and $f_S(x)$ are piecewise continuous or continuous functions on $\mathbb{R}$.

If the density function $f_S(x)$ describes the beta distribution, then operator (13) is the Kober fractional integral up to numerical factor. In general, we can use other type of distributions (for example, see [26,27,28,29]). For example, it is possible to use the Weibull distribution with pdf of the form

$$f_W(x)=\begin{cases} a\,\lambda x^{a-1}\exp(-\lambda\, x^a) & x>0,\\ 0 & x\leq 0.\end{cases} \quad (15)$$

In function (15), the parameter $a>0$ describes the shape of distribution and, $\theta=\lambda^{-1/a}$ $(\lambda>0)$ describes the scale. Note that this distribution is applied to describe a particle-size distribution in [30]. We can use the gamma distribution, for which the probability density function is

$$f_G(x)=\begin{cases} \frac{\lambda^a\, x^{a-1}}{\Gamma(a)}\,exp(-\lambda\, x) & x>0,\\ 0 & x\leq 0,\end{cases} \quad (16)$$

where the parameters $a>0$ and $\theta=1/\lambda$ describe the shape and scale respectively.

Note that one example of operator (13), which is a generalization of the Kober integral, is the Erdelyi-Kober fractional integral [4], p. 105, such that

$$\left(I^{\alpha}_{EK;0+;\sigma,\eta}\varphi\right)(t)=\frac{\sigma\,t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_0^t \tau^{\sigma(\eta+1)-1}\,(t^{\sigma}-\tau^{\sigma})^{\alpha-1}\varphi(\tau)d\tau, \quad (17)$$

where $\alpha>0$ is the order of integration. The operator (17) is bounded for $\varphi(\tau)\in L_p(\mathbb{R}_+)$ where ≥ 1, $\eta>(p\sigma-1)/(p\sigma)$, [1], p.323. In the caser $\sigma=1$, operator (17) is represented in the form of the Kober operator (4). Operator (17) can be represented by equation (13) up to a constant factor in the form

$$\left(I^{\alpha}_{K;0+;\eta}\varphi\right)(t)=\frac{\Gamma(\eta+\alpha+1)}{\Gamma(\eta+1)}\int_0^1 f_{EK}(x)\,(S_x\varphi)(t)dx, \quad (18)$$

where the probability density function is

$$f_{EK}(x) = \frac{\sigma}{B(\eta+1,\alpha)} x^{\sigma(\eta+1)-1} (1-x^{\sigma})^{\alpha-1}. \quad (19)$$

For $\sigma = 1$, the density function (19) describes the beta distribution (9).

Using the suggested integral operator (13), we propose new fractional differential operators derivatives of arbitrary orders, in which the scaling variable has continuous probability distributions. The differentiation of integer order $n \in \mathbb{N}$ with distributed scaling is defined be the expression

$$(D_S^n \varphi)(t) = \left(I_S \varphi^{(n)}\right)(t) = \int_0^{\infty} f_S(x)\, \varphi^{(n)}(x\,t)\, dx, \quad (20)$$

where $f_S(x) \geq 0$ is the probability density function. The derivatives and integrals of non-integer orders, in which dilation is described by continuous probability distributions, can be defined analogously (for details see Section 9 of [31]). For example, the Caputo fractional derivative $\left(D^{\alpha}_{C,0+}\varphi\right)(t)$ (or another type of fractional derivatives) of the function $\varphi(t)$ can be used [31] instead of the integer derivative $\varphi^{(n)}(x\,t)$ in equation (20).

4 Conclusion

We assume that the proposed differential and integral operators (including the Kober integral and differential operators) can be applied to describe scale phenomena in economics, physics and other sciences. For eample, the suggested operators can be used to describe and generalize scalle phenomena that are considered in [32,33,34,35,36,37,38,39]. Such applications of fractional integrals and derivatives and their generalizations, which include continuously-distributed scaling, can give new interesting results and lead to the development of the fractional calculus.

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