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Transmuted Ishita Distribution and Its Applications

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Abstract: In this paper, we use quadratic rank transmutation map to propose a new distribution called Transmuted Ishita Distribution (TID). The proposed distribution is a generalization of Ishita distribution. Many properties of this distribution are investigated such as: the reliability, hazard rate and cumulative hazard functions, rth moment, moment-generating function, order statistics, generalized entropy, quantile function. The maximum likelihood method is used to estimate the unknown parameters of the TID. The proposed distribution is used for modeling a real-life data set. It is found that the TID is a better fit for this data set than some other available distributions.

Keywords: Transmuted Ishita distribution, Moments, Skewness, Kurtosis, Entropy, Order statistics, Quantile function, Reliability.

1 Introduction

Classical families of distributions might not be adequate for modeling many real data. Therefore, generalizing the existing distributions by adding one or more parameters allows the resulted distributions to be more appropriate to fit real-life data. Shaw and Buckley (2007) employed the quadratic rank transmutation map to generate a general and flexible family of distributions called transmuted family of distributions.

Transmuted distributions have received a lot of attention in the past years. Aryal and Tsokos (2011, 2013) used transmutation with Weibull and log-logistic distributions. Merovci (2013a,b) considered the quadratic rank transmutation map to develop the transmuted Lindley distribution and transmuted Rayleigh distribution. Cordeiro et al. (2013) derived the transmuted exponentiated generalized G family as an extension of the exponentiated generalized G class of distributions. Bourguignon et al. (2016) gave a simple representation for the transmuted G-family density function as a linear mixture of the G and the exponentiated-G densities. A transmuted Lomax distribution is introduced by Ashour and Eltehiwy (2013). In addition to the aforementioned work, transmutation have been used to generalize several more distributions such as the transmuted Lindley-Geometric distribution (Merovci and Elbatal, 2014), transmuted exponentiated Fréchet distribution (Elbatal et al., 2014), transmuted new modified weibull distribution (Vardhan and Balaswamy, 2016), transmuted Burr Type XII distribution (Khazaleh, 2016), transmuted Weibull Fréchet distribution (Afify et al., 2016), transmuted Mukherjee-Islam distribution (Al-zou′bi, 2017), and transmuted Janardan distribution (Al-Omari et al., 2017). We use the quadratic rank transmutation map to introduce a new distribution namely, Transmuted Ishita Distribution (TID). This proposed distribution is a generalization of the Ishita distribution (Shanker and Shukla, 2017).

The rest of this paper is organized as follows: the definition of Ishita distribution and its basic properties are given in Section 2. In Section 3, we define the probability density function (pdf) and cumulative distribution function (cdf) of the TID. In Section 4, the reliability, hazard rate, cumulative hazard, reversed hazard and odds functions of the proposed distribution are obtained. The distribution of order statistics is presented in Section 5. Some properties including the r^{th} moment, mean, variance, skewness, kurtosis, coefficient of variation and the moment generating function of the TID are studied in Section 6. In Section 7, the maximum likelihood estimates of the distribution parameters are discussed. The generalized entropy is presented in Section 8. Section 9 provides the q^{th} quantile of the TID. Application of the TID is considered in Section 10. The paper is concluded in Section 11.

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2 Ishita Distribution

Ishita distribution was proposed by Shanker and Shukla (2017). This distribution is a two-component mixture of exponential distribution with parameter η and a gamma distribution with parameters (3, η) using mixing proportion $\frac{\eta^3}{\eta^3+2}$.

The probability density function (pdf) of the Ishita distribution is given by:

$$f(x) = \frac{\eta^3}{\eta^3+2}(\eta + x^2)e^{-\eta x}; x > 0, \eta > 0, \tag{1}$$

with a corresponding cumulative distribution function (cdf) defined as:

$$F(x) = 1 - \left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}; x > 0, \eta > 0. \tag{2}$$

The moment-generating function and r^{th} moment of the Ishita distribution random variable are, respectively, given by

$$M_X(t) = \sum_{k=0}^{\infty} \frac{\eta^3 + (k+1)(k+2)}{\eta^3+2}\left(\frac{t}{\eta}\right)^k, \tag{3}$$

and

$$E(X^r) = \frac{r!\left(\eta^3 + (r+1)(r+2)\right)}{\eta^r(\eta^3+2)}; \quad r = 1, 2, 3, \ldots \tag{4}$$

3 Transmuted Ishita Distribution

Definition 1. *A random variable X is said to have a transmuted distribution (see [1]) if its cdf is given by*

$$F_T(x) = (1+\lambda)F(x) - \lambda[F(x)]^2, \quad -1 \le \lambda \le 1, \tag{5}$$

where $F(x)$ is the cdf of the base distribution.

The pdf of the transmuted random variable is given by

$$f_T(x) = f(x)\left(1 + \lambda - 2\lambda F(x)\right), \tag{6}$$

where $f(x)$ is the pdf of the base distribution.

Therefore, by plugging equation (2) in (5), the cdf of the TID random variable, X, is defined as:

$$\begin{aligned} F_{TID}(x) &= (1+\lambda)\left[1 - \left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right] - \lambda\left[1 - \left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]^2 \\ &= 1 - \left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\left[1 - \lambda + \lambda\left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right], \end{aligned} \tag{7}$$

with corresponding pdf given by

$$f_{TID}(x) = \frac{\eta^3}{\eta^3+2}(\eta + x^2)e^{-\eta x}\left[1 - \lambda + 2\lambda\left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]. \tag{8}$$

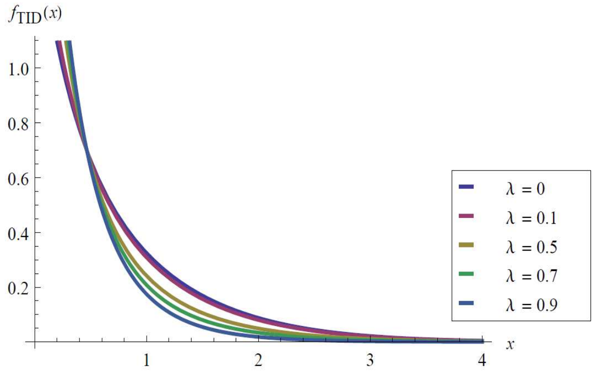

Fig. 1: The pdf of the TID distribution with $\eta = 2$ and $\lambda = 0, 0.1, 0.5, 0.7, 0.9$

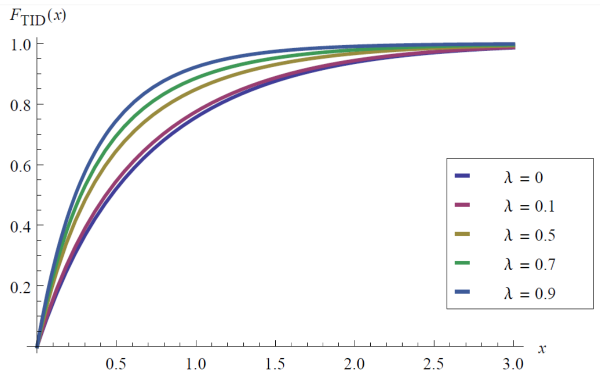

Fig. 2: The cdf of the TID distribution with $\eta = 2$ and $\lambda = 0, 0.1, 0.5, 0.7, 0.9$

Figures 1 and 2 show the pdf and cdf of the TID with different values of the distribution parameters. It is clear that the TID is skewed to the right.

4 Reliability analysis

The reliability or survival function, $R(x)$, is the probability that an object of interest survives beyond a specified time x. Using (7), the reliability function of the TID is given by

$$R_{TID}(x) = 1 - F_{TID}(x) = \left(1 + \frac{\eta x(\eta x + 2)}{\eta^3 + 2}\right) e^{-\eta x} \left[1 - \lambda + \lambda \left(1 + \frac{\eta x(\eta x + 2)}{\eta^3 + 2}\right) e^{-\eta x}\right]. \tag{9}$$

The hazard or failure rate function $H(x)$ is defined as the ratio of the probability density function and the survival function. Using (8) and (9), the hazard rate function of the TID is defined as

$$H_{TID}(x) = \frac{f_{TID}(x)}{1 - F_{TID}(x)} = \frac{\eta^3(\eta + x^2)\left[1 - \lambda + 2\lambda \left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right) e^{-\eta x}\right]}{\left[\eta^3 + 2 + \eta x(\eta x + 2)\right]\left[1 - \lambda + \lambda \left(1 + \frac{\eta x(\eta x+2)}{\eta^3+2}\right) e^{-\eta x}\right]}. \tag{10}$$

The cumulative hazarad function $H_{TIDcum}(x)$ is

$$\begin{aligned} H_{TIDcum}(x) &= -ln\left(1 - F_{TID}(x)\right) \\ &= \eta x - ln\left(1 + \frac{\eta x(\eta x + 2)}{\eta^3 + 2}\right) - ln\left[1 - \lambda + \lambda \left(1 + \frac{\eta x(\eta x + 2)}{\eta^3 + 2}\right) e^{-\eta x}\right]. \end{aligned} \tag{11}$$

The reliability and hazard functions of the TID for some values of the distribution parameters are shown in Figures 3 and 4.

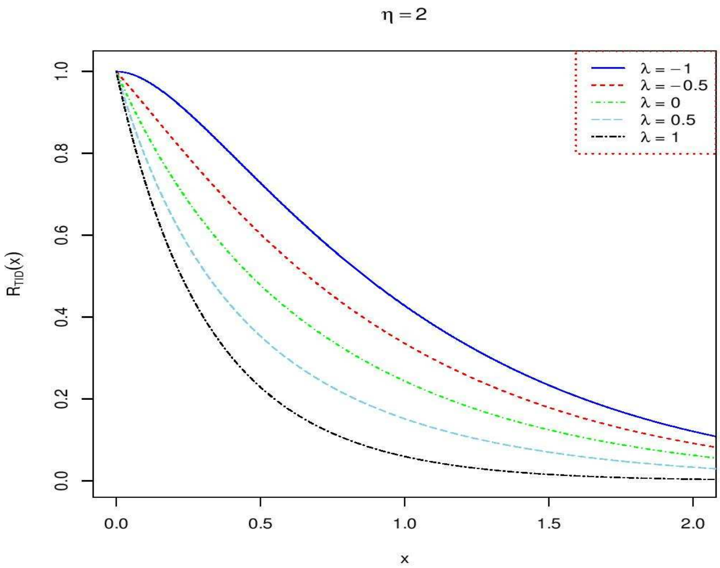

Fig. 3: The reliability of the TID with $\eta = 2$ and $\lambda = -1, -0.5, 0, 0.5, 1$

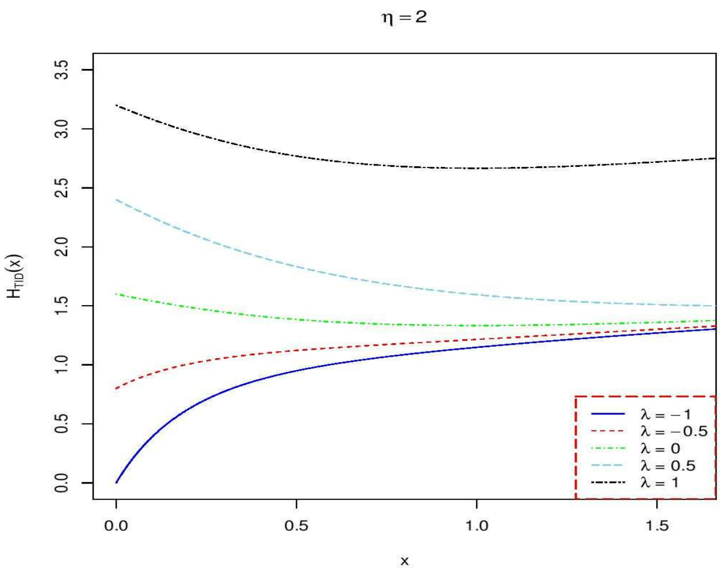

Fig. 4: The hazard of the TID with $\eta = 2$ and $\lambda = -1, -0.5, 0, 0.5, 1$

The reversed hazard rate function and odds function of the TID are

$$RH_{TID}(x) = \frac{f_{TID}(x)}{F_{TID}(x)} = \frac{\frac{\eta^3}{\eta^3+2}(\eta+x^2)\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]}{e^{\eta x}-\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)\left[1-\lambda+\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]}, \tag{12}$$

and

$$O_{TID}(x) = \frac{F_{TID}(x)}{1-F_{TID}(x)} = \frac{e^{\eta x}}{\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)\left[1-\lambda+\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]} - 1. \tag{13}$$

The cumulative hazard, reversed hazard, and odds functions of the TID for some values of the distribution parameters are shown in Figures 5, 6, and 7. From Figurs 3 - 7, we can see that hazard, cumulative hazard and odds functions increase as the value of λ increases, while reliability and reversed hazard decrease.

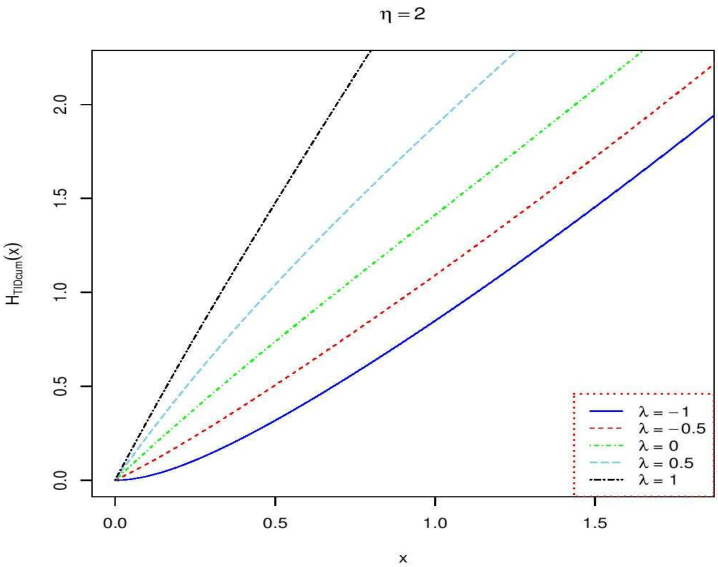

Fig. 5: The cumulative hazard function of the TID with $\eta = 2$ and $\lambda = -1, -0.5, 0, 0.5, 1$

5 Order Statistics

In this section, the pdf of the order statistics of the TID is derived. Let $X_1, X_2, \ldots X_n$ be a random sample with pdf $f_{TID}(x)$ and cdf $F_{TID}(x)$. If $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ are the order statistics of this sample, where $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Then, the pdf of the j^{th} order statistics, $X_{(j)}$ (see David and Nagaraja (2005)) is given by:

$$f_{TID(j)}(x) = \frac{n!}{(n-j)!(j-1)!} f_{TID}(x)[F_{TID}(x)]^{j-1}[1-F_{TID}(x)]^{n-j} \tag{14}$$

By using binomial series and substituting (7) and (8) in (14), we have

$$\begin{aligned} f_{TID(j)}(x) &= \frac{n!}{(n-j)!(j-1)!} f_{TID}(x) \sum_{l=0}^{j-1} \binom{j-1}{l} (-1)^l [1-F_{TID}(x)]^{n+l-j} \\ &= \frac{n!}{(n-j)!(j-1)!} \left[\frac{\eta^3}{\eta^3+2}(\eta+x^2)e^{-\eta x}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]\right] \\ &\times \sum_{l=0}^{j-1} \binom{j-1}{l} (-1)^l \left[\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\left[1-\lambda+\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]\right]^{n+l-j} \end{aligned} \tag{15}$$

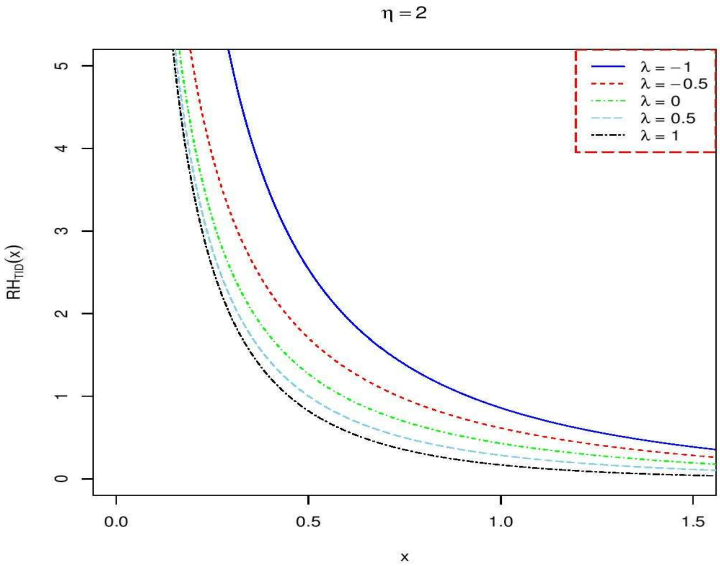

Fig. 6: The reversed hazard function of the TID with $\eta = 2$ and $\lambda = -1, -0.5, 0, 0.5, 1$

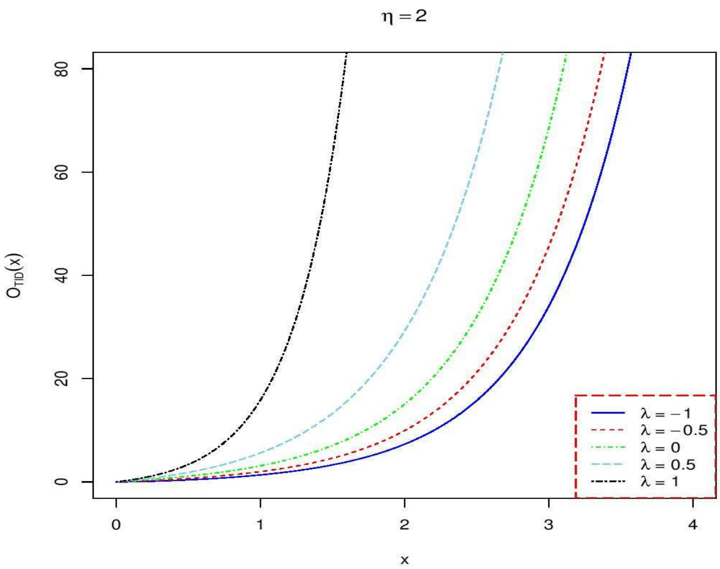

Fig. 7: The odds function of the TID with $\eta = 2$ and $\lambda = -1, -0.5, 0, 0.5, 1$

Using binomial series, we can write

$$\left[1-\lambda+\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]^{n+l-j} = \sum_{k=0}^{n+l-j}\binom{n+l-j}{k}(1-\lambda)^{n+l-j-k}\left(\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right)^k \tag{16}$$

Therefore, by plugging (16) in (15), the pdf of the j^{th} order statistics, $f_{TID(j)}(x)$, can be written as

$$f_{TID(j)}(x) = \frac{n!}{(n-j)!(j-1)!}\frac{\eta^3(\eta+x^2)}{\eta^3+2}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]$$
$$\times\sum_{l=0}^{j-1}\sum_{k=0}^{n+l-j}\binom{n+l-j}{k}\binom{j-1}{l}(-1)^l(1-\lambda)^{n+l-j-k}\lambda^k\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)^{n+l-j+k}e^{-\eta x(n+l-j+k+1)} \quad (17)$$

Thus, the pdf of the first-order statistics $X_{(1)} = min(X_1, X_2, \dots X_n)$ is defined as:

$$f_{TID(1)}(x) = \frac{n\eta^3(\eta+x^2)}{\eta^3+2}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]$$
$$\times\sum_{k=0}^{n-1}\binom{n-1}{k}(1-\lambda)^{n-1-k}\lambda^k\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)^{n-1+k}e^{-\eta x(n+k)}$$

Furthermore, the pdf of the n^{th} order statistic $X_{(n)} = max(X_1, X_2, \dots X_n)$, is given by:

$$f_{TID(n)}(x) = \frac{n\eta^3(\eta+x^2)}{(\eta^3+2}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]$$
$$\times\sum_{l=0}^{n-1}\sum_{k=0}^{l}\binom{l}{k}\binom{n-1}{l}(-1)^l(1-\lambda)^{l-k}\lambda^k\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)^{l+k}e^{-\eta x(l+k+1)}.$$

6 Moments

The rth moment of the TID random variable is derived in this section. Also, the mean, variance, coefficient of kurtosis, coefficient of skewness, coefficient of variation, and moment-generating function are presented.

6.1 r^{th} Moment

Theorem 6.1. The r^{th} moment of the TID random variable is defined as:

$$E(X^r)=\frac{1}{2^{r+4}\eta^r(\eta^3+2)^2}\Bigg\{(\eta^3+2)\left(2^{r+4}(1-\lambda)(\eta^3\Gamma(r+1)+\Gamma(r+3))+4\lambda(4\eta^3\Gamma(r+1)+\Gamma(r+3))\right)$$
$$+\lambda\left(4\eta^3(4\Gamma(r+2)+\Gamma(r+3))+4\Gamma(r+4)+\Gamma(r+5)\right)\Bigg\}, \quad (18)$$

where $\Gamma(r) = (r-1)!$.

Proof. The r^{th} moment of the TID random variable, with pdf $f_{TID}(x)$ in (8), can be proved as:

$$E(X^r) = \int_0^\infty x^r f_{TID}(x)dx = \int_0^\infty x^r\frac{\eta^3}{\eta^3+2}(\eta+x^2)e^{-\eta x}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]dx$$
$$= \frac{\eta^3}{\eta^3+2}(1-\lambda)\int_0^\infty x^r(\eta+x^2)e^{-\eta x}dx + \frac{\eta^3}{\eta^3+2}(2\lambda)\int_0^\infty x^r(\eta+x^2)e^{-2\eta x}dx$$
$$+\frac{\eta^3}{\eta^3+2}(2\lambda)\int_0^\infty x^r\frac{\eta x(\eta x+2)}{\eta^3+2}(\eta+x^2)e^{-2\eta x}dx$$
$$= \frac{\eta^3(1-\lambda)}{\eta^3+2}\left(\frac{\Gamma(r+1)}{\eta^r}+\frac{\Gamma(r+3)}{\eta^{r+3}}\right)+\frac{\eta^3(2\lambda)}{\eta^3+2}\left(\frac{\eta\Gamma(r+1)}{(2\eta)^{r+1}}+\frac{\Gamma(r+3)}{(2\eta)^{r+3}}\right)$$
$$+\frac{\eta^5(2\lambda)}{(\eta^3+2)^2}\left(\frac{\eta\Gamma(r+3)}{(2\eta)^{r+3}}+\frac{\Gamma(r+5)}{(2\eta)^{r+5}}\right)+\frac{\eta^4(4\lambda)}{(\eta^3+2)^2}\left(\frac{\eta\Gamma(r+2)}{(2\eta)^{r+2}}+\frac{\Gamma(r+4)}{(2\eta)^{r+4}}\right)$$

Then, we have

$$E(X^r) =\frac{1}{2^{r+4}\eta^r(\eta^3+2)^2}\Bigg\{(\eta^3+2)\left(2^{r+4}(1-\lambda)(\eta^3\Gamma(r+1)+\Gamma(r+3))+4\lambda(4\eta^3\Gamma(r+1)+\Gamma(r+3))\right)$$
$$+\lambda\left(4\eta^3(4\Gamma(r+2)+\Gamma(r+3))+4\Gamma(r+4)+\Gamma(r+5)\right)\Bigg\}.$$

6.1.1 Mean, Variance, Skewness, Kurtosis and Coefficient of Variation

From (18), the mean and second moment can be obtained as follows:

$$E(X)=\frac{(\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]}{4\eta(\eta^3+2)^2} \tag{19}$$

$$E(X^2)=\frac{(\eta^3+2)\left[4(1-\lambda)(2\eta^3+24)+\lambda(2\eta^3+6)\right]+\lambda\left[12\eta^3+75\right]}{4\eta^2(\eta^3+2)^2} \tag{20}$$

Thus, the variance of the TID random variable is defined as

$$\begin{aligned}\sigma^2=var(X)=E(X^2)-(E(X))^2\\ &=\frac{1}{16\eta^2(\eta^3+2)^4}\Big[4(\eta^3+2)^2\Big((\eta^3+2)\left[4(1-\lambda)(2\eta^3+24)+\lambda(2\eta^3+6)\right]+\lambda\left[12\eta^3+75\right]\Big)\\ &\qquad-\Big((\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big)^2\Big].\end{aligned} \tag{21}$$

The coefficient of variation $(C.V)$ is defined as $C.V=\frac{\sigma}{E(X)}$. Therefore,

$$C.V=\frac{\begin{bmatrix}4(\eta^3+2)^2\Big((\eta^3+2)\left[4(1-\lambda)(2\eta^3+24)+\lambda(2\eta^3+6)\right]+\lambda\left[12\eta^3+75\right]\Big)\\ -\Big((\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big)^2\end{bmatrix}^{1/2}}{(\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]}.$$

Using (18), the third and fourth moments of the TID random variable X are given, respectively, by

$$E(X^3)=\frac{(\eta^3+2)\left[48(1-\lambda)(\eta^3+20)+6\lambda(\eta^3+5)\right]+9\lambda\left[6\eta^3+55\right]}{8\eta^3(\eta^3+2)^2}, \tag{22}$$

$$E(X^4)=\frac{3(\eta^3+2)\left[32(1-\lambda)(\eta^3+30)+\lambda(2\eta^3+15)\right]+15\lambda\left[5\eta^3+63\right]}{4\eta^4(\eta^3+2)^2}. \tag{23}$$

The skewness and the kurtosis of a random variable are defined as:

$$sk(X)=\frac{E(X^3)-3E(X)E(X^2)+2(E(X))^3}{\sigma^3}$$

$$ku(X)=\frac{E(X^4)-4E(X)E(X^3)+6(E(X))^2E(X^2)-3(E(X))^4}{\sigma^4}$$

Based on these formulas, the skewness and the kurtosis of the TID random variable are given, respectively, by:

$$Sk_{TID}(X)=\frac{\begin{bmatrix}8(\eta^3+2)^4\Big((\eta^3+2)\left[48(1-\lambda)(\eta^3+20)+6\lambda(\eta^3+5)\right]+9\lambda\left[6\eta^3+55\right]\Big)\\ -12(\eta^3+2)^2\Big((\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big)\\ \times\Big((\eta^3+2)\left[4(1-\lambda)(2\eta^3+24)+\lambda(2\eta^3+6)\right]+\lambda\left[12\eta^3+75\right]\Big)\\ +2\Big[(\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big]^3\end{bmatrix}}{\begin{bmatrix}4(\eta^3+2)^2\Big((\eta^3+2)\left[4(1-\lambda)(2\eta^3+24)+\lambda(2\eta^3+6)\right]+\lambda\left[12\eta^3+75\right]\Big)\\ -\Big((\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big)^2\end{bmatrix}^{3/2}},$$

$$ku_{TID}(X)=\frac{\begin{bmatrix}64(\eta^3+2)^6\Big(3(\eta^3+2)\left[32(1-\lambda)(\eta^3+30)+\lambda(2\eta^3+15)\right]+15\lambda\left[5\eta^3+63\right]\Big)\\ -32(\eta^3+2)^6\Big((\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big)\\ \times\Big((\eta^3+2)\left[48(1-\lambda)(\eta^3+20)+6\lambda(\eta^3+5)\right]+9\lambda\left[6\eta^3+55\right]\Big)\\ +24(\eta^3+2)^2\Big[(\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big]^2\\ \times\Big[(\eta^3+2)\left[4(1-\lambda)(2\eta^3+24)+\lambda(2\eta^3+6)\right]+\lambda\left[12\eta^3+75\right]\Big]\\ -3\Big[(\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big]^4\end{bmatrix}}{\begin{bmatrix}4(\eta^3+2)^2\Big((\eta^3+2)\left[4(1-\lambda)(2\eta^3+24)+\lambda(2\eta^3+6)\right]+\lambda\left[12\eta^3+75\right]\Big)\\ -\Big((\eta^3+2)\left[4(1-\lambda)(\eta^3+6)+\lambda(2\eta^3+3)\right]+\lambda\left[7\eta^3+27\right]\Big)^2\end{bmatrix}^{2}}.$$

The mean, variance, skewness, kurtosis and the coefficient of variation of the TID for different values of λ and η are given in Tables 1 and 2.

It can be seen from Table 1 that the mean increases as the value of λ decreases. The other values in the table depend on λ and η. Table 2 indicates that the mean and variance decrease as η increases. The positive values of skewness in both tables mean that the TID is skewed to the right.

Table 1: The mean, variance, skewness, kurtosis and the coefficient of variation of the TID for $\eta = 1$, 3, and variant values of λ

$\eta = 1$						$\eta = 3$					
λ	$E(X)$	$var(X)$	$C.V$	$sk_{TID}(X)$	$ku_{TID}(X)$	λ	$E(X)$	$var(X)$	$C.V$	$sk_{TID}(X)$	$ku_{TID}(X)$
0.9	1.4583	1.5149	0.8440	1.4390	6.3699	0.9	0.2046	0.0516	1.1103	2.8179	17.3188
0.8	1.5556	1.7803	0.8577	1.5274	6.7492	0.8	0.2240	0.0661	1.1477	2.9114	17.4041
0.7	1.6528	2.0267	0.8613	1.5333	6.6425	0.7	0.2434	0.0799	1.1608	2.8490	16.1948
0.6	1.7500	2.2542	0.8579	1.5014	6.3728	0.6	0.2628	0.0928	1.1593	2.7407	14.8413
0.5	1.8472	2.4627	0.8496	1.4519	6.0615	0.5	0.2823	0.1051	1.1485	2.6218	13.6058
0.4	1.9444	2.6525	0.8376	1.3948	5.7566	0.4	0.3017	0.1166	1.1318	2.5053	12.5370
0.3	2.0417	2.8233	0.8230	1.3353	5.4770	0.3	0.3211	0.1273	1.1112	2.3956	11.6276
0.2	2.1389	2.9752	0.8064	1.2762	5.2295	0.2	0.3405	0.1373	1.0882	2.2945	10.8573
0.1	2.2361	3.1081	0.7884	1.2193	5.0153	0.1	0.3599	0.1465	1.0635	2.2021	10.205
-0.1	2.4306	3.3174	0.7493	1.1154	4.6826	-0.1	0.3987	0.1627	1.0116	2.0425	9.1841
-0.2	2.5278	3.3937	0.7288	1.0698	4.5607	-0.2	0.4181	0.1697	0.9851	1.9746	8.7880
-0.3	2.6250	3.4510	0.7077	1.0291	4.4661	-0.3	0.4376	0.1759	0.9585	1.9141	8.4543
-0.4	2.7222	3.4895	0.6862	0.9940	4.3973	-0.4	0.4570	0.1813	0.9319	1.8607	8.1752
-0.5	2.8194	3.5091	0.6644	0.9651	4.3532	-0.5	0.4764	0.1860	0.9054	1.8145	7.9445
-0.6	2.9167	3.5097	0.6423	0.9434	4.3327	-0.6	0.4958	0.1900	0.8791	1.7752	7.7573
-0.7	3.0139	3.4915	0.6200	0.9300	4.3354	-0.7	0.5152	0.1932	0.8531	1.7430	7.6099
-0.8	3.1111	3.4543	0.5974	0.9263	4.3608	-0.8	0.5346	0.1956	0.8273	1.7181	7.4995
-0.9	3.2083	3.3983	0.5746	0.9346	4.4084	-0.9	0.5540	0.1973	0.8018	1.7009	7.4238

Table 2: The mean, variance, skewness, kurtosis and the coefficient of variation of the TID for $\lambda = 0.3$, 0.8 and variant values of η

$\lambda = 0.3$						$\lambda = 0.8$					
η	$E(X)$	$var(X)$	$C.V$	$sk_{TID}(X)$	$ku_{TID}(X)$	η	$E(X)$	$var(X)$	$C.V$	$sk_{TID}(X)$	$ku_{TID}(X)$
0.1	27.1769	264.0670	0.5979	1.2999	5.6021	0.1	22.4885	168.8420	0.5778	1.4102	6.6754
0.2	13.5517	66.1934	0.6004	1.2951	5.5889	0.2	11.2042	42.3695	0.5810	1.4029	6.6510
0.3	8.9689	29.6232	0.6069	1.2831	5.5552	0.3	7.3983	19.0127	0.5894	1.3851	6.5892
0.4	6.6337	16.8661	0.6191	1.2639	5.4971	0.4	5.4483	10.8719	0.6052	1.3576	6.4864
0.5	5.1907	10.9747	0.6382	1.2415	5.4203	0.5	4.2339	7.1093	0.6298	1.3288	6.3591
0.6	4.1919	7.7623	0.6646	1.2241	5.3414	0.6	3.3867	5.0465	0.6633	1.3119	6.2442
0.7	3.4496	5.7945	0.6978	1.2205	5.2849	0.7	2.7532	3.7676	0.7050	1.3196	6.1891
0.8	2.8729	4.4769	0.7365	1.2371	5.2766	0.8	2.2603	2.8966	0.7530	1.3587	6.2376
0.9	2.4131	3.5328	0.7789	1.2760	5.3369	0.9	1.8689	2.2621	0.8048	1.4296	6.4197
1	2.0417	2.8233	0.8230	1.3353	5.4770	1	1.5556	1.7803	0.8577	1.5274	6.7492
2	0.5924	0.4224	1.0972	2.1545	9.4830	2	0.4130	0.2249	1.1484	2.6645	14.2412
3	0.3211	0.1273	1.1112	2.3956	11.6276	3	0.2240	0.0661	1.1477	2.9114	17.4041
4	0.2249	0.0606	1.0949	2.3765	11.6775	4	0.1577	0.0317	1.1290	2.8611	17.1760
5	0.1751	0.0361	1.0846	2.3366	11.3982	5	0.1231	0.0190	1.1188	2.8062	16.6390
6	0.1442	0.0242	1.0790	2.3094	11.1800	6	0.1015	0.0128	1.1136	2.7723	16.2717
7	0.1228	0.0175	1.0759	2.2928	11.0395	7	0.0866	0.0092	1.1107	2.7523	16.0466
8	0.1070	0.0132	1.0741	2.2825	10.9507	8	0.0755	0.0070	1.1091	2.7402	15.9076
9	0.0949	0.0104	1.0730	2.2760	10.8933	9	0.0670	0.0055	1.1080	2.7326	15.8190
10	0.0853	0.0084	1.0722	2.2716	10.8550	10	0.0602	0.0044	1.1074	2.7276	15.7604

6.2 Moment-Generating Function

Theorem 6.2. The moment-generating function of the TID random variable is given by

$$M_X(t) = \frac{(1-\lambda)\eta^3\left[\eta(\eta-t)^2+2\right]}{(\eta^3+2)(\eta-t)^3} + \frac{2\lambda\eta^3\left[\eta(2\eta-t)^2+2\right]}{(\eta^3+2)(2\eta-t)^3}$$
$$+\frac{4\lambda\eta^4\left[\eta^2(2\eta-t)^2+12\eta+\eta(2\eta-t)^3+6(2\eta-t)\right]}{(\eta^3+2)^2(2\eta-t)^5}$$

Proof. The moment-generating function can be proved as

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_0^\infty e^{tx} f_{TID}(x)dx \\
&= \int_0^\infty e^{tx} \frac{\eta^3}{\eta^3+2}(\eta+x^2)e^{-\eta x}\left[1-\lambda+2\lambda\left(1+\frac{\eta x(\eta x+2)}{\eta^3+2}\right)e^{-\eta x}\right]dx \\
&= \frac{\eta^3(1-\lambda)}{\eta^3+2}\left[\eta\int_0^\infty e^{tx}e^{-\eta x}dx+\int_0^\infty x^2e^{tx}e^{-\eta x}dx\right] \\
&+ \frac{\eta^3(2\lambda)}{\eta^3+2}\left[\eta\int_0^\infty e^{tx}e^{-2\eta x}dx+\int_0^\infty x^2e^{tx}e^{-2\eta x}dx\right] \\
&+ \frac{\eta^3(2\lambda)}{(\eta^3+2)^2}\left[\eta^3\int_0^\infty x^2e^{tx}e^{-2\eta x}dx+\eta^2\int_0^\infty x^4e^{tx}e^{-2\eta x}dx\right] \\
&+ \frac{\eta^3(4\lambda)}{(\eta^3+2)^2}\left[\eta^2\int_0^\infty xe^{tx}e^{-2\eta x}dx+\eta\int_0^\infty x^3e^{tx}e^{-2\eta x}dx\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M_X(t) &= \frac{\eta^3(1-\lambda)}{\eta^3+2}\left[\eta\int_0^\infty e^{-(\eta-t)x}dx+\int_0^\infty x^2e^{-(\eta-t)x}dx\right] \\
&+ \frac{\eta^3(2\lambda)}{\eta^3+2}\left[\eta\int_0^\infty e^{-(2\eta-t)x}dx+\int_0^\infty x^2e^{-(2\eta-t)x}dx\right] \\
&+ \frac{\eta^3(2\lambda)}{(\eta^3+2)^2}\left[\eta^3\int_0^\infty x^2e^{-(2\eta-t)x}dx+\eta^2\int_0^\infty x^4e^{tx}e^{-2\eta x}dx\right] \\
&+ \frac{\eta^3(4\lambda)}{(\eta^3+2)^2}\left[\eta^2\int_0^\infty xe^{-(2\eta-t)x}dx+\eta\int_0^\infty x^3e^{-(2\eta-t)x}dx\right]
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
M_X(t) &= \frac{\eta^3(1-\lambda)}{\eta^3+2}\left[\eta\left(\frac{1}{\eta-t}\right)+\frac{2}{(\eta-t)^3}\right]+\frac{\eta^3(2\lambda)}{\eta^3+2}\left[\eta\left(\frac{1}{2\eta-t}\right)+\frac{2}{(2\eta-t)^3}\right] \\
&+ \frac{\eta^3(2\lambda)}{(\eta^3+2)^2}\left[\eta^3\left(\frac{2}{(2\eta-t)^3}\right)+\eta^2\left(\frac{24}{(2\eta-t)^5}\right)\right]+\frac{\eta^3(4\lambda)}{(\eta^3+2)^2}\left[\eta^2\left(\frac{1}{(2\eta-t)^2}\right)+\eta\left(\frac{6}{(2\eta-t)^4}\right)\right] \\
&= \frac{(1-\lambda)\eta^3\left[\eta(\eta-t)^2+2\right]}{(\eta^3+2)(\eta-t)^3}+\frac{2\lambda\eta^3\left[\eta(2\eta-t)^2+2\right]}{(\eta^3+2)(2\eta-t)^3} \\
&+ \frac{4\lambda\eta^4\left[\eta^2(2\eta-t)^2+12\eta+\eta(2\eta-t)^3+6(2\eta-t)\right]}{(\eta^3+2)^2(2\eta-t)^5}
\end{aligned}
$$

7 Maximum Likelihood Estimates

Let $X_1, X_2, \ldots, X_n$ be a random sample from the TID with pdf $f_{TID}(x)$ in (8) and parameters η and λ, then the likelihood function is given by

$$
\begin{aligned}
L(\eta,\lambda|x_1,x_2,\ldots,x_n) &= \prod_{i=1}^n f_{TID}(x_i|\eta,\lambda) \\
&= \prod_{i=1}^n\left[\frac{\eta^3}{\eta^3+2}(\eta+x_i^2)e^{-\eta x_i}\left[1-\lambda+2\lambda\left(1+\frac{\eta x_i(\eta x_i+2)}{\eta^3+2}\right)e^{-\eta x_i}\right]\right] \\
&= \left(\frac{\eta^3}{\eta^3+2}\right)^n\left[\prod_{i=1}^n(\eta+x_i^2)\right]e^{-\eta\sum_{i=1}^n x_i}\prod_{i=1}^n\left[1-\lambda+2\lambda\left(1+\frac{\eta x_i(\eta x_i+2)}{\eta^3+2}\right)e^{-\eta x_i}\right].
\end{aligned}
$$

Hence, the log-likelihood function is given by

$$L^* = ln\, L(\eta,\lambda|x_1,x_2,\ldots,x_n)$$
$$= n\, ln\left(\frac{\eta^3}{\eta^3+2}\right) + \sum_{i=1}^{n} ln(\eta+x_i^2) - \eta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} ln\left[1-\lambda+2\lambda\left(1+\frac{\eta x_i(\eta x_i+2)}{\eta^3+2}\right)e^{-\eta x_i}\right].$$

The derivatives of the log-likelihood function with respect to the parameters η and λ are:

$$\frac{\partial L^*}{\partial \eta} = \frac{6n}{\eta(\eta^3+2)} + \sum_{i=1}^{n}\frac{1}{(\eta+x_i^2)} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{-2\lambda\eta^2 x_i e^{-\eta x_i}\left(\eta^4+\eta^3 x_i^2+3\eta^2 x_i+8\eta+2x_i^2\right)}{(\eta^3+2)^2\left[1-\lambda+2\lambda\left(1+\frac{\eta x_i(\eta x_i+2)}{\eta^3+2}\right)e^{-\eta x_i}\right]}, \tag{24}$$

$$\frac{\partial L^*}{\partial \lambda} = \sum_{i=1}^{n}\frac{-1+2\left(1+\frac{\eta x_i(\eta x_i+2)}{\eta^3+2}\right)e^{-\eta x_i}}{1-\lambda+2\lambda\left(1+\frac{\eta x_i(\eta x_i+2)}{\eta^3+2}\right)e^{-\eta x_i}}. \tag{25}$$

The maximum likelihood estimators for the distribution parameters η and λ can be found by equating the derivatives in Equations (24) and (25) to zero and solving the resulting equations simultaneously by using numerical methods.

8 Generalized Entropy

The entropy of a random variable X is a measure of variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. For more details about entropy, see Zamanzade (2015), Zamanzade and Arghami (2012), and Zamanzade and Arghami (2011). In this section, the Generalized Entropy (GE) of the TID is given. It is defined as

$$GE(\alpha) = \frac{\Delta_\alpha \mu^{-\alpha}-1}{\alpha(\alpha-1)}, \alpha \neq 0,1, \tag{26}$$

where $\Delta_\alpha = \int_{-\infty}^{\infty} x^\alpha f(x)dx$ and $\mu = E(X)$, (see Biewen and Jenkins (2003)).

By plugging (19) and (18) in (26), the generalized entropy of the TID is given by

$$GE(\alpha) = (\alpha(\alpha-1))^{-1}\Bigg(\Bigg[\Big((\eta^3+2)\left[32(1-\lambda)(\eta^3+6)+4\lambda(4\eta^3+6)\right]+\lambda\left[56\eta^3+216\right]\Big)^{-\alpha}$$
$$\times\Bigg\{(\eta^3+2)\Big(2^{\alpha+4}(1-\lambda)(\eta^3\Gamma(\alpha+1)+\Gamma(\alpha+3))+4\lambda(4\eta^3\Gamma(\alpha+1)+\Gamma(\alpha+3))\Big)$$
$$+\lambda\Big(4\eta^3(4\Gamma(\alpha+2)+\Gamma(\alpha+3))+4\Gamma(\alpha+4)+\Gamma(\alpha+5)\Big)\Bigg\}(4(\eta^3+2))^{2(\alpha-1)}\Bigg]-1\Bigg)$$

Table 3 provides generalized entropy values of the TID with variant values of the distribution parameters. From Table 3, we can note that the generalized entropy, $GE(\alpha)$, increases in α for $\eta = 0.1$ and $\lambda = 0.5, -0.5, -0.9$. No such conclusion can be deduced for $\eta = 3,4,5,6,7,8,9,10$ with $\alpha = 3$ and $\lambda = 0.5, -0.5, -0.9$.

9 Quantile Function

The q^{th} quantile value, x_q, is a value of the random variable, X, with cdf $F(x)$ such that

$$F(x_q) = p(X \leq x_q) = q \qquad ;0 < q < 1. \tag{27}$$

The following lemma gives the q^{th} quantile of the TID.

Lemma 1. *The q^{th} quantile ,x_q, of the TID is the solution of*

$$\left(1+\frac{\eta x_q(\eta x_q+2)}{\eta^3+2}\right)e^{-\eta x_q} = \frac{\lambda-1+\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda} \qquad ;x_q > 0. \tag{28}$$

Table 3: Generalized entropy values with variant values of the distribution parameters

α	$\eta=0.1, \lambda=0.5$	$\eta=0.1, \lambda=-0.5$	$\eta=0.1, \lambda=-0.9$
3	0.2310	0.1582	0.1222
4	0.3482	0.2091	0.1557
5	0.6237	0.3129	0.2207
6	1.3305	0.5333	0.3506
7	3.3540	1.0363	0.6259
8	9.8396	2.2872	1.2620
9	33.0082	5.6912	2.8484
10	124.5950	15.8115	7.1581
η	$\alpha=3, \lambda=0.5$	$\alpha=3, \lambda=-0.5$	$\alpha=3, \lambda=0.9$
3	1.0363	0.5734	1.2591
4	1.2646	0.6343	1.1953
5	1.2246	0.5969	1.1539
6	1.2024	0.5866	1.1398
7	1.1899	0.5807	1.1294
8	1.1826	0.5771	1.1234
9	1.1780	0.5749	1.1196
10	1.1750	0.5734	1.1172

proof. Using the cdf of TID in (7) and plugging it in (27), we have

$$(1+\lambda)\left[1-\left(1+\frac{\eta x_q(\eta x_q+2)}{\eta^3+2}\right)e^{-\eta x_q}\right]-\lambda\left[1-\left(1+\frac{\eta x_q(\eta x_q+2)}{\eta^3+2}\right)e^{-\eta x_q}\right]^2=q.$$

Let $y=\left[1-\left(1+\frac{\eta x_q(\eta x_q+2)}{\eta^3+2}\right)e^{-\eta x_q}\right]$, then we have

$$(1+\lambda)y-\lambda y^2=q,$$
$$\lambda y^2-(1+\lambda)y+q=0.$$

To solve the above quadratic equation for $0\le y\le 1$, the general formula can be used to get

$$y=\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}.$$

By replacing y by its value $1-\left(1+\frac{\eta x_q(\eta x_q+2)}{\eta^3+2}\right)e^{-\eta x_q}$, we have

$$1-\left(1+\frac{\eta x_q(\eta x_q+2)}{\eta^3+2}\right)e^{-\eta x_q}=\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}$$

Therefore, the q^{th} quantile x_q is the positive solution of

$$\left(1+\frac{\eta x_q(\eta x_q+2)}{\eta^3+2}\right)e^{-\eta x_q}=\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda},$$

which can be found by numerical methods.

10 Applications

In this section, the TID is used to fit a real-life data set. The data set (given in Table 4) is from Andrews and Herzberg (1985) which represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. The goodness of fit of the TID to this data set is compared with Ishita distribution (given in (1)) in addition to the following distributions:

– Rama distribution (see Shanker (2017)): $f(x)=\frac{\alpha^4}{\alpha^3+6}(1+x^3)e^{-\alpha x}\quad ;x>0,\alpha>0.$

– Akash distribution (see Shanker (2015)): $f(x)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}\quad ;x>0,\theta>0.$

Table 4: The life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566	0.6748	0.6751	0.6753
0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.9120	0.9836	1.0483	1.0596	1.0773	1.1733
1.2570	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595	1.4880	1.5728	1.5733	1.7083	1.7263	1.7460
1.7630	1.7746	1.8275	1.8375	1.8503	1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903
2.1093	2.1330	2.2100	2.2460	2.2878	2.3203	2.3470	2.3513	2.4951	2.5260	2.9911	3.0256	3.2678
3.4045	3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.0960		

Table 5: $-2logL$, AIC, AICC, K-S Statistics and its p-value of the fitted distributions.

Distribution	*-2 log L*	*AIC*	*CAIC*	*KS Statistic*	*P-value*
Rama	254.9094	256.9093	256.9634	0.1431	0.0805
Akash	249.1510	251.1510	251.2050	0.1231	0.1836
Ishita	249.6816	251.6815	251.7356	0.1293	0.1440
Transmuted Ishita	246.3972	250.3971	250.5615	0.1181	0.2214

based on $-2logL$, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Kolmogorov Smirnov (KS) statistic and its p-value. The results are presented in Table 5. It turns out that the Transmuted Istha distribution has the lowest values of the $-2logL$, AIC, CAIC, and KS statistic in comparison with the other fitted distributions. Therefore, the TID is better than Rama, Akash, and Ishita distributions for fitting this real-life data set. The Maximum Likelihood Estimates (MLEs) of the parameters of the fitted distributions and their Confidence Intervals (CI) are computed and given in Table 6.

Table 6: The MLEs of the parameters of the fitted distributions and their confidence intervals

Distribution	*Parameters*	*MLE*	*Standard Error*	*95% CI* Lower Limit	*95% CI* Upper Limit
Rama	α	1.4944	0.0767	1.3441	1.6447
Akash	θ	1.1324	0.0729	0.9894	1.2754
Ishita	η	1.1050	0.0621	0.9833	1.2266
Transmuted Ishita	η	0.8702	0.0668	0.7392	1.0012
	λ	0.8152	0.1875	0.4477	1.1827

11 Conclusion

In this paper, the Ishita distribution is modified using the quadratic rank transmutation map to suggest the Transmuted Ishita Distribution (TID). We have studied several properties of this distribution including moments, mean, variance, skewness, kurtosis, coefficient of variation, moment-generating function, order statistics and maximum likelihood estimates of the distribution parameters. Also, we have obtained the reliability, hazard rate, cumulative hazard, reversed hazard rate, odds functions, generalized entropy and the quantile function. The proposed distribution (TID) is used to fit a real lifetime data set. The results of this application have revealed that the proposed distribution can be a better fit than Ishita distribution and some other competitive distributions considered in this study.

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