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Approximate Analytical and Numerical Solutions for Time Fractional Generalized Nonlinear Huxley Equation

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Abstract: In this work, multiple traveling wave solutions for one kind of nonlinear partial differential equations of fractional order using the tanh-function method are investigated. Namely, time fractional generalized nonlinear Huxley equation is explored. The proposed method benefits in handling other related general forms of fractional nonlinear partial differential equations. The analytic solutions behavior is illustrated graphically. In addition, a numerical treatment for the same problem is proposed using the cubic spline function method. Stability of the method is investigated based on the Von Neumann concept. The method proved to be conditionally stable. A numerical example is presented to assert that the proposed algorithm is effective. The results confirmed the effectiveness and accuracy of the proposed technique.

Keywords: Tanh-function method, time fractional Huxley equation, cubic parametric spline method, Von Neumann method.

1 Introduction

In many fields in mathematical physics, such as optical fibers, plasma physics, solid state physics, chemical kinetics geochemistry, quantum technologies and fluid mechanics, the nonlinear partial differential equations appear as mathematical models [1,2,3,4,5]. Different powerful analytical methods have recently been considered to such mathematical models. The analytical traveling wave solutions help engineers and physicists to better understand the behavior and mechanism that govern such mathematical models and possible applications. Some powerful methods involve the tanh-coth method and the sine - cosine method [6,7,8,9,10,11]. Moreover, the extended tanh-function method is the most recently used to find exact solutions to some nonlinear partial differential equations, see for example [12,13,14].

Recently, analytical and numerical solutions for different types of fractional differential equations, including Burger, Burgers-Huxley, regularized long–wave, Fokker-Plank and other fractional equations, are considered fractional partial differential equations and which several authors investigate, see for example [15,16,17,18,19,20,21,22,23].

The present paper aims to implement the tanh-function method to obtain multiple traveling wave solutions of the generalized nonlinear Huxley equation with time-fractional derivative of the form:

$$u_t^{\alpha} - u_{xx} - u(1-u^{\delta})(u^{\delta}-\lambda) = 0,\ 0<\alpha\leq 1,$$

subject to boundary conditions

$$u(a,t) = g_1(t),\ \ u(b,t) = g_2(t),\ t\geq 0,$$

and the initial condition $u(x,0) = f(x)$, $a\leq x\leq b$, where λ and δ are parameters, $\delta>0, \lambda\in(0,1)$.

To show its effectiveness and convenience, solution procedure of this method is obtained with the help of Mathematica.

Applying non-polynomial spline (NPS) functions to solve some partial differential equations is not regarded as a new subject because one can pursue this subject in the pieces of literature using NPS in solving Burgers' equation, cubic nonlinear Schrödinger equation, nonlinear Klein-Gordon equation, variable coefficient fourth-order wave equations and Bratu's problem [24,25,26,27,28]. A large number of non-polynomial splines based methods that investigate approximate

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solutions of boundary value problems of different orders are recently considered, see for example, [29,30,31,32]. The existence and uniqueness of the solution of partial differential equations can be found in [33].

The paper is organized as follows: In section Two, the tanh- function method is first described and the analytic solution is then obtained. In section Three, the cubic non-polynomial spline function method is presented. Stability analysis of the proposed numerical method is derived in section Four and it proved that the method is conditionally stable. In section Five, numerical results are discussed and presented to illustrate the applicability and accuracy of both methods. Section Six is devoted to conclusion.

2 The Use of the Tanh-Function Method

2.1 *Description of the Method*

The use of the tanh-function method is a good solution technique to compute exact traveling wave solutions for fractional nonlinear partial differential equations. The technique depends on introducing a power series in tanh form to obtain analytical solutions of traveling wave type. The wave variable $\eta = (x - c\frac{t^{\alpha}}{\Gamma(1+\alpha)})$ or $\eta = (x + c\frac{t^{\alpha}}{\Gamma(1+\alpha)})$ [34,35] carries the fractional nonlinear partial differential equation

$$P(u, u_t u_x, u_{xx}, u_{xxx}, \ldots) = 0, \tag{1}$$

to a nonlinear ordinary differential equation

$$O(u, u', u'', u''', \ldots) = 0. \tag{2}$$

Equation (2) is integrated as long as all terms contain derivatives where integration constants are neglected. Then, we introduce a new independent variable

$$y = Tanh(\mu\eta), \tag{3}$$

that leads to

$$\frac{d}{d\eta} = \mu(1-y^2)\frac{d}{dy}, \tag{4}$$

$$\frac{d^2}{d\eta^2} = \mu(1-y^2)\frac{d^2}{dy^2}\frac{d}{d\eta} + (-2\mu y)\frac{d}{dy}\frac{d}{d\eta}. \tag{5}$$

Substituting (4) in (5), we obtain

$$\frac{d^2}{d\eta^2} = \mu^2(1-y^2)(-2y\frac{d}{dy} + (1-y^2)\frac{d^2}{dy^2}), \tag{6}$$

where higher derivatives can be derived in a similar way. We then propose the following series expansion:

$$u(\mu\eta) = f(y) = \sum_{i=0}^{L} a_i y^i, \tag{7}$$

and L is a positive integer in most cases that will be defined. Applying Equations (6) and (7) into Equation (2), we get an equation in power of y. After L is defined, coefficients of powers of y are obtained in the resulting equation where these coefficients have to vanish. Then, a system of algebraic equations for the parameters a_i, $i\,(i = 0, 1, \ldots .L)$, c *and* μ is given. Deterring these parameters to obtain the value of L and using (7), then an analytical solution $u(x,t)$ is obtained in a closed form. For non-integer values of *L, an* approximate transformation formula will be used so that an integer value can be obtained. This will be introduced in the forthcoming problems.

2.2 *Tanh-Function Method for Generalized Nonlinear Huxley Equation with Time Fractional Derivative*

In this subsection, the tanh-function method is implemented to obtain analytical solution for the generalized nonlinear Huxley equation with time fractional derivative given as:

$$u_t^{\alpha} - u_{xx} - u(1-u^{\delta})(u^{\delta}-\lambda) = 0\,.$$

Or equivalently,

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^2 u}{\partial x^2} - (1+\lambda)u^{\delta+1} + u^{2\delta+1} + \lambda u = 0 \tag{8}$$

Equation (8) describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals. Using the transformation $\eta = (x - c\frac{t^{\alpha}}{\Gamma(1+\alpha)})$, we obtain

$$-cu' - u'' - (1+\lambda)u^{\delta+1} + u^{2\delta+1} + \lambda u = 0. \tag{9}$$

We define the degree of u as $D[u] = L$ to balance the highest order linear term with nonlinear term which gives rise to the degree of other expressions as:

$$D\left[\frac{d^q u}{d\xi^q}\right] = L+q,\; D\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right] = Lp + s(L+q). \tag{10}$$

Using (10), balancing $u^{2\delta+1}$ with u'' gives

$$2+L = (2\delta+1)L \qquad \text{so that} \qquad L = \frac{1}{\delta}. \tag{11}$$

It is normal to use the transformation

$$u(x,t) = v^{\frac{1}{\delta}}\,, \tag{12}$$

then substituting Equation (12) into Equation (9), we get:

$$-c\frac{1}{\delta}v^{\frac{1-\delta}{\delta}}v' - \frac{1}{\delta}(\frac{1-\delta}{\delta})v^{\frac{1-2\delta}{\delta}}v'^2 - \frac{1}{\delta}v^{\frac{1-\delta}{\delta}}v'' - (1+\lambda)v^{\frac{1+\delta}{\delta}} + v^{\frac{1+2\delta}{\delta}} + \lambda v^{\frac{1}{\delta}} = 0. \tag{13}$$

Simplifying Equation (13) yields

$$c\delta vv' - \delta vv'' - (1-\delta)v'^2 - (1+\lambda)\delta^2 v^3 + \delta^2 v^4 + \delta^2\lambda v^3 = 0. \tag{14}$$

Balancing vv'' with v^4, we find

$$4L = L+2+L \qquad \text{so that} \qquad L = 1. \tag{15}$$

Now, using the tanh-function method, we set

$$v(x,t) = f(y) = a_0 + a_1 y, \tag{16}$$

and

$$y' = \mu - \mu\, y^2. \tag{17}$$

With the help of Mathematica, we get the system of algebraic equations

$$\begin{aligned}
&-c\delta\mu a_0 a_1 - \delta^2 a_0^3 + \delta^2\lambda a_0^2 - \mu^2 a_1^2 - \delta^2\lambda a_0^3 + \delta^2 a_0^4 + \mu^2 a_1^2\delta = 0,\\
&2\delta\mu^2 a_0 a_1 + 4\delta^2 a_0^3 a_1 + 2\delta^2\lambda a_0 a_1 - 3\delta^2 a_0^2 a_1 - c\delta\mu a_1^2 - 3\delta^2\lambda a_0^2 a_1 = 0,\\
&k\delta^2\lambda a_1^2 - 3\delta^2\lambda a_0 a_1^2 - 3\delta^2 a_0 a_1^2 + 6\delta^2 a_0^2 a_1^2 + c\delta\mu a_1 a_0 + 2\mu^2 a_1^2 = 0,\\
&-\delta^2\lambda a_1^3 - \delta^2 a_1^3 + 4\delta^2 a_0 a_1^3 + c\delta\mu a_1^2 - 2\delta\mu^2 a_1 a_0 = 0,\\
&\delta^2 a_1^4 - \upsilon\mu^2 a_1^2 - \mu^2 a_1^2\delta = 0,
\end{aligned}$$

from Equation (16) into Equation (13) and calculating the coefficients of y.

Solving this system, we get the coefficients a_0, a_1, c and μ which are given by:

$$a_0 = \frac{\lambda}{2},\quad a_1 = \pm\frac{\lambda}{2},\quad c = \pm\delta\sqrt{\frac{\lambda}{\delta+1}},\quad \mu = \frac{\delta}{2\sqrt{\delta+1}}, \tag{18}$$

$$a_0 = \frac{1}{2},\quad a_1 = \mp\frac{1}{2},\quad c = \mp\delta\sqrt{\frac{1}{\delta+1}},\quad \mu = \frac{\delta}{2\sqrt{\delta+1}}. \tag{19}$$

Now, we have two sets of traveling wave solutions where c is a free parameter. Recalling that $u = v^{\frac{1}{\delta}}$ and using (18) and (19), the following forms of traveling wave solutions are obtained. The first form of the analytical wave solution is

$$u(x,t) = \left\{\frac{\lambda}{2} \pm \frac{\lambda}{2}\ \tanh\Big(\frac{\delta\lambda}{2\sqrt{\delta+1}}\big(x \mp \frac{1+\delta-\lambda}{1+\delta}\sqrt{1+\delta}\big)\frac{t^\alpha}{\Gamma(1+\alpha)}\Big)\right\}^{\frac{1}{\delta}}, \tag{20}$$

and the second is of the form

$$u(x,t) = \left\{\frac{1}{2} \mp \frac{1}{2}\ \tanh\Big(\frac{\delta}{2\sqrt{(\delta+1)}}\left(x \pm \delta\sqrt{1+\delta}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}\Big)\right\}^{\frac{1}{\delta}}. \tag{21}$$

Note: Taking $\delta = 1$ the solution of time fractional Huxley equation takes the form

$$u(x,t) = \left\{\frac{\lambda}{2} \pm \frac{\lambda}{2}\ \tanh\Big(\frac{\lambda}{2\sqrt{2}}(x \mp (2-\lambda)\sqrt{2})\frac{t^\alpha}{\Gamma(1+\alpha)}\Big)\right\}. \tag{22}$$

3 Derivation of the Proposed Cubic the Non-polynomial Spline Functions Method

In this section, to get a new numerical method that helps get numerical solutions, the cubic non-polynomial spline functions that have a polynomial and trigonometric part is applied to the generalized nonlinear Huxley equation with time fractional derivative of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} = u(1-u^\delta)(u^\delta - \lambda),\ 0 \le \alpha \le 1, \tag{23}$$

subject to boundary conditions

$$u(a,t) = g_1(t),\ \ u(b,t) = g_2(t),\ t \ge 0, \tag{24}$$

and initial conditions

$$u(x,0) = f(x),\ a \le x \le b, \tag{25}$$

where λ and δ are parameters, $\delta > 0, \lambda \in (0,1)$.

The proposed spline function takes the form [36], $T_3 = span\{1, x, \sin\omega x, \cos\omega x\}$ with ω as a frequency of the trigonometric part of the spline functions used to improve the method accuracy.

Now, the region $R = [a,b] \times [0,\infty[$ is discretized by a set of points $R_{h,k}$ which are the vertices of a grid of points(x_i,t_j), where $x_i = a + ih$, $i = 0,1,...,N+1$, and $t_j = jk$, $j = 0,1,\ldots$.

The quantities h and k are mesh sizes in the space and time directions.

Let $Z_i^j \equiv Z(x_i,t_j)$ be an approximate to $u_i^j \equiv u(x_i,t_j)$, that forms by the segment $P_i(x,t_j)$ of the spline function passing through the points (x_i, Z_i^j) and (x_{i+1}, Z_{i+1}^j). Each of these segments takes the form

$$P_i(x,t_j) = a_i(t_j)\ \cos\omega(x-x_i) + b_i(t_j)\ \sin\omega(x-x_i) + c_i(t_j)\ (x-x_i) + d_i(t_j), \tag{26}$$

for each $i = 0,1,\ldots,N$. To obtain expressions for the coefficients of (4) in terms of Z_i^j, Z_{i+1}^j, S_i^j and S_i^{j+1}, we first define

$$P_i(x_i,t_j) = Z_i^j,\ \ P_i(x_{i+1},t_j) = Z_{i+1}^j,\ \ P_i^{(2)}(x_i,t_j) = S_i^j, and\ P_i^{(2)}(x_{i+1},t_j) = S_{i+1}^j. \tag{27}$$

Using Equations (4) and (5), we obtain

$$a_i + d_i = Z_i^j,$$

$$a_i \cos\theta + b_i \sin\theta + c_i h + d_i = Z^j_{i+1}, \tag{28}$$
$$-a_i \omega^2 = S^j_i,$$
$$-a_i \omega^2 \cos\theta - b_i \omega^2 \sin\theta = S^j_{i+1},$$

where $a_i \equiv a_i(t_j)$, $b_i \equiv b_i(t_j)$, $c_i \equiv c_i(t_j)$, $d_i \equiv d_i(t_j)$ and $\theta = \omega\, h$. Solving the last four equations, we obtain the following expressions

$$a_i = -\frac{h^2}{\theta^2} S^j_i, b_i = \frac{h^2\left(\cos\theta\, S^j_i - S^j_{i+1}\right)}{\theta^2 \sin\theta}, c_i = \frac{\left(Z^j_{i+1} - Z^j_i\right)}{h} + \frac{h\left(S^j_{i+1} - S^j_i\right)}{\theta^2}, d_i = \frac{h^2}{\theta^2} S^j_i + Z^j_i. \tag{29}$$

From continuity condition of the first derivative at $x = x_i$, that is $P^1_i(x_i,t_j) = P^1_{i-1}(x_i,t_j)$, we obtain

$$b_i \omega + c_i = -a_{i-1}\omega \sin\theta + b_{i-1}\omega\cos\theta + c_{i-1}. \tag{30}$$

Using expressions in (29), Eq. (30) becomes

$$\begin{aligned}&\frac{h^2\omega\left(\cos\theta\, S^j_i - S^j_{i+1}\right)}{\theta^2\sin\theta} + \frac{\left(Z^j_{i+1}-Z^j_i\right)}{h} + \frac{h\left(S^j_{i+1}-S^j_i\right)}{\theta^2}\\ &= \frac{h^2\omega}{\theta^2} S^j_{i-1}\sin\theta + \frac{h^2\omega\left(\cos\theta\, S^j_{i-1} - S^j_i\right)}{\theta^2\sin\theta}\cos\theta + \frac{\left(Z^j_i - Z^j_{i-1}\right)}{h} + \frac{h\left(S^j_i - S^j_{i-1}\right)}{\theta^2},\end{aligned}$$

or

$$\begin{aligned}&Z^j_{i+1} - 2Z^j_i + Z^j_{i-1} = \left(\tfrac{h^2}{\theta\sin\theta} - \tfrac{h^2}{\theta^2}\right) S^j_{i+1} + \left(\tfrac{-h^2\cos\theta}{\theta\sin\theta} + \tfrac{h^2}{\theta^2} - \tfrac{h^2\cos\theta}{\theta\sin\theta} + \tfrac{h^2}{\theta^2}\right) S^j_i\\ &+ \left(\tfrac{h^2\sin\theta}{\theta} + \tfrac{h^2\cos^2\theta}{\theta\sin\theta} - \tfrac{h^2}{\theta^2}\right) S^j_{i-1},\ \ where\ \ \theta = h\omega\,.\end{aligned}$$

After some manipulations, we get

$$Z^j_{i+1} - 2Z^j_i + Z^j_{i-1} = \gamma S^j_{i+1} + \beta S^j_i + \gamma S^j_{i-1}, \quad i = 1,2,\ldots,N, \tag{31}$$

where $\gamma = \frac{h^2}{\theta\sin\theta} - \frac{h^2}{\theta^2}$, $\beta = -\frac{2h^2\cos\theta}{\theta\sin\theta} + \frac{2h^2}{\theta^2} and\ \theta = \omega h.$

Remarks

1- Truncation error for Equation (31), that is

$$T^{*j}_i = \left(u^j_{i-1} + u^j_{i+1}\right) - 2u^j_i - \gamma\left(D^2_x u^j_{i-1} + D^2_x u^j_{i+1}\right) - \beta D^2_x u^j_i,$$

can be achieved by expanding this equation in Taylor series in terms of $u(x_i,t_j)$ and its derivatives are as follows:

$$T^{*j}_i = \left(h^2 - (\beta + 2\gamma)\right) D^2_x u^j_i + h^2\left(\frac{h^2}{12} - \gamma\right) D^4_x u^j_i + h^4\left(\frac{h^2}{360} - \frac{\gamma}{12}\right) D^6_x u^j_i + \ldots.$$

Using that formula, with $\beta + 2\gamma = h^2$ the schema is off $O\left(h^2\right)$, however with $\beta + 2\gamma = h^2$ and $\gamma = \frac{h^2}{12}$ the schema is of $O\left(h^4\right)$.

2- As $\omega \to 0$, that is $\theta(\omega) \to 0$, then $(\gamma,\beta) \to \left(\frac{h^2}{6}, \frac{4h^2}{6}\right)$, $\beta + 2\gamma = h^2$. The system (31) becomes a normal cubic spline, that is

$$Z^j_{i+1} - 2Z^j_i + Z^j_{i-1} = \frac{h^2}{6}(S^j_{i+1} + 4S^j_i + S^j_{i-1}), \quad i = 1,2,\ldots,N.$$

Using the generalized Huxley Equation (23), we can write S^j_i in the form

$$S^j_{i+1} = \frac{\partial^2 Z^j_{i+1}}{\partial x^2} = \left(\frac{\partial^\alpha Z^j_{i+1}}{\partial t^\alpha} - (Z^j_{i+1})\left(1 - (Z^j_{i+1})^\delta\right)\left((Z^j_{i+1})^\delta - \lambda\right)\right),$$
$$S^j_i = \frac{\partial^2 Z^j_i}{\partial x^2} = \left(\frac{\partial^\alpha Z^j_i}{\partial t^\alpha} - (Z^j_i)\left(1 - (Z^j_i)^\delta\right)\left((Z^j_i)^\delta - \lambda\right)\right), \tag{32}$$

$$S_{i-1}^{j}=\frac{\partial^{2}Z_{i-1}^{j}}{\partial x^{2}}=\left(\frac{\partial^{\alpha}Z_{i-1}^{j}}{\partial t^{\alpha}}-(Z_{i-1}^{j})\left(1-(Z_{i-1}^{j})^{\delta}\right)\left((Z_{i-1}^{j})^{\delta}-\lambda\right)\right).$$

Applying the partial fractional formula of Caputo derivative, we get

$$\frac{\partial^{\alpha}Z\left(x_{i},t_{j}\right)}{\partial t^{\alpha}}=\frac{1}{\Gamma\left(1-\alpha\right)}\int_{0}^{t_{j}}\frac{\partial Z\left(x_{i},s\right)}{\partial t}\left(t_{j}-s\right)^{-\alpha}ds,\ \ t_{j}=jk,\quad 0<\alpha<1. \tag{33}$$

Applying the technique of piecewise, Equation (33) takes the form

$$\frac{\partial^{\alpha}Z\left(x_{i},t_{j}\right)}{\partial t^{\alpha}}=\frac{1}{\Gamma\left(1-\alpha\right)}\sum_{q=0}^{j-1}\int_{qk}^{(q+1)k}\frac{\partial Z\left(x_{i},s\right)}{\partial t}\left(t_{j}-s\right)^{-\alpha}ds,\quad 0<\alpha<1. \tag{34}$$

Because of the non-negative of $(t_j-s)^{-\alpha}$ cannot be negative on $[qk,\ (q+1)k]$, the Theorem of Weighted Mean Value for Integrals [37], given [38,39],

$$\int_{qk}^{(q+1)k}\frac{\partial Z\left(x_{i},s\right)}{\partial t}\left(t_{j}-s\right)^{-\alpha}ds=\frac{\partial Z\left(x_{i},s^{*}\right)}{\partial t}\int_{qk}^{(q+1)k}\left(t_{j}-s\right)^{-\alpha}ds,\quad qk<s^{*}<(q+1)k.$$

The above-mentioned can be discretized as

$$\begin{aligned}\int_{qk}^{(q+1)k}\frac{\partial Z\left(x_{i},s\right)}{\partial t}\left(t_{j}-s\right)^{-\alpha}ds&\approx\frac{Z_{i}^{q+1}-Z_{i}^{q}}{k}\int_{k}^{(q+1)k}\left(t_{j}-s\right)^{-\alpha}ds\\&=\left[\frac{Z_{i}^{q+1}-Z_{i}^{q}}{k}\right]\left[\frac{\left(t_{j}-qk\right)^{1-\alpha}-\left(t_{j}-qk-k\right)^{1-\alpha}}{1-\alpha}\right]\\&=\left[\frac{Z_{i}^{q+1}-Z_{i}^{q}}{k}\right]\left[\frac{\left(jk-qk\right)^{1-\alpha}-\left(jk-qk-k\right)^{1-\alpha}}{1-\alpha}\right]\\&=\frac{1}{k^{\alpha}\left(1-\alpha\right)}\left[Z_{i}^{q+1}-Z_{i}^{q}\right]\left[(j-q)^{1-\alpha}-(j-q-1)^{1-\alpha}\right].\end{aligned}$$

Hence, the partial fractional derivative approaches (12) takes the form

$$\frac{\partial^{\alpha}Z\left(x_{i},t_{j}\right)}{\partial t^{\alpha}}\approx\sigma\sum_{q=0}^{j-1}\varphi_{j,q}^{\alpha}\left[Z_{i}^{q+1}-Z_{i}^{q}\right],\ 0<\alpha<1, \tag{35}$$

where $\varphi_{j,q}^{\alpha}=(j-q)^{1-\alpha}-(j-q-1)^{1-\alpha}$ and $\sigma=\frac{1}{(1-\alpha)\Gamma(1-\alpha)k^{\alpha}}$. Formula (35) allows us to express S_i^j in the form

$$S_{i}^{j}\approx\sigma\sum_{q=0}^{j-1}\varphi_{j,q}^{\alpha}\left[Z_{i}^{q+1}-Z_{i}^{q}\right]+\ \rho_{i}^{j}\left(Z_{i}^{j}\right), \tag{36}$$

which may be written as:

$$\begin{aligned}S_{i-1}^{1}&=\sigma\left(Z_{i-1}^{1}-Z_{i-1}^{0}\right)+\rho_{i-1}^{1}\left(Z_{j}^{1}\right),\\S_{i}^{1}&=\sigma\left(Z_{i}^{1}-Z_{i}^{0}\right)+\rho_{i}^{1}\left(Z_{j}^{1}\right),\\S_{i+1}^{1}&=\sigma\left(Z_{i+1}^{1}-Z_{i+1}^{0}\right)+\rho_{i+1}^{1}\left(Z_{j}^{1}\right),\end{aligned} \tag{37}$$

$$\begin{aligned}S_{i-1}^{j}&=\sigma\left(Z_{i-1}^{j}-Z_{i-1}^{j-1}\right)+\sigma\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left[Z_{i-1}^{q+1}-Z_{i-1}^{q}\right]+\rho_{i-1}^{j}\left(Z_{i-1}^{j}\right),\\S_{i}^{j}&=\sigma\left(Z_{i}^{j}-Z_{i}^{j-1}\right)+\sigma\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left[Z_{i}^{q+1}-Z_{i}^{q}\right]+\rho_{i}^{j}\left(Z_{i}^{j}\right),\end{aligned} \tag{38}$$

$$S_{i+1}^{j} = \sigma\left(Z_{i+1}^{j} - Z_{i+1}^{j-1}\right) + \sigma\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left[Z_{i+1}^{q+1} - Z_{i+1}^{q}\right] + \rho_{i+1}^{j}\left(Z_{i+1}^{j}\right),$$

where $\rho_i^j = \eta\left(1-(Z_i^j)^{\delta}\right)\left((Z_i^j)^{\delta} - \lambda\right)$. Using (16) in Equation (31), we get the following system

$$A_iZ_{i-1}^{1} + B_iZ_i^{1} + C_iZ_{i+1}^{1} = A_i^{*}Z_{i-1}^{0} + B_i^{*}Z_i^{0} + C_i^{*}Z_{i+1}^{0}, \tag{39}$$

and

$$A_iZ_{i-1}^{j} + B_iZ_i^{j} + C_iZ_{i+1}^{j} = A_i^{*}Z_{i-1}^{j-1} + B_i^{*}Z_i^{j-1} + C_i^{*}Z_{i+1}^{j-1} + \mu_i^{j}, \quad i = 1,2,...,N \; and \; j = 1,2,..., \tag{40}$$

where

$$A_i = \sigma - \gamma + \sigma\gamma\rho_{i-1}^{j},\; A_i^{*} = -\gamma,$$

$$B_i = -2\sigma - \beta + \sigma\beta\rho_i^{j},\; B_i^{*} = -\beta,$$

$$C_i = \sigma - \gamma + \sigma\gamma\rho_{i-1}^{j},\; C_i^{*} = -\gamma,$$

and

$$\mu_i^{j} = \sigma\gamma\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left[Z_{i-1}^{q+1} - Z_{i-1}^{q}\right] + \sigma\beta\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left[Z_i^{q+1} - Z_i^{q}\right] + \sigma\gamma\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left[Z_{i+1}^{q+1} - Z_{i+1}^{q}\right], \quad j \geq 2.$$

Or

$$\mu_i^{j} = \sigma\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left(\gamma Z_{i-1}^{q+1} + \beta Z_i^{q+1} + \gamma Z_{i+1}^{q+1}\right) - \sigma\sum_{q=0}^{j-2}\varphi_{j,q}^{\alpha}\left(\gamma Z_{i-1}^{q} + \beta Z_i^{q} + \gamma Z_{i+1}^{q}\right), \quad j \geq 2. \tag{41}$$

System (41) represents of N equations on unspecified variables Z_i, $i = 0,...,N+1$. Toget a system solution, one needs 2-more equations. Such equations are obtained when applying conditions in (24)

Remarks In order to cope with the nonlinear terms in (41), we follow the following steps:

1- At $j = 1$, we approximate ρ_i^1 by $\rho_i^{1\&}$ obtained from Z_i^0 and get a first approximation to Z_i^1. Hence, we get δ_i^1 from Z_i^1 to refine the approximation to Z_i^1.

2- At $j = m$, we approximate ρ_i^m by remarks $\rho_i^{m\&}$ obtained from Z_i^{m-1} and get a first approximation to Z_i^m. Now, we calculate δ_i^m from Z_i^m to refine the approximation to Z_i^m.

4 Stability Analysis of the Proposed Numerical Method

In this section, the Von-Neumann concept is applied to study the stability analysis of the suggested scheme. To carry out this, we linearise the nonlinear term $u(1-u^{\delta})(u^{\delta}-\lambda)$ of Huxley equation (8) by making the quantity $\Psi(u) = u(1-u^{\delta})(u^{\delta}-\lambda)$ a locally constant which is equivalent to assuming values $\rho_{i+1}, \rho_i \; and \; \rho_{i-1}$ are equal to a local constant d^{*} in difference equation (41). According to the Von Neumann method, we have [6]

$$Z_i^{j} = \zeta^{j}\exp(q\varphi ih), \tag{42}$$

with φ is the mode number, $q = \sqrt{-1}$, h is element size and ζ is the amplification factor. Substituting Equation (42) into Equation (40), we obtain

$$\begin{aligned}\zeta^{j+1}\{A_i\exp((i-1)q\varphi h) + B_i\exp(iq\varphi h) + C_i\exp((i+1)q\varphi h)\} = \\ \zeta^{j}\{A_i^{*}\exp((i-1)q\varphi h) + B_i^{*}\exp(iq\varphi h) + C_i^{*}\exp((i+1)q\varphi h)\},\end{aligned} \tag{43}$$

where

$$A_i = \sigma - \gamma + \sigma\gamma\rho_{i-1}^{j},\; A_i^{*} = -\gamma,$$

$$B_i = -2\sigma - \beta + \sigma\beta\rho_i^j,\ B_i^* = -\beta,$$

$$C_i = \sigma - \gamma + \sigma\gamma\rho_{i-1}^j,\ C_i^* = -\gamma.$$

After simple calculations, Equation (43) becomes

$$\zeta = \frac{A_i^* \exp(-q\phi) + B_i^* + C_i^* \exp(q\phi)}{A_i \exp(-q\phi) + B_i + C_i \exp(q\phi)}, \tag{44}$$

where $\phi = \varphi h$. Using Euler's formula, that is $\exp[q\phi] = \cos\phi + q\sin\phi$, Equation (44) can be rewritten in the form

$$\zeta = \frac{(C_i^* + A_i^*)\cos\phi + B_i^* + q(C_i^* - A_i^*)\sin\phi}{(C_i + A_i)\cos\phi + B_i + q(C_i - A_i)\sin\phi}, \tag{45}$$

which can be rewritten as

$$\zeta = \frac{(-2\gamma)\cos\phi + (-\beta)}{(2\sigma - 2\gamma + 2\sigma\gamma d^*)\cos\phi + (-2\sigma - \beta + \sigma\beta d^*)},$$

or

$$\zeta = \frac{-(\beta + 2\gamma\cos\phi)}{-2\sigma(1-\cos\phi) - (\beta + 2\gamma\cos\phi) + \sigma d^*(\beta + 2\gamma\cos\phi)}.$$

After slight rearrangement, this equation becomes

$$\zeta = \frac{(\beta + 2\gamma\cos\phi)}{(\beta + 2\gamma\cos\phi) + 2\sigma(1-\cos\phi) - d^*(\sigma\beta + 2\sigma\gamma\cos\phi)}.$$

If we take $\beta > 0$ *and* $\gamma > 0$ such that $\beta > 2\gamma$ but $(1-\cos\phi)$ is non-negative then we will be sure that $\beta + 2\gamma\cos\phi > 0$. If choosing $\sigma,\ \gamma$ *and* β small enough to make $d^*(\sigma\beta + 2\sigma\gamma\cos\phi) \to 0$ then, the last equation is close to

$$\zeta = \frac{(\beta + 2\gamma\cos\phi)}{(\beta + 2\gamma\cos\phi) + 2\sigma(1-\cos\phi)}.$$

For stability, we must have $|\zeta| \le 1$ (otherwise ζ^j in (42) would grow in an unbounded manner). This condition is satisfied for $\beta > 0,\ \gamma > 0$ and $\beta > 2\gamma$. Finally, we can say that our system is stable for $\beta > 0,\ \gamma > 0$ and $\beta > 2\gamma$ such that $\sigma,\ \gamma$ and β are chosen to be small enough such that σ depends on $\alpha \to 1$.

5 Numerical Results

Here, using the method presented above by applying it to the generalised Huxley Equation (23), we present the numerical results obtained. The exact solution of this equation is of the form

$$u(x,t) = \left\{\frac{\lambda}{2} + \frac{\lambda}{2}\tanh\left[\frac{\delta\lambda}{2}\sqrt{\frac{\eta}{(1+\delta)}}\left(x - \frac{1+\delta-\lambda}{1+\delta}\sqrt{(1+\delta)}\,\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\right]\right\}^{1/\delta}, \quad a \le x \le b,\ t \ge 0,$$

where λ and δ are parameters, $\delta > 0, \lambda \in (0,1)$. Using the following conditions

$$u(x,0) = \left\{\frac{\lambda}{2} + \frac{\lambda}{2}\tanh\left[\frac{\delta\lambda}{2}\sqrt{\frac{1}{(1+\delta)}}\,x\right]\right\}^{1/\delta},$$

$$u(0,t) = \left\{\frac{\lambda}{2} + \frac{\lambda}{2}\tanh\left[\frac{\delta\lambda}{2}\sqrt{\frac{1}{(1+\delta)}}\left(-\frac{1+\delta-\lambda}{1+\delta}\sqrt{(1+\delta)}\,\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\right]\right\}^{1/\delta},$$

$$u(1,t) = \left\{\frac{\lambda}{2} + \frac{\lambda}{2}\tanh\left[\frac{\delta\lambda}{2}\sqrt{\frac{1}{(1+\delta)}}\left(1 - \frac{1+\delta-\lambda}{1+\delta}\sqrt{(1+\delta)}\,\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\right]\right\}^{1/\delta},$$

Table 1: $\Delta t = 0.005, h = 0.025,\ \ \lambda = 0.1,\ \gamma = h^2/12,\ and\ \ \beta = h^2 - 2\gamma, \alpha = 0.5$

Time	**2.00**	**2.5**	**3.00**	**3.5**
$L_\infty - error$	2.3199×10^{-5}	2.37462×10^{-5}	2.37462×10^{-5}	2.413868^{-5}

Table 2: $\Delta t = 0.005, h = 0.025,\ \ \gamma = 0.1,\ \ \alpha = h^2/12,\ and\ \ \beta = h^2 - 2\gamma, \alpha = 0.7$

Time	**2.00**	**2.5**	**3.00**	**3.5**
$L_\infty - error$	2.28919×10^{-6}	2.34831×10^{-6}	2.3905×10^{-6}	2.42073^{-6}

Table 3: $\Delta t = 0.005, h = 0.025,\ \ \lambda = 0.1,\ \gamma = h^2/12,\ and\ \ \beta = h^2 - 2\gamma, \alpha = 0.9$

Time	**2.00**	**2.5**	**3.00**	**3.5**
$L_\infty - error$	1.63149×10^{-7}	1.692×10^{-7}	1.7371×10^{-7}	1.77103^{-7}

Table 4: $\Delta t = 0.005, h = 0.025,\ \ \gamma = 0.1,\ \ \alpha = h^2/12,\ and\ \ \beta = h^2 - 2\gamma, \alpha = 1$

Time	**2.00**	**2.5**	**3.00**	**3.5**
$L_\infty - error$	7.40148×10^{-8}	7.376×10^{-8}	7.34161×10^{-8}	7.29844^{-8}

where $a = 0\ and\ b = 1$. Accuracy between the analytic and numerical solutions at each mesh point is measured by computing the absolute error, then compute the L_∞ - error norm. The numerical results are summarized in the following tables for $\Delta x = 0.025$ and $\delta = 1$. The numerical results for $\lambda = 0.1$ are presented in Tables (5.1-5.6).

Table 5: $t = 3.7, \Delta t = 0.005,\ \lambda = 0.1,\ \ \gamma = h^2/12, \beta = h^2 - 2\gamma,\ and\ \alpha = 0.5$

x	**Numerical Solution**	**Analytic Solution**
0.0	0.041377294947890371	0.041377294947890374
0.1	0.04154302242954488	0.0415536570733319
0.2	0.04171042542357709	0.041727794914301106
0.3	0.04187949977788283	0.0419012287026391
0.4	0.04205024186321015	0.042074273460570896
05	0.04222264856776379	0.04224707575652421
0.6	0.042396717290171554	0.04241971985128984
07	0.042572445930789016	0.04259225983232278
0.8	0.04274983288132666	0.042764732840558
0.9	0.042928877012791196	0.042937165539863355
1.0	0.043109577661742414 3	0.043109577661742414 2

Table 6: $t = 3.7, \Delta t = 0.005,\ \lambda = 0.1,\ \ \gamma = h^2/12, \beta = h^2 - 2\gamma,\ and\ \alpha = 0.9$

x	**Numerical Solution**	**Exact Solution**
0.0	0.04130262685316931	0.04130262685316924
0.1	0.041474124464810726	0.04147420624927227
0.2	0.04164583842893575	0.04164596887606904
0.3	0.041817764808917364	0.041817925573029824
0.4	0.041989899654741314	0.041989075522075225
0.5	0.04216223900333557	0.04216241624011511
0.6	0.04233477887888606	0.042334944614419806
0.7	0.042507515293137774	0.042507657217380035
0.8	0.04268044424568129	0.0426805504365799
0.9	0.042853561724224394	0.042853620538718176
1.0	0.043026863704848862 2	0.043026863704848863 2

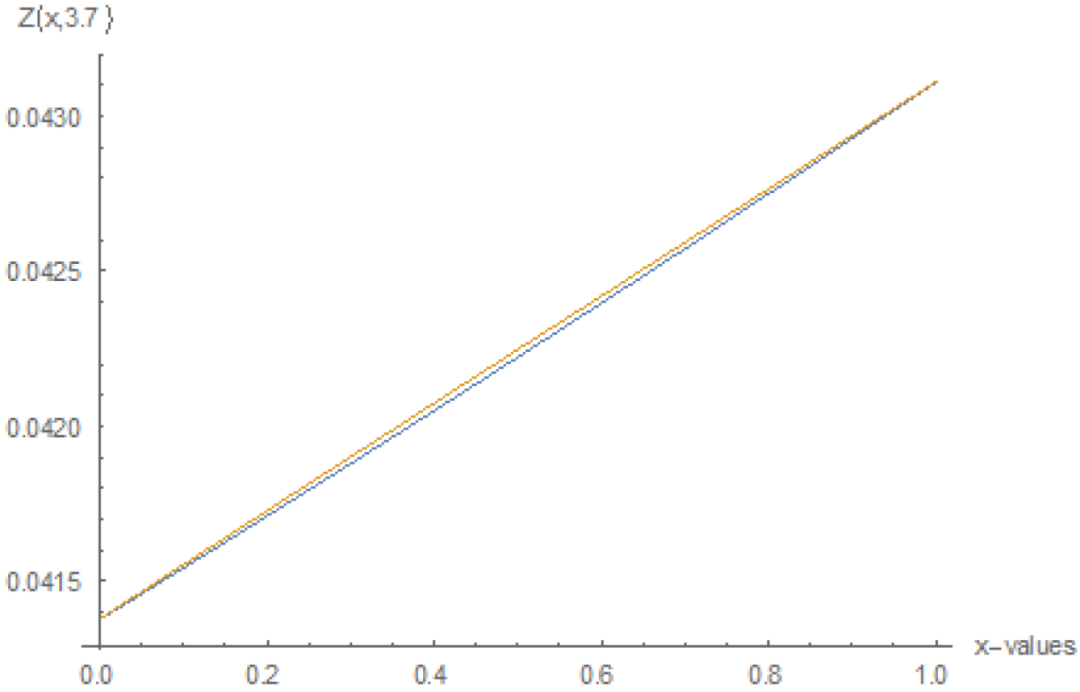

Fig. 1: illustrates the behavior of the exact and numerical solution at $\alpha = 0.5$ for $t = 3.7,\ \Delta t = 0.005,\ \lambda = 0.1, h = 0.025, \gamma = \frac{h^2}{12}$ *and* $\beta = h^2 - 2\gamma$

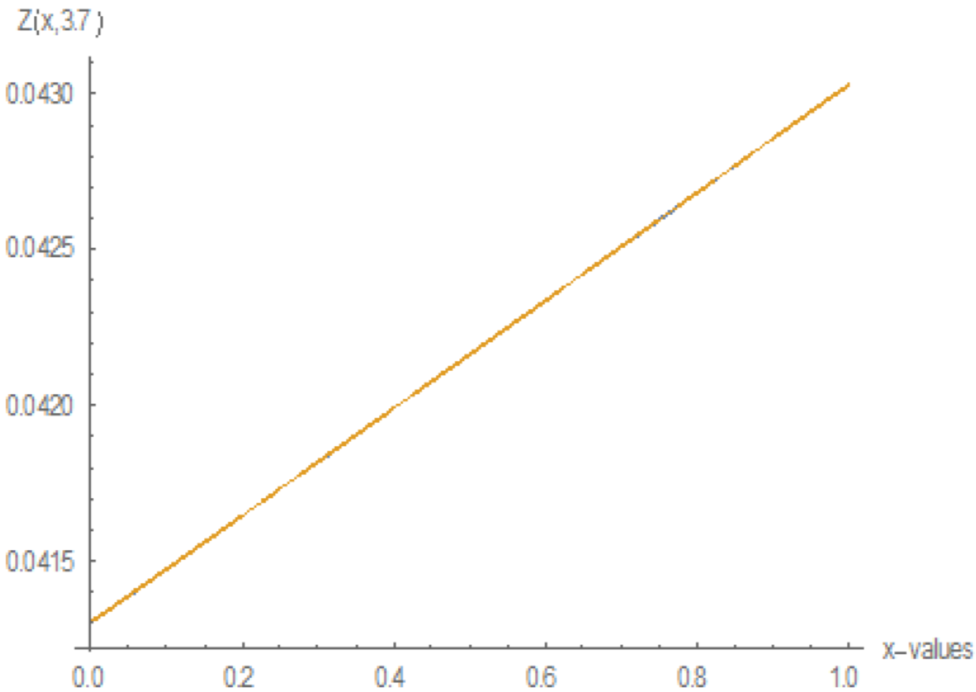

Fig. 2: illustrates the behavior of the exact and numerical solution at $\alpha = 0.9$ for $t = 3.7,\ \Delta t = 0.005,\ \lambda = 0.1, h = 0.025, \gamma = \frac{h^2}{12}$ *and* $\beta = h^2 - 2\gamma$

6 Conclusion

In this paper, the tanh-function method is successfully used to obtain multiple traveling wave solutions to the problem under consideration. Also, new numerical method for solving the generalized nonlinear Huxley equation with fractional time derivative based on non-polynomial splines was proposed. Applying the Von-Neumann stability analysis, the developed method conditionally stable. The obtained approximate numerical solutions maintained good accuracy compared with the exact solutions at $\alpha \to 1$.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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