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Solution of Black-Scholes Fractional Partial Differential Equation with two Assets by Aboodh Decomposition Method

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Abstract: In this paper, we present the solution of Black-Scholes fractional partial differential equation with two assets in the Caputo fractional derivative sense by the Aboodh decomposition method. This method is a mixture between new transform called Aboodh transform and the Adomian decomposition method.

Keywords: Fractional differential equation, Aboodh transform, Adomian decomposition method, Black-Scholes equation.

1 Introduction, motivation and preliminaries

Partial differential equations of fractional order arise in many fields of applied sciences such as applied mathematics, biology, physics, electrical circuits, industrial,etc. Therefore several authors have turned their attentions to solve such equations. In literature, several numerical methods used to solve linear and nonlinear fractional partial differential equations such as Variation iteration method (VIM) [1, 2], Homotopy perturbation method (HPM) [3, 4], Adomian decomposition method (ADM) [5, 6], reduce differential iteration method (RDIM) [7], Natural decomposition method (NDM) [8, 9], Laplace decomposition method (LDM) [10], Aboodh decomposition method (ADM) [11] and, so on. Laplace, Natural, Ezaki, Sumudu, Aboodh, Shehu, and many others [12–17], are different types of integral transforms that are used to solve linear ordinary differential equations, partial differential equation of integer and fractional order. Nonlinear differential equations cannot be solved these integral transforms. Therefore, researchers are combining these transformations with many other methods [1, 3], among them, the Adomian decomposition method (ADM). The idea of this approach is to solve differential equation by expressing the solution in terms of an infinite series, moreover separate the linear and nonlinear terms. The nonlinear parts can be expressed in terms of Adomian polynomials, and the initial approximation solution can be obtained from the initial condition and the terms of independent variables, then by a recurrence relation, we can find other terms of the series. Now we feel compelled to combine the Adomian decomposition method with a relatively new integral transform called Aboodh transform, in what is known the Aboodh decomposition method (ADM). The Black-Scholes partial differential equation was firstly proposed by F. Black and M. Scholes, this model was used to explain the financial derivative and applied to investigate and describe the conductance to option pricing in market, for more about Black-Scholes models see [18–20]. In recent years, many researchers have been discussing and studying the Black-Scholes model of fractional order with one and two assets. For example, in [21] the author studied and used the Black-Scholes equation of fractional order to call option price for bank foreign exchange in China. In [22] the authors used an arithmetic Brownian motion to propose an option pricing equation. However, there are so many different methods developed to obtain the exact and approximate solutions of the Black-Scholes model. For example finite difference method [23], Homotopy perturbation method [24], Mellin transform [25], the Quantic B-spline function [26]. In [27, 28] the authors used the Laplace- homotopy perturbation method and the Laplace- decomposition method (LDM) which are a combination of Laplace transform and homotopy perturbation method, Adomian decomposition method respectively.

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The classical Black-Scholes differential equation with one asset, which describes the price of the option over time [18, 29] is defined as follows:

$$\frac{\partial f}{\partial \tau}+0.5\delta^2 S^2\frac{\partial^2 f}{\partial S^2}+rS\frac{\partial f}{\partial S}-rf=0,\quad S\in\mathscr{R}^+,\tau\in[0,T],$$

with terminal condition:

$$f(S,T)=\max\{0,S-k\},$$

and the classical two assets Black-Scholes PDE for European-style option has been defined as follows:

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\delta_i\delta_j\rho_{ij}S_iS_j\frac{\partial^2 f}{\partial S_iS_j}+\sum_{i=1}^{2}S_i(r-q_i)\frac{\partial f}{\partial S_i}=rf,\quad S_1,S_2\in\mathscr{R}^+,\tau\in[0,T],\tag{1}$$

with terminal condition:
$f(S_1,S_2,T)=\max\{0,C_1S_1+C_2S_2-k\}$, and boundary conditions:

$$\begin{cases} f(S_1,S_2,\tau)=0, & S_1 \ and \ S_2\to 0,\\ f(S_1,S_2,\tau)=C_1S_1+C_2S_2-ke^{-r(T-\tau)}, & S_1 \ or \ S_2\to\infty,\end{cases}$$

where $f=f(S_1,S_2,\tau)$ denotes the call option which depends on the underlying stock prices S_1,S_2 at time τ, for more about the model 1 and the definitions of the parameters, see [30].
The two dimensional pricing problem based on the Black-Scholes equation (1) is presenting which is defined as follows:

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}\delta_1^2S_1^2\frac{\partial^2 f}{\partial {S_1}^2}+\frac{1}{2}\delta_2^2S_2^2\frac{\partial^2 f}{\partial {S_2}^2}+\rho\delta_1\delta_2S_1S_2\frac{\partial^2 f}{\partial S_1\partial S_2}+r\left[S_1\frac{\partial f}{\partial S_1}+S_2\frac{\partial f}{\partial S_2}\right]-rf=0\tag{2}$$

with terminal condition:
$f(S_1,S_2,T)=\max\{0,C_1S_1+C_2S_2-k\}$,
and boundary conditions:

$$\begin{cases} f(S_1,S_2,\tau)=0, & S_1 \ and \ S_2\to 0,\\ f(S_1,S_2,\tau)=C_1S_1+C_2S_2-ke^{-r(T-\tau)}, & S_1 \ or \ S_2\to\infty,\end{cases}$$

following the steps given in [30], we get the fractional Black -Scholes model of two assets which can be written as:

$$D_t^{\alpha}u=\frac{1}{2}\delta_1^2\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\delta_2^2\frac{\partial^2 u}{\partial y^2}+\rho\delta_1\delta_2\frac{\partial^2 u}{\partial x\partial y},\tag{3}$$

with initial condition :

$$u(x,y,0)=\max\{0,c_1e^x+c_2e^y-k\},\tag{4}$$

and boundary conditions:

$$\begin{cases} u=0,\\ u=c_1e^x+\frac{1}{2}\delta_1^{x+\frac{1}{2}\delta_1^2t}t+c_2e^{y+\frac{1}{2}\delta_2^2t}-k, \quad x\to\infty \ or \ y\to\infty,\end{cases}$$

where $D_t^{\alpha}.$ is the time fractional derivative in the Caputo sense, with $\alpha \in (0,1]$ and $c_1 = C_1 e^{\left(r-\frac{1}{2}\delta_1^2\right)T}$, $c_2 = C_2 e^{\left(r-\frac{1}{2}\delta_2^2\right)T}$.

In the present article, we apply the Aboodh decomposition method (ADM) to calculate the explicit solutions of the Black-Scholes model of fractional order with two assets given in (2), this method is an efficient and easy to implement to solve such problems. This approach constructs an approximate solutions in term of an infinite series.
The article is formed as indicated below: In Section 1, we present the mathematical model of the standard Black-Scholes PDE. In Section 2, the definitions of Caputo derivative and the Riemann-Liouville fractional integral and some basic properties are given. In Section 3, we present basic definitions of Aboodh transform, and Aboodh transform of some functions and derivatives. In Section 4, the basic idea of the Aboodh decomposition method is presented. In Section 5, numerical examples are given . This paper ends in Section 6 with conclusions.

2 Preliminaries

Herein, basic definitions and some of the main properties of the Caputo derivative and the Riemann-Liouville fractional integral are presented.

Definition 2.1 [31, 32] A real valued function $g(t), t > 0$, belongs to the space $C_{\alpha}, \alpha \in \Re$ if there exists a real number $q > \alpha$ s.t $g(t) = t^q g_1(t)$, where $g_1(t) \in C[0,\infty)$.

Definition 2.2 [12] A real valued function $g(t), t > 0$ is in $C_{\alpha}^{m}, m \in \cup\{0\}$, if $g^m \in C_{\alpha}$.

Definition 2.3 [31] Let $g(x,t) \in C_{\alpha}$, $(\alpha \geq -1)$ be a continuous function. Then the Riemann-Liouville integral can be defined by

$$I^{\alpha} g(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-p)^{\alpha-1} g(x,p)\, dp,$$

where,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt, \forall \alpha > 0,$$

is the Euler Gamma function.
Definition 2.4 The Caputo derivative of order $\alpha > 0$ of $f(x,t)$ can be defined as:

$$D_t^{\alpha}(g(x,t)) = \begin{cases} \int_0^t \frac{(t-s)^{n-\alpha-1} g^{(n)}(x,s)}{\Gamma(n-\alpha)} ds, & n-1 < \alpha < n, \\ g^{(n)}(x,t), & \alpha \in N. \end{cases}$$

Depending on the previous definitions the well known properties are :

1.$D_t^{\alpha} I_t^{\alpha} g = g, \ \alpha > 0,$

2.$I_t^{\alpha} D_t^{\alpha} g = g - \sum_{k=0}^{n-1} g^k(x,0)\frac{t^k}{k!}, \ \alpha > 0, \ \alpha \in (n-1,n] \ , n \in \mathrm{N},$

3.$I_t^{\alpha} t^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}(t)^{\nu+\alpha-1}, \ \alpha > 0, \ \nu > 0 \ , t > 0,$

4.$D_t^{\alpha} t^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)}(t)^{\nu-\alpha-1}, \ \alpha > 0, \ \nu > 0, \ t > 0.$

3 Aboodh transform

In this section, we present the definition of the Aboodh transform, and we give same fundamental properties of this transform.
Let B be a set of functions defined by:

$$B=\{g(t):\exists C,r_1,r_2>0,|g(t)|<Ce^{-vt}\},$$

where, C is a finite constant, and r_1,r_2 are finite or infinite. Then, the Aboodh integral transform [15] is defined by:

$$A[g]=\frac{1}{v}\int_0^{\infty} ge^{-vt}dt, v>0. \tag{5}$$

By simple calculation, one can obtain the following results:

3.1 The Aboodh transform of some elementary functions:

$$A[1]=\frac{1}{v^2},$$
$$A[t^{\mu}]=\frac{\Gamma(\mu+1)}{v^{\mu+2}},\ \ 0\leq\mu,$$
$$A\left[e^{-at}\right]=\frac{1}{v^2+av},$$
$$A[\sin(at)]=\frac{a}{v(v^2+a^2)}.$$

Furthermore, the Aboodh transform is linear, i.e.

$$A[af+bg]=aAf+bAg.$$

3.2 Aboodh transform of some partial derivatives of $g(x,y,t)$:

$$A\left[\frac{\partial g}{\partial t}\right]=vA(g)-\frac{g(x,y,0)}{v},$$
$$A\left[\frac{\partial^2 g}{\partial t^2}\right]=v^2A(g)-\frac{1}{v}\frac{\partial g(x,y,0)}{\partial t}-g(x,y,0),$$
$$A\left[\frac{\partial^n g}{\partial x^n}\right]=\frac{\partial^n}{\partial x^n}A(g).$$

3.3 Aboodh transform of fractional order of $g(x,y,t)$:

(i) For Riemann-Liouville derivative:

$$A[D_t{}^{\alpha}g]=v^{\alpha}\left[A(g)-\sum_{k=1}^{n}\frac{D^{\alpha-k}g(x,y,0)}{v^{k+2}}\right],-1<n-1\leq\alpha<n.$$

(ii) For Caputo derivative:

$$A[D_t{}^{\alpha}g]=v^{\alpha}\left[A(g)-\sum_{k=0}^{n-1}\frac{D^{\alpha-k}g(x,y,0)}{v^{k+2}}\right],n-1<\alpha<n.$$

4 Analysis of (ABDM)

In this part, we explain how the (ABDM) works, thus, we consider the nonlinear fractional differential equation in the form:

$$D_t^{\alpha} u(x,y,t) = L(u(x,y,t)) + N(u(x,y,t)) + g(x,y,t) \tag{6}$$

with initial conditions $D_0^k u(x,y,0) = f_k, k = 0, \ldots .n-1,$, where $D_t^{\alpha} u(x,y,t)$ is the Caputo fractional operator, g is known function, L represents the linear differential operator, and N is the nonlinear differential operator.
First, by taking the Aboodh transform for both side of equation (6), we get:

$$A[D_t^{\alpha} u] = A[L(u)] + A[N(u)] + A[g(x,y,t)], \tag{7}$$

depending on the properties of Aboodh transform given in section(3) and the initial conditions, we get:

$$A[u] = v^{-\alpha} A[L(u)] + v^{-\alpha} A[N(u)] + \sum_{k=0}^{m-1} v^{-k-2} u^{(k)}(x,y,0) + v^{-\alpha} A[(g(x,y,t))], \tag{8}$$

now, by applying the inverse Aboodh transform for both side of equation (8), we obtain:

$$u = G + A^{-1}[v^{-\alpha} A[L(u)] + v^{-\alpha} A[N(u)]], \tag{9}$$

where, $G(x,y,t)$ represents the terms arising from the non-homogeneous term and the initial conditions. Now, assume that the solution of (3) can be expressed by the following infinite series:

$$u = \sum_{n=0}^{\infty} u_n, \tag{10}$$

and, $Nu(x,y,t)$ is replaced by the series of the Adomian polynomials [6] given by:

$$Nu(x,y,t) = \sum_{m=0}^{\infty} A_m(u_0,u_1,u_2,\ldots), \tag{11}$$

where,

$$A_m = \frac{1}{m!} \frac{d^m}{d\lambda^m} \left[N\left(\sum_{m=0}^{\infty} \lambda^m u_m \right) \right]_{\lambda=0}, m = 0,1,2,\ldots,$$

substituting equations (10),(11) we obtain:

$$\sum_{n=0}^{\infty} u_n = G + A^{-1}\left[A\left[L\left(\sum_{n=0}^{\infty} u_n \right) v^{-\alpha} \right] + v^{-\alpha} A\left[\sum_{n=0}^{\infty} A_n(u_0,u_1,u_2,\ldots) \right] \right], \tag{12}$$

equation (12) is the coupling of Aboodh transform and (ABDM).
Comparing both sides of equation (12) we obtain the following general recursive relations:

$$u_0 = G,$$

$$u_1 = A^{-1}[v^{-\alpha} A[L(u_0)] + v^{-\alpha} A[A_0]],$$

$$u_2 = A^{-1}[v^{-\alpha} A[L(u_1)] + v^{-\alpha} A[A_1]],$$

$$u_{n+1} = A^{-1}\left[v^{-\alpha} A[L(u_n)] + v^{-\alpha} A[A_n] \right], n \geq 0.$$

.

4.1 Applying Aboodh decomposition method on fractional Black-Scholes model

Herein, we apply Aboodh decomposition method to Black -Scholes model given in (3) with respect to the condition(4). Taking the Aboodh transform, we get:

$$A\left(D_t^{\alpha}u(x,y,t)\right) = A\left(\frac{1}{2}\delta_1^2\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\delta_2^2\frac{\partial^2 u}{\partial y^2}+\rho\delta_1\delta_2\frac{\partial^2 u}{\partial x\partial y}\right), \tag{13}$$

by substituting Aboodh transform of the Caputo fractional derivative with respect to t, then using the inverse Aboodh transform equation,(13) becomes,

$$u(x,y,t) = u(x,y,0)+A^{-1}\left[v^{-\alpha}A\left(\frac{1}{2}\delta_1^2\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\delta_2^2\frac{\partial^2 u}{\partial y^2}+\rho\delta_1\delta_2\frac{\partial^2 u}{\partial x\partial y}\right)\right], \tag{14}$$

applying the (ADM), we represent the approximated solution as:

$$u=\sum_{n=0}^{\infty}u_n, \tag{15}$$

next, we substitute equation (15) in equation (14) to obtain:

$$\sum_{n=0}^{\infty}u_n = u_0+A^{-1}\left[v^{-\alpha}A\left(\frac{1}{2}\delta_1^2\frac{\partial^2\sum_{n=0}^{\infty}u_n}{\partial x^2}+\frac{1}{2}\delta_2^2\frac{\partial^2\sum_{n=0}^{\infty}u_n}{\partial y^2}+\rho\delta_1\delta_2\frac{\partial^2\sum_{n=0}^{\infty}u_n}{\partial x\partial y}\right)\right] \tag{16}$$

from equation (16) we have:

$$\begin{cases} u_0 = u(x,y,0) \\ u_{n+1} = A^{-1}\left[v^{-\alpha}A\left(\frac{1}{2}\delta_1^2\frac{\partial^2 u_n}{\partial x^2}+\frac{1}{2}\delta_2^2\frac{\partial^2 u_n}{\partial y^2}+\rho\delta_1\delta_2\frac{\partial^2 u_n}{\partial x\partial y}\right)\right], n\geq 0, \end{cases} \tag{17}$$

then, we can write $u_0,u_1,...,u_n$ in the general form as:

$$u_0 = u(x,y,0) = \max\{c_1e^x+c_2e^y-k,0\},$$

$$u_1(x,y,t) = A^{-1}\left[v^{-\alpha}A\left(\frac{1}{2}\delta_1^2\frac{\partial^2 u_0}{\partial x^2}+\frac{1}{2}\delta_2^2\frac{\partial^2 u_0}{\partial y^2}+\rho\delta_1\delta_2\frac{\partial^2 u_0}{\partial x\partial y}\right)\right]$$

$$= A^{-1}\left[v^{-\alpha}\left(\frac{1}{2}\delta_1^2\max\{c_1e^x,0\}+\frac{1}{2}\delta_2^2\max\{c_1e^y,0\}\right)\frac{1}{v^{\alpha+2}}\right]$$

$$= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{1}{2}\delta_1^2\max\{c_1e^x,0\}+\frac{1}{2}\delta_2^2\max\{c_1e^y,0,\}\right),$$

$$u_n(x,y,t) = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\left(\frac{1}{2^n}\delta_1^{2n}\max\{b_1e^x,0\}+\frac{1}{2^n}\delta_2^{2n}\max\{b_1e^y,0\}\right),$$

finally, the solution of equation (3) and (4) is given by the following series:

$$u = \max\{c_1e^x+c_2e^y-k,0\}+\sum_{n=1}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\left(\frac{1}{2^n}\delta_1^{2n}\max\{c_1e^x,0\}+\frac{1}{2^n}\delta_2^{2n}\max\{c_1e^y,0\}\right) \tag{18}$$

choosing the initial condition in the form:

$$u(x,y,0) = \{0, c_1 e^x + c_2 e^y - k\} + e^{x+y} t^{\alpha}, \tag{19}$$

the solution of equation (3) will be:

$$\begin{aligned} u =& \max\{0, c_1 e^x + c_2 e^y - k\} \\ &+ e^{x+y} t^{\alpha} + \sum_{n=1}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \left(\frac{1}{2^n} \delta_1^{2n} \max\{c_1 e^x, 0\} + \frac{1}{2^n} \delta_2^{2n} \max\{c_1 e^y, 0\} \right) \\ &+ e^{x+y} \sum_{n=1}^{\infty} \frac{t^{(n+1)\alpha} \Gamma(1+\alpha)}{\Gamma(1+\alpha+n\alpha)} \left(\frac{\delta_1^2}{2} + \frac{\delta_2^2}{2} + \rho \delta_1 \delta_2 \right)^n \\ &- e^{x+y} \sum_{n=1}^{\infty} \frac{t^{n\alpha} \Gamma(1+\alpha)}{\Gamma(1+n\alpha)} \left(\frac{\delta_1^2}{2} + \frac{\delta_2^2}{2} + \rho \delta_1 \delta_2 \right)^{n-1}. \end{aligned} \tag{20}$$

5 Application

In this part, we present a numerical example of the solutions to illustrate the explicit solutions, we use the following parameters given in table (1), to compute the solution of the European call option.

Table 1: Financial parameters.

parameters	value
k(dollars)	50
r	0.03
T	2
σ_1	0.10
σ_2	0.15
ρ	0.50
C_1	3
C_2	2

Depending on the values in Table 1, we get $c_1 = 3.15$ and $c_2 = 2.07$, for $\alpha = 1$, the solution of equation (3) subject to initial conditions (4) ,(19) will be equal, and given by:

$$u(x,y,t) = \max\{3.15e^x + 2.07e^y - 50, 0\} + \max\{3.15e^x, 0\} \left[e^{0.00005t} - 1 \right] + \max\{2.07e^y, 0\} \left[e^{0.011t} - 1 \right], \tag{21}$$

Figure 1, shows the surface plot of equation (21) with $t = 1$ and $0 \le x, y \le 100$.

Note that the solution (21) is the solution of the classical Black-Scholes partial differential equation given in (2), if we choose $\alpha \neq 1$, say 0.9, then the solution of equation (3) with respect to condition (4) can be calculated as :

$$\begin{aligned} u =& \max\{3.15e^x + 2.07e^y - 50, 0\} + \max\{3.15e^x, 0\} \sum_{n=1}^{\infty} \frac{(0.005t)^n}{\Gamma(0.9n+1)} \\ &+ \max\{2.07e^y, 0\} \sum_{n=1}^{\infty} \frac{(0.011t)^n}{\Gamma(0.9n+1)}, \end{aligned} \tag{22}$$

Figure 2, shows 3D plot of equation (21) with $t = 1$ and $0 \le x, y \le 100$.

and, the solution of equation (3) subject to initial condition (19) can be calculated by:

$$\begin{aligned} u(x,y,t) =& \max\{3.15e^x + 2.07e^y - 50, 0\} + e^{x+y} t^{0.9} + \max\{3.15e^x, 0\} \sum_{n=1}^{\infty} \frac{(0.005t)^n}{\Gamma(0.9n+1)} \\ &+ \max\{2.07e^y, 0\} \sum_{n=1}^{\infty} \frac{(0.011t)^n}{\Gamma(0.9n+1)} + \Gamma(1.9) e^{x+y} t^{0.9} \sum_{n=1}^{\infty} \frac{(0.024t^{0.9})^n}{\Gamma(0.9n+1)} \\ &- \frac{\Gamma(1.9)}{0.024} e^{x+y} t^{0.9} \sum_{n=1}^{\infty} \frac{(0.024t^{0.9})^n}{\Gamma(0.9n+1)}. \end{aligned} \tag{23}$$

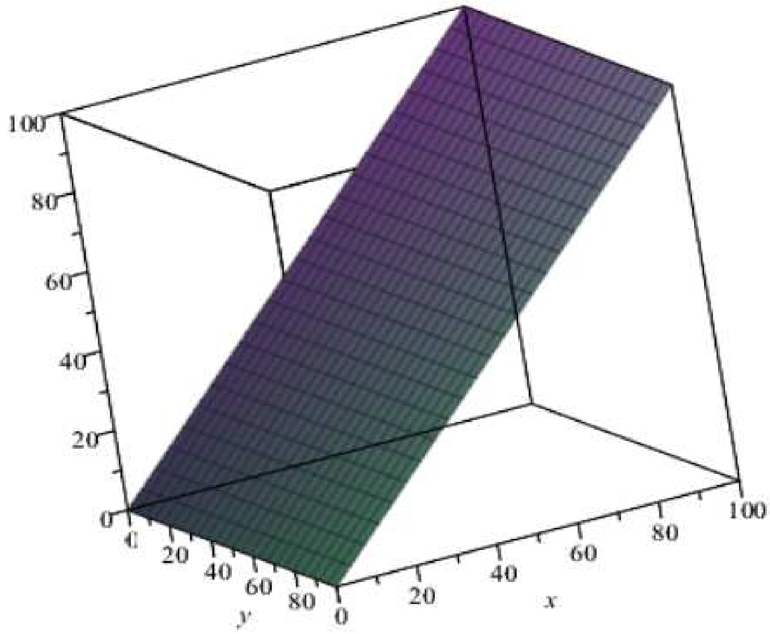

Fig. 1: The explicit solution u with $0 \leq x, y \leq 100$ and $t = 1$.

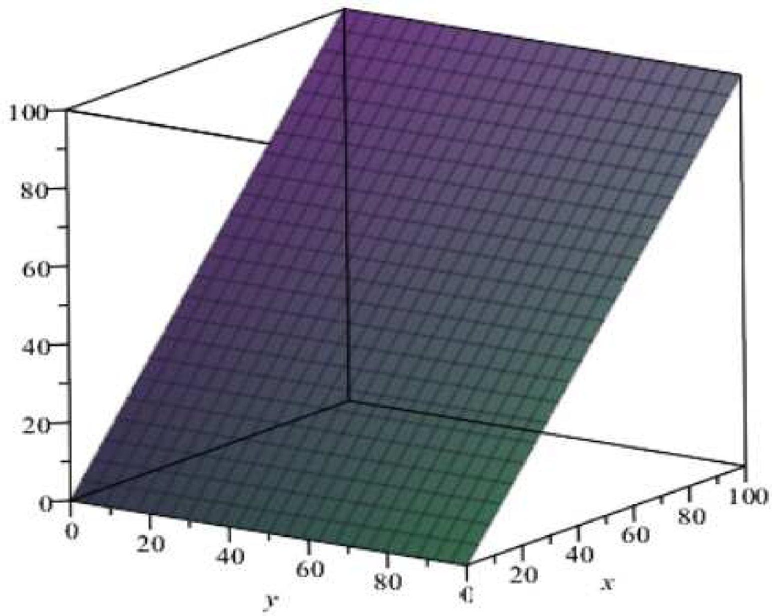

Fig. 2: The explicit solution u with $0 \leq x, y \leq 100$ and $t = 1$

Figure 3, shows 3D plot of equation (5) with $t = 1$ and $0 \leq x, y \leq 100$.

6 Conclusion

In this study, the mixture of the Aboodh transform and the Adomian decomposition method is applied to solve fractional Black-Scholes differential Equation with two Assets, to test the efficiency of the method, we apply this method on

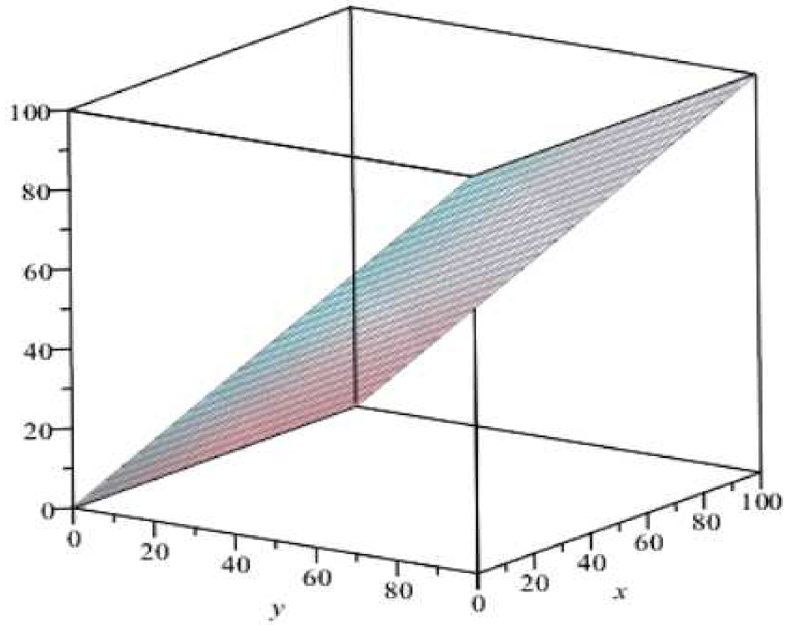

Fig. 3: The explicit solution u with $0 \leq x, y \leq 100$ and $t = 1$

numerical examples, by using the initial condition only this combination of two methods is successfully implemented, this method is simple and accurate to find the approximate and analytical solutions of initial value fraction partial differential equations, the obtained solution is coincide with the solutions obtained by published papers, thus we strongly recommend this approach for solving linear and nonlinear fractional differential equations.

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