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The Approximative Hamiltonian for Dicke Model Defined in Terms of the One-Zone Potential

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1 Introduction

The method of approximative Hamiltonians is one of the most helpful method in statistical physics. It has been shown within the framework of the method that, for a number of nonlinear problems, model Hamiltonians are equivalent to approximative ones in terms of the thermodynamic variables [5-8], [18]. This method consists of substituting some operator-expressions by functions from nonlinear equations arised from the conditions of self-consistency. The problem is regarded as being solvable if the solution of this nonlinear equation exists.

A new approach to the approximative Hamiltonian method (AHM) was developed by D. Ya. Petrina and E. D. Belokolos [4]. The new AHM was used to investigate the Fröhlich Hamiltonian, which gives an account of interactions of electrons having a countable set of phonon modes. The results, obtained for the first time by applying the inverse scattering method, revealed a powerful way of improving the AHM. The authors have succeeded in proving that a) the model Fröhlich type Hamiltonian is equivalent to the approximative one and b) the self-consistency agrees well with the equations of the Peierls-Fröhlich problem, which may be exactly solvable by the finite zone potentials [2], [12],[13] derived from the inverse scattering method [1], [15],[17] in one-dimensional case.

A large number of articles devoted to study of the Dicke model is available [9], [11], [16]. Here, we investigate the Dicke model with the Petrins-Belokolos's method. This method offers the advantage that it is easy to simplify the construction of the self-consistent equation and the structure of approximative Hamiltonians [7], [9], [10]. In addition, the AHM allows the exact solution of the self-consistent equation to be found and, the approximative Hamiltonian for the Dicke model to be defined in terms of one-zone potential.

2 Construction of Approximative Hamiltonian for the Dicke Model

Consider a model system confined in a glass $\Lambda \in R^1$ of a finite volume $|\Lambda| < \infty$. We assume that the Hamiltonian [7] describes the system given by

$$H = \sum_{k=1}^{M} \omega_k b_k^+ b_k^- + \varepsilon S^z + \frac{1}{\sqrt{|\Lambda|}} \sum_{k=1}^{M} \lambda_k (b_k^- J_k^+ + b_k^+ J_k^-), \tag{2.1}$$

where $\omega_k \geq \omega_0 > 0,\ S^z = \sum_{j=1}^{N} \sigma_j^z,\ J_k^{\pm} = \sum_{j=1}^{N} (\sigma_j^{\pm} + \mu_k \sigma_j^{\mp}),\ \sigma_j^{\pm} = \sigma_j^x + i\sigma_j^y$ and $\{\sigma_j^x, \sigma_j^y, \sigma_j^z\}_{j=1}^N$ are the Pauli matrices.

$$\sigma_j^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_j^y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_j^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

correspond to spin $1/2$, that is, $\dim H_L = 2^N;\ \varepsilon \in R^1,\ |\mu_{jk}| \leq C,\ k = 1, 2, \ldots, M;\ \lambda_k \equiv \lambda(k)$ are real numbers (the constants of interaction, involving k-mode).

The model describes the interaction of electromagnetic field with N two-level molecules placed in a resonator Λ, having M normal frequencies $\{\omega_k\}_k^M$.

The system under study consists of two interacting subsystems B and L. A state space of the system H is governed by the tensor product of proper Hilbert spaces

$$\mathcal{H} = \mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{L}}$$

of pairs of functions $\Phi = \Phi_B \otimes \Phi_L \in \mathcal{H}$, , whose scalar product is given by

$$(\Phi_1, \Phi_2) = (\Phi_{B_1}, \Phi_{B_2})(\Phi_{L_1}, \Phi_{L_2}).$$

The subsystem L contains N particles and satisfies the stability condition

$$H_{(\Lambda^N)} \geq -NC,$$

where C is the constant for any Λ and N. The subsystem B is a free boson field with finite number of mode M and energies $\omega_\alpha \geq \omega_0 > 0 \qquad (\alpha = 1, 2, \ldots, M)$,

$$[b_{\alpha_1}, b_{\alpha_2}] = \delta_{\alpha_1, \alpha_2}, \qquad \alpha_1, \alpha_2 = (1, 2, \ldots, M),$$

where b_α^+ and b_α^- are the creation operator and the annihilation operator for bosons in α mode. For $M < \infty$

$$B = \otimes_{i=1}^{M} B_i, \qquad B_i = B, \qquad i = 1, 2, \ldots, M \tag{2.2}$$

and, consequently, $\Phi = \Phi_B \otimes \Phi_L = \otimes_{i=1}^{M} \Phi_B^i \otimes \Phi_L^i$,

$$b_{\alpha_1}^+ = I \otimes I \otimes \cdots \otimes b^+ \otimes \cdots \otimes I, \tag{2.3}$$

and

$$b_{\alpha_2}^- = I \otimes I \otimes \cdots \otimes b^- \otimes \cdots \otimes I, \tag{2.4}$$

where the operators b^+ and b^- stand for α_1 and α_2 in the tensor product, respectively.

The domain

$$T_B = \sum_{k=1}^{M} \omega_k b_k^+ b_k^- \otimes I$$

is defined by the domains of infinite operators b^+ and b^-. The operators b^+ and b^- have dense domains $D(b^+)$ and $D(b^-)$. So the domains $D(b_k^+)$ and $D(b_k^-)$ are also dense in the space $\mathcal{H}_{\mathcal{B}}$, i.e. the operators b_k^+ and b_k^- are self conjugated [7].

The operator H_L on H takes the form

$$H_L = I \otimes H_L(\Lambda^N)$$

and, consequently, the operator H_0 given by

$$H_0 = H_B + H_L$$

is selfconjugated [7] and has the domain

$$D(H_0) = D\left(\sum_{k=1}^{M} \omega_k b_k^+ b_k^-\right) \otimes D(H_L(\Lambda^N)).$$

Here H_L is equal to

$$H_l = \sum_{j=1}^{N} \varepsilon \sigma_j^z + \sum_{k=1}^{M} \lambda_k (b_k^- J_k^+ b_k^+ J_k^-).$$

The operators $(J_k)_{k=1}^M$ are operators acting on space $\mathcal{H}$ and satisfying the condition:

$$(1/|\Lambda|)|J_k|_{\mathcal{H}_{\mathcal{L}}(*^{\mathcal{N}})} \leq C.$$

Performing the following change of variables in (2.1):

$$\begin{gathered} b_k^+ \longrightarrow \frac{(2\pi)^{3/2}}{(2\pi)^{3/2}} \frac{\sqrt{|\Lambda|}}{\sqrt{|\Lambda|}} b_k^+ = \frac{(2\pi)^{3/2}}{\sqrt{|\Lambda|}} b^+(k), \qquad b_k^- \longrightarrow \frac{(2\pi)^{3/2}}{\sqrt{|\Lambda|}} b^-(k), \\ J_k^+ \longrightarrow \frac{(2\pi)^{3/2}}{\sqrt{|\Lambda|}} J^+(k), \qquad J_k^- \longrightarrow \frac{(2\pi)^{3/2}}{\sqrt{|\Lambda|}} J^-(k), \qquad \omega_k \longrightarrow \omega(k). \end{gathered} \tag{2.5}$$

and using (2.5), the Hamiltonian (2.1) takes the form

$$H(\Lambda, N) = \sum_{k=1}^{N} \omega(k) \frac{(2\pi)^3}{|\Lambda|} b^+(k) b^-(k) + \varepsilon \sum_{j=1}^{N} \sigma_j^z$$

$$+\frac{1}{\sqrt{|\Lambda|}}\sum_{k=1}^{M}\lambda(k)\frac{(2\pi)^3}{|\Lambda|}(b^-(k)J^+(k)+b^+(k)J^-(k)).$$

We substitute the summation by the integration according to the convention

$$\frac{(2\pi)^3}{|\Lambda|}\sum_k \to \int dk.$$

Next we define the Hamiltonian by

$$H(\Lambda,N)=\int\omega(k)b^+(k)b^-(k)dk+\varepsilon\sum_{j=1}^{N}\sigma_j^z$$
$$+\frac{1}{\sqrt{|\Lambda|}}\int\lambda(k)(b^-(k)J^+(k)+b^+(k)J^-(k))dk,$$

where $b^+(k)$ and $b^-(k)$ depend on time through

$$b^-(t,k)=e^{i(H-\tilde{\mu}N)t}b^-(0,k)e^{-(H-\tilde{\mu}N)t},$$
$$b^+(t,k)=e^{-i(H-\tilde{\mu}N)t}b^+(0,k)e^{(H-\tilde{\mu}N)t},$$
$$b^\pm(t,k)|_{t=0}=b^\pm(0,k)\equiv b^\pm(k),$$

where $\tilde{\mu}$ - the chemical potential.

These operators satisfy the commutation relation

$$b^-(k)b^+(k')-b^+(k')b^-(k)=\delta(k-k'). \tag{2.6}$$

Consider the motion equation

$$i\frac{\partial}{\partial t}b^-(t,k)=[H,b^-(t,k)].$$

Using (2.6) we can rewrite it as

$$(i\frac{\partial}{\partial t}b^-(t,k))_{t=0}=\omega(k)b^-(k)+\frac{\lambda(k)}{\sqrt{|\Lambda|}}J^-(k).$$

By similar arguments we find

$$(i\frac{\partial}{\partial t}b^+(t,k))_{t=0}=\omega(k)b^+(k)+\frac{\lambda(k)}{\sqrt{|\Lambda|}}J^+(k).$$

Denote

$$< b^-(t,k) >=\lim_{\Lambda\to\infty}\frac{Sp(b^-(t,k)e^{-\beta(H-\tilde{\mu}N)})}{Spe^{-\beta(H-\tilde{\mu}N)}},$$

where β is the reverse temperature. We also assume that the correlation function exists in an infinite volume.

Then the equation for $< b^+(k) >_{\mathcal{H}_\mathcal{B}}$ and $< b^-(k) >_{\mathcal{H}_\mathcal{B}}$ are of the form

$$< b^+(k) >_{\mathcal{H}_\mathcal{B}} = -\frac{\lambda(k) < J^+(k) >_{\mathcal{H}_\mathcal{B}}}{\omega(k)\sqrt{|\Lambda|}},$$
$$< b^-(k) >_{H_B} = -\frac{\lambda(k) < J^-(k) >_{\mathcal{H}_\mathcal{B}}}{\omega(k)\sqrt{|\Lambda|}}, \quad k = 1, 2, \ldots, M,$$

where $\lambda(k)/\omega(k) = \text{const}$, As the operators $J^+(k)$ and $J^-(k)$ are related to σ via

$$J^+(k) = \sum_{j=1}^{N}(\sigma_j^+ + \mu(k)\sigma_j^-) = \sum_{j=1}^{N}\left[\begin{array}{cc} 0 & 2 \\ 0 & 0 \end{array}\right]_j + \mu(k)\left[\begin{array}{cc} 0 & 0 \\ 2 & 0 \end{array}\right]_j = \sum_{j=1}^{N}\left[\begin{array}{cc} 0 & 2 \\ 2\mu(k) & 0 \end{array}\right]_j,$$
$$J^-(k) = \sum_{j=1}^{N}(\sigma_j^- + \mu(k)\sigma_j^+) = \sum_{j=1}^{N}\left[\begin{array}{cc} 0 & 0 \\ 2 & 0 \end{array}\right]_j + \mu(k)\left[\begin{array}{cc} 0 & 2 \\ 0 & 0 \end{array}\right]_j = \sum_{j=1}^{N}\left[\begin{array}{cc} 0 & 2\mu(k) \\ 2 & 0 \end{array}\right]_j,$$

where

$$\sigma_j^+ = \sigma_j^x + i\sigma_j^y = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]_j + i\left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right]_j = \left[\begin{array}{cc} 0 & 2 \\ 0 & 0 \end{array}\right]_j,$$
$$\sigma_j^- = \sigma_j^x - i\sigma_j^y = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]_j - i\left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right]_j = \left[\begin{array}{cc} 0 & 0 \\ 2 & 0 \end{array}\right]_j,$$

it follows that

$$J^-(k) = J^+(k)\det A = J^+(k)|A| = \sum_{j=1}^{N}\left[\begin{array}{cc} 0 & 2 \\ 2\mu(k) & 0 \end{array}\right]_j \left[\begin{array}{cc} 1/\mu(k) & 0\mu(k) \\ 0 & \mu(k) \end{array}\right]_j$$
$$= \sum_{j=1}^{N}\left[\begin{array}{cc} 0\cdot 1/\mu(k) + 2\cdot 0 & 0\cdot 0 + 2\mu(k) \\ 2\mu(k)\cdot 1/\mu(k) + 0 & 2\mu(k)\cdot 0 + 0\cdot\mu(k) \end{array}\right]_j$$
$$= \sum_{j=1}^{N}\left[\begin{array}{cc} 0 & 2\mu(k) \\ 2 & 0 \end{array}\right]_j \equiv J^-(k).$$

Here

$$|A| = \left[\begin{array}{cc} 1/\mu(k) & 0 \\ 0 & \mu(k) \end{array}\right]$$

and

$$\det A = 1.$$

From these conditions, we can find that

$$< J^+(k) >_{\mathcal{H}_\mathcal{L}} = < J^-(k) >_{\mathcal{H}_\mathcal{L}}$$

and

$$< b^+(k) >_{\mathcal{H}_\mathcal{B}} = < b^-(k) >_{\mathcal{H}_\mathcal{B}} = -\frac{\lambda(k)}{\omega(k)\sqrt{|\Lambda|}} = < J^-(k) >_{\mathcal{H}_\mathcal{L}}, \tag{2.7}$$

where

$$< J^-(k) >_{\mathcal{H}_\mathcal{L}} =< \sum_{j=1}^{N} \begin{bmatrix} 0 & 2\mu(k) \\ 2 & 0 \end{bmatrix}_j > .$$

Furthermore, by denoting

$$\eta^\#(k) = \frac{< J^\pm(k) >_{\mathcal{H}_\mathcal{L}}}{|\Lambda|},$$

it follows from (2.7)

$$\eta^\#(k) = -\frac{\omega(k)}{\lambda(k)\sqrt{|\Lambda|}} < b^\#(k) >_{\mathcal{H}_\mathcal{B}} .$$

For

$$\tilde{W}(k) = -\frac{\lambda(k)\sqrt{\Lambda}}{\omega(k)}\eta(k),$$

we can express Eq.(2.7) in the form

$$< b^+(k) >_{\mathcal{H}_\mathcal{B}} =< b^-(k) >_{\mathcal{H}_\mathcal{B}} = \tilde{W}(k), \qquad k = 1, 2, \ldots, M.$$

We introduce the approximative Hamiltonian by

$$H_{app}(\eta, \eta^*) = \int \omega(k) b^+(k) b^-(k) dk + \varepsilon \sum_{j=1}^{N} \sigma_j^z - \int \frac{\lambda^2(k)}{\omega(k)} (J^+(k)\eta(k) + J^-(k)\eta^*(k)) dk,$$

$$\eta(k) = \{\eta_k\}_{k=1}^M \in C^M.$$

It is not difficult to prove that the equation for correlation function on approximative Hamiltonian defined as

$$b^+(t,k)_{H_{app}} = \lim_{\Lambda\to\infty} \frac{Sp(b^-(t,k)e^{-\beta(H_{app}-\tilde{\mu}N)})}{Spe^{-\beta(H_{app}-\tilde{\mu}N)}},$$

where $< b^+(t,k) >_{H_{app}}$ is assumed to exist as $\Lambda \to \infty$, satisfies the same motion equation as that on model Hamiltonian.

It should be noted that the approximative Hamiltonian is also equivalent to the approximative one discussed in [5].

$$\begin{aligned} \tilde{H}_{app}(\eta, \eta^*) = \int & \omega(k)(b^+(k) + \frac{\lambda(k)\sqrt{|\Lambda|}}{\omega(k)}\eta^*(k))b^-(k) \\ & + \frac{\lambda(k)\sqrt{|\Lambda|}}{\omega(k)}\eta(k)) + \varepsilon \sum_{j=1}^{N} \sigma_j^z - \int \frac{\lambda^2(k)}{\omega(k)}(J^+(k)\eta(k) \\ & + J^-(k)\eta^*(k))dk - |\Lambda| \frac{\lambda^2(k)}{\omega(k)} |\eta(k)|^2 dk, \end{aligned} \tag{2.8}$$

where $\eta(k) = \{\eta_k\}_{k=1}^M \in C^M$. This fact can be directly proved by deriving the equation for correlation function defined on $H_{app}(\eta, \eta^*)$.

Theorem 2.1. *If the equation for the correlation function involving approximative Hamiltonian $H_{app}(\eta, \eta^*)$ has a solution, then the equation for the correlation function involving the model Hamiltonian H has the same solution.*

3 Proof of the Thermodynamics Equality

As a starting point for our proof we consider reduced free energies $f_\lambda(H)$, $f_\Lambda(H_{app}(\eta,\eta^{\#}))$ and show that the densities of thermodynamic potentials $f_\Lambda(H)$ and $f_\Lambda(H_{app}(\eta,\eta^*))$ are close. For this let us introduce terms with "source" $\nu = \{\nu_1,\nu_2,\dots,\nu_M\} \in C$ in Hamiltonian (2.1). In so doing, the self conjugation and simplification of the Hamiltonian (2.1) are not broken [7].

$$H(\nu) = \sum_{k=1}^{M}\omega_k b_k^+ b_k^- + \varepsilon\sum_{j=1}^{N}\sigma_j^z + \frac{1}{\sqrt{|\Lambda|}}\sum_{k=1}^{M}\lambda_k(b_k^- J_k^+ + b_k^+ J_k^-)$$
$$- \sqrt{|\Lambda|}\sum_{k=1}^{M}(\nu_k b_k^+ + \nu_k^* b_k^-). \tag{3.1}$$

Introducing ν_k and ν_k^* via

$$\nu_k = \frac{(2\pi)^{3/2}}{\sqrt{|\Lambda|}}\nu(k),\ \nu_k^* = \frac{(2\pi)^{3/2}}{\sqrt{|\Lambda|}}\nu^*(k),$$

and using (2.5), the Hamiltonian (3.1) takes the form

$$H(\nu) = \int\omega(k)b^+(k)b^-(k)dk + \varepsilon\sum_{j=1}^{N}\sigma_j^z + \frac{1}{\sqrt{|\Lambda|}}\int\lambda(k)(b^-(k)J^+(k) + b^+(k)J^-(k))dk$$
$$- \sqrt{|\Lambda|}\int(\nu(k)b^+(k) + \nu^*(k)b^-(k))dk.$$

The above considered procedure is useful to obtain the relation between $< b^{\#}(k) >_{\mathcal{H}(\nu)_{\mathcal{L}}}/\sqrt{|\Lambda|}$ and $< J^{\#}(k) >_{\mathcal{H}(\nu)_{\mathcal{L}}}/\sqrt{|\Lambda|}$. So, consider the motion equation for

$$b^-(t,k) = e^{i(H(\nu)-\tilde{\mu}N)t}b^-(0,k)e^{-i(H(\nu)-\tilde{\mu}N)t},$$
$$(i\frac{\partial}{\partial t}b^-(t,k))|_{t=0} = [H(\nu), b^-(k)].$$

Using the commutation relation (2.6) we get

$$(i\frac{\partial}{\partial t}b^-(t,k))|_{t=0} = \omega(k)b^-(k) + \frac{\lambda(k)}{\sqrt{|\Lambda|}}J^-(k) + \sqrt{|\Lambda|}\nu(k)$$

or

$$(i\frac{\partial}{\partial t} < b^-(t,k) >_{\mathcal{H}(\nu)_{\mathcal{L}}})|_{t=0} = \omega(k) < b^-(k) >_{\mathcal{H}(\nu)_{\mathcal{L}}}$$
$$+ \frac{\lambda(k)}{\sqrt{|\Lambda|}} < J^-(k) >_{\mathcal{H}(\nu)_{\mathcal{L}}} + \sqrt{|\Lambda|}\nu(k).$$

For $b^-(k)$ and $< b^-(k) >$ we find that

$$-\frac{b^-(k)}{\sqrt{|\Lambda|}} = \frac{\lambda(k)J^-(k)}{\omega(k)|\Lambda|} - \frac{\nu(k)}{\omega(k)},$$

$$-\frac{< b^-(k) >_{\mathcal{H}(\nu)\mathcal{L}}}{\sqrt{|\Lambda|}} = \frac{\lambda(k) < J^-(k) >_{\mathcal{H}(\nu)\mathcal{L}}}{\omega(k)|\Lambda|} - \frac{\nu(k)}{\omega(k)}.$$

Following the Hamiltonian (2.8) we construct the approximative Hamiltonian

$$\begin{aligned} H_{app}(\nu,\eta) = & \sum_{k=1}^{M} \omega_k (b_k^+ - \frac{v_k^*}{\omega_k}\sqrt{|\Lambda|})(b_k^- - \frac{v_k}{\omega_k}\sqrt{|\Lambda|}) + H_L - |\Lambda| \sum_{k=1}^{M} \frac{\lambda_k^2}{\omega_k} \\ & \times (\frac{J_k^+}{|\Lambda|}\eta_k + \frac{J_k^-}{|\Lambda|}\eta_k^*) + |\Lambda| \sum_{k=1}^{M} \frac{\lambda_k}{\omega_k}(\nu_k^* \frac{J_k^-}{|\Lambda|} + \nu_k \frac{J_k^+}{|\Lambda|}) - |\Lambda| \sum_{k=1}^{M} \frac{|v_k|^2}{\omega_k}. \end{aligned}$$

Next we estimate the difference $H(\nu) - H_{app}(\nu, \eta)$.

$$\begin{aligned} H_I(\nu,\eta) &= H(\nu) - H_{app}(\nu,\eta) \\ &= \sum_{k=1}^{M} \omega_k b_k^+ b_k^- + H_L + \frac{1}{\sqrt{|\Lambda|}} \sum_{k=1}^{M} \lambda_k (b_k^- J_k^+ + b_k^+ J_k^-) \\ &\quad - \sqrt{|\Lambda|} \sum_{k=1}^{M} (\nu_k b_k^+ - \nu_k^* b_k^-) - \sum_{k=1}^{M} \omega_k (b_k^+ - \frac{\nu_k^*}{\omega_k}\sqrt{|\Lambda|})(b_k^- - \frac{\nu_k}{\omega_k}\sqrt{|\Lambda|}) \\ &\quad - H_L - |\Lambda| \sum_{k=1}^{M} \frac{\lambda_k^2}{\omega_k} (\frac{J_k^+}{|\Lambda|}\eta_k + \frac{J_k^-}{|\Lambda|}\eta_k^*) - |\Lambda| \sum_{k=1}^{M} \frac{\lambda_k}{\omega_k}(\nu_k^* \frac{J_k^-}{|\Lambda|} - \nu_k \frac{J_k^+}{|\Lambda|}) \\ &\quad + |\Lambda| \sum_{k=1}^{M} \frac{|\nu_k|^2}{\omega_k} \\ &= |\Lambda| \sum_{k=1}^{M} \lambda_k \Big\{ \Big(\frac{b_k^+}{\sqrt{|\Lambda|}} + \frac{\lambda_k}{\omega_k}\eta_k^* - \frac{\nu_k^*}{\omega_k} \Big) \Big(\frac{J_k^-}{|\Lambda|} - \eta_k \Big) \\ &\quad + \Big(\frac{b_k^-}{\sqrt{|\Lambda|}} + \frac{\lambda_k}{\omega_k}\eta_k - \frac{\nu_k}{\omega_k} \Big) \Big(\frac{J_k^+}{|\Lambda|} - \eta_k^* \Big) \Big\}. \end{aligned}$$

As known [7], for the difference operator $H_I(v, \eta)$, near-asymptotics between $f_\Lambda[H(\nu)]$ and $f_\Lambda[H_{app}(\nu, \eta)]$ has been proved by Bogolyubov's theorem. For all $\nu \in K$, where $K = \{(\nu_1, \ldots, \nu_M) : V_k \in C\},\ |\nu_k| \le K,\ k = 1, \ldots, M,\ k \ge 0,$

$$0 \le f_\Lambda[H_{app}(\nu,\eta)] - f_\Lambda[H(\nu)] \le \varepsilon(|\Lambda|^{-1/3}).$$

Here

$$\begin{aligned} \varepsilon(|\Lambda|^{-1/3}) = & \frac{1}{|\Lambda|^{-1/3}} \left\{ \sum_{k=1}^{M} C^2 \frac{\lambda_k^2}{\omega_k} + \frac{4}{\beta_0} \sum_{k=1}^{M} \left(\frac{\lambda_k}{\sqrt{\omega_k}} C + \frac{K}{\sqrt{\omega_k}} \right) \right\} \\ & + \frac{1}{|\Lambda|^{1/2}} \sum_{k=1}^{M} \frac{\omega_k^{3/4}}{\sqrt{\theta_0}} \sqrt{\lambda_k C + K} \beta_0^{1/4}, \end{aligned}$$

where θ_0 is a fixed temperature, β_0 is a fixed $\beta = 1/\theta_0$. On the basis of the results obtained we can formulate the next theorem.

Theorem 3.1. *Let operators $H_L, b_k^{\#}, J_k^{\#}$ in (2.1) satisfy the following conditions:*

$$H_L = \sum_{j=1}^{N} \sigma_j^z, \qquad H_L = H_L^+, \qquad \frac{1}{|\Lambda|}||J_k^{\#}|| \le C,$$

$$b_k^- b_{k'}^+ - b_{k'}^+ b_k^- = \delta_{kk'},\ b_k^{\#} b_{k'}^{\#} - b_{k'}^{\#} b_k^{\#} = 0,$$

$$\omega_k \ge \omega_0 > 0, \qquad \lambda_k \ge 0, \qquad k = 1, \ldots, M$$

and let the reduced free energy for H_L be bounded by constant

$$|f_\Lambda(H_L)| \le C_L = \text{const}.$$

The approximative Hamiltonian in terms of the operators can be constructed as

$$H_{app}(\eta) = \sum_{k=1}^{M} \omega_k b_k^+ b_k^- - \varepsilon \sum_{j=1}^{N} \sigma_j^z - |\Lambda| \sum_{k=1}^{M} \frac{\lambda_k^2}{\omega_k} (\frac{J_k^+}{|\Lambda|}\eta_k + \frac{J_k^-}{|\Lambda|}\eta_k^*).$$

That is, in terms of (2.5) and $\eta_k = (2\pi)^{3/2}\sqrt{|\Lambda|}\eta(k)$, $H_{app}(\eta)$ takes the form

$$H_{app}(\eta) = \int \omega(k) b^+(k) b^-(k) dk - \varepsilon \sum_{j=1}^{N} \sigma_j^z - \int \frac{\lambda^2(k)}{\omega(k)} (J^+(k)\eta(k) + J^-(k)\eta^*(k)) dk,$$

where $\eta(k) = \{\eta_k\}_{k=1}^{M} \in C^M$). Then the following inequality

$$0 \le \min_{\eta \in C^M} f_\Lambda[H_{app}] - f_\Lambda[H] \le 0(|\Lambda|^{-1/3})$$

is valid if and only if $0(|\Lambda|^{-1/3}) \to 0$ as $|\Lambda| \to \infty$ is uniform on temperature θ in the range of $0 \le \theta \le \theta_0$, where θ_0 is a fixed temperature.

4 Approximative Hamiltonian in Terms of One-Zone Potential

The expression for approximative Hamiltonian is defined in terms of finite-zone potential, which is supposed to stand for strong boson interactions. It should be mentioned that the interaction is considered as being strong only under condition that particles have spins of certain positioned momenta $k = nk'$ (for this reason all different spin boson interactions are ignored).

The Hamiltonian

$$H = \sum_{k=1}^{M} \omega_k b_k^+ b_k^- + \varepsilon \sum_{j=1}^{N} \sigma_j^z + \frac{1}{|\Lambda|} \sum_{k=1}^{M} \sum_{n} \delta_{kk'_n} \lambda_k (b_k^- J_k^+ + b_k^+ J_k^-),$$

corresponds to the Dicke model Hamiltonian [7] at M=1, $k = k'$ is

$$H_D = \omega b^+ b^- + \varepsilon \sum_{j=1}^{N} \sigma_j^z + \frac{\lambda}{\sqrt{|\Lambda|}} (b^- J^+ + b^+ J^-)) \tag{4.1}$$

within the framework of our method and we can define the following transformations

$$\eta(x) = \frac{1}{(2\pi)^{1/3}} \sum_k \eta(k) e^{ikx}, \quad \eta^*(x) = \frac{1}{(2\pi)^{1/3}} \sum_k \eta^*(k) e^{-ikx},$$

$$< b^+(x) b^-(x) >= \frac{1}{(2\pi)^{2/3}} \iint < b^+(k + k'^-(k') > dk'^{ikx} dk.$$

Here, $< b^+(x)b^-(x) >$ is the boson density in suitably chosen configuration space. The above values are determined from the selfconsistency condition by

$$< b^*(k) >= -\frac{\lambda(k)\sqrt{|\Lambda|}}{\omega(k)} \eta^*(k) = i\sqrt{æ'}\eta^*(k) \tag{4.2}$$

and from the independency of subsystems $æ_B, æ_L$ by

$$< b^+(x) b^-(x) >= -æ'^2 = æ|\eta(x)|^2. \tag{4.3}$$

By using the formula (4.2), we can obtain

$$\sqrt{æ'} = -i\sqrt{æ} = i\frac{\lambda(k)}{\omega(k)}\sqrt{|\Lambda|},$$

as according to the Dicke model (4.1)

$$\sqrt{æ} = -\frac{\lambda(k)}{\omega(k)}\sqrt{|\Lambda|} = \text{const}, \qquad k = 1, 2, \ldots, M.$$

Now, by using the formula (4.3), we can show that the boson density $< b^+(x)b^-(x) >$ is an elliptic function, and a periodical potential formed by bosons $U(x) = |\eta(x)|^2$ is one-zone potential (see, for example [3]):

$$U(x) = C + 2\gamma(x + \omega/\omega, \omega_x'^2 \ln \theta_3(T^{-1}x + l, q),$$

where

$$C = (1/3)(E_1 + E_2 + E_3); \qquad U = E_2 + (E_3 - E_1)[1 - 2E(k')|K(k')|].$$

Here $E_1 \le E_2 \le E_3$ are the spectrum boundaries, $T = -2i\omega'$ is a real period of the potential, $T = 2i\omega$ is an imaginary period of potential. For all the values, except for the boundaries of spectrum, we will use the standard notations of elliptic function theory [18].

To prove the ellipticity and one-zone of the boson density, we write the operator $b^{\#}(x)$ as

$$b^{\#}(x) = \sum_E b_E^{\#} \varphi(x, E), \tag{4.4}$$

where the functions $\varphi(x, E)$

$$\varphi(x, E) = \text{const}\ \chi^{-1/2}(x, E) e^{i\int dx \chi(x,E)}$$

are the solutions of the Schrodinger equation.

$$\partial_x^2\varphi(x,E)+(E-U(x))\varphi(x,E)=0.$$

The function $\chi(x,E)$ is real on the spectrum and satisfies the following differential equation

$$\frac{1}{2}\frac{\chi_{xx}}{\chi}-\frac{3}{4}\left(\frac{\chi_x}{\chi}\right)^2+\chi^2+U-E=0.$$

Hence, $|\varphi(x,E)|^2$ takes the form

$$|\varphi(x,E)|^2=\overline{[\chi^{-1}(x,E)]^{-1}}\chi^{-1}(x,E), \tag{4.5}$$

where the bar ($^-$) denotes the mean value on coordinate x.

By using the representation (4.4) and the condition that $< b_E^+b_{E'}^- >= \delta_{E,E'}$ where $i(E)$ is the boson distribution function, and by the aid of formula (4.2), we can obtain the following condition

$$\sum_E f(E)|\varphi(x,E)|^2=-æU(x). \tag{4.6}$$

Combining the relations from [8] and [12]

$$N(U(.),E)=\int_{x_0}^{x_0+T}\chi(x,E)dx;\quad \frac{dN}{dE}=\overline{\frac{1}{\chi(x,E)}};\quad \frac{\delta N}{\delta E}=-\frac{1}{\chi(x,E)}$$

and using the formula (4.5), we can define $|\varphi(x,E)|^2$ through dN/dE and $\delta N/\delta E$

$$|\varphi(x,E)|^2=-\left(\frac{dN}{dE}\right)^{-1}\left[\left(\frac{\delta N}{\delta E}\right)(x,E)\right].$$

Substituting the latter formula into (4.6) and representing the sum by the integral, we get equation (4.4) as the variation of equation of the Peierls-Fröhlich problem

$$\int dEf(E)\left(\frac{\delta N}{\delta E}\right)(x,E)=æU(x).$$

Thus, the variational derivative of the member of states is a linear function of the potential, which is equivalent to one-zone of the potential $U(x)$. Following the description from [4] we can show the ellipticity of one-zone potential.

Theorem 4.1. *Consider, in the model Hamiltonian Dicke, bosons interact strongly only with spins at exact momenta $k=nk'=k_n'\in Z$, and suppose that the bosons interaction with the rest momenta can be neglected, that is*

$$H=\sum_{k=1}^{M}\omega_k b_k^+b_k^-+\varepsilon S^z+\frac{1}{|\Lambda|}\sum_{k=1}^{M}\sum_n\delta_{kk_n'}\lambda_k(b_k^-J_k^++b_k^+J_k^-), \tag{4.7}$$

which is represented in the configuration space as

$$H = \int \omega(k) b^+(k) b^-(k) dk + \varepsilon S^z + \frac{1}{|\Lambda|} \iint \delta(k - k_n'^-(k) J^+(k) + b^+(k) J^-(k)) dk dk_n'$$

then the form of Dicke approximative Hamiltonian is defined in terms of one-zone potential by

$$\begin{aligned} H_{app}(\eta, \eta^*) &= \int \omega(k) b^+(k) b^-(k) dk + \varepsilon \sum_{j=1}^{N} \sigma_j^z \\ &- \iint \delta(k - k_n') \frac{\lambda^2(k)}{\omega(k)} (J^+(k)\eta(k) + J^-(k)\eta^*(k)) dk dk_n'), \\ H_{app}(\eta, \eta^*) &= \int \omega(k) (b^+(k) + \frac{\lambda(k)\sqrt{|\Lambda|}}{\omega(k)} \eta^*(k))(b^-(k) + \frac{\lambda(k)\sqrt{|\Lambda|}}{\omega(k)} \eta(k)) \\ &+ \varepsilon \sum_{j=1}^{N} \sigma_j^z - \iint \delta(k - k_n') \frac{\lambda^2(k)}{\omega(k)} (J^+(k)\eta(k) + J^-(k)\eta^*(k)) dk dk_n' \\ &- |\Lambda| \iint \delta(k - k_n') \frac{\lambda^2(k)}{\omega(k)} |\eta(k)|^2 dk dk_n', \end{aligned}$$

where

$$\delta_k \to \frac{(2\pi)^3}{|\Lambda|} \delta(k),$$

and

$$|\eta(k)|^2 = U(k) = \frac{1}{(2\pi)^{1/3}} \int e^{ikx} U(x) dx$$

is one-zone potential.

Putting in the Hamiltonian (4.5) that $M = 1$, $k = k'$, we can obtain the Dicke model Hamiltonian discussed in [7] as

$$H_D = \omega b^+ b^- + \varepsilon \sum_{j=1}^{N} \sigma_j^z + \frac{\lambda}{\sqrt{|\Lambda|}} (b^- J^+ + b^+ J^-).$$

Thus, in this paper, for the model Hamiltonian H approximative Hamiltonians $H_{app}(\eta, \eta^*)$ and $H_{app}\tilde{(}\eta, \eta^*)$ in terms of one-zone potential on the space $\mathcal{H}$ have been constracted and discussed by using the method developed by D. Ya. Petrina and E. D. Belokolos.

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