Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/080307

Testing Exponentiality Against $RNBU_{mgf}$ based on Laplace Transform Technique

*S. M. El-Arishy, L. S. Diab and E. S. El-Atfy**

College of Science for (girls), Dept. of Mathematics, Al-Azhar University, Nasr City, 11884, Egypt.

Received: 20 Jan. 2019, Revised: 9 Apr. 2019, Accepted: 19 Apr. 2019
Published online: 1 Nov. 2019

Abstract: In this paper, a new hypothesis test is constructed to test exponentiality against Renewal New Better than Used in moment-generating function $(RNBU_{mgf})$ based on Laplace transform order and another test based on goodness of fit approach follows as a special case. Pitman Asymptotic Efficiency (PAE) are studied, the critical values of the tests are tabulated for sample sizes $n = 5(5)50$, and the power estimates are calculated to assess the performance of the tests. Also a test of exponentiality versus $(RNBU_{mgf})$ for right censored data is considered. The power estimates of the tests are simulated for some commonly used distributions in reliability. Finally, sets of real data are used as examples to elucidate the use of the proposed test statistic for practical problems in case of complete and uncomplete data in the reliability analysis.

Keywords: Moment generating function, $RNBU$, $RNBU_{mgf}$, laplace transform, goodness of fit, efficiency, monte Carlo method, power and censored data

1 Introduction

The concept of ageing is very important in reliability analysis, it describes how a component or system improves or deteriorates with age. Many classes of life distributions are categorized or defined in literature according to their ageing properties. An important aspect of such classifications is that the exponential distribution is nearly always a member of each class. The notion of stochastic ageing plays an important role in any reliability analysis and many test statistics have been developed in the literature for testing exponentiality against different ageing alternatives. Consider a device (system or component) with life time T and a continuous life distribution $F(t)$, is put on operation. When the failure occurs the device will be replaced by a sequence of mutually independent devices. The spare devices are independent of the first device and are identically distributed with the same life distribution $F(t)$. In the long run, the remaining life distribution of the system under operation at time t is given by stationary renewal distribution as follows:

$$W_F(x) = \frac{1}{\mu}\int_0^x \overline{F}(t)\,dt, \qquad 0 \le t < \infty.$$

with renewal survival function,

$$\overline{W}_F(x) = \frac{1}{\mu}\int_x^\infty \overline{F}(t)\,dt, \qquad 0 \le t < \infty.$$

Where $\mu = \int_0^\infty \overline{F}(u)\,du$.

For extra details, see (Abouammoh, and Ahmed [1,2]), (Barlow and Proschan [10]).
A non-negative random variable X is said to be renewal new better than used (denoted by $X \in RNBU$) if, and only, if

$$\overline{W}_F(x+t) \le \overline{W}_F(x)\,\overline{W}_F(t), \qquad \forall\ x,t \ge 0$$

Statisticians and reliability analysts studied renewal new better than used classes of life distributions from various points of view. Related papers dealing with $RNBU$ see Abu-Youssef [4], Diab et al. [11], EL-Arishy et al. [15] and Elbatal [18].

* Corresponding author e-mail: d_ emy83@yahoo.com

Definition 1.1: [Klar and Muller [22]] Given two non-negative random variables X and Y, with survival functions $\overline{F}$ and $\overline{G}$, respectively, X is said to be smaller than Y in the moment-generating function ordering $\left(\text{denoted by} X <_{mgf} Y\right)$ if and only if,

$$\int_0^\infty e^{\lambda x}\overline{F}(x)\,dx \le \int_0^\infty e^{\lambda y}\overline{G}(y)\,dy, \qquad \forall\ \lambda > 0.$$

Definition 1.2: We say that X is renewal new better than used in the moment-generating function order $\left(\text{denoted by } X \in RNBU_{mgf}\right)$ if $X_t \le_{mgf} Y$ for all $t > 0$, where Y is an exponential random variable with the same mean as X. Equivalently, $X \in RNBU_{mgf}$ if, and only if,

$$\int_0^\infty e^{\lambda x}\overline{W}_F(x+t)\,dx \le \overline{W}_F(t)\int_0^\infty e^{\lambda x}\overline{W}_F(x)\,dx, \qquad \forall\ \lambda > 0 \tag{1}$$

In literature, many papers proposed tests for testing exponentiality against some classes of life distributions based on Laplace transform see Abu youssef et al. [6], Al-Gashgari et al. [8], Atallah et al. [9] and Mahmoud et al. [26]. Also, testing exponentiality against some classes of life distributions based on goodness of fit approach is studied by Abu youssef and Bakr [5], Ahmad [7], Diab and Mahmoud [12], Diab [13,14], El-Arishy et al. [16], El-Atfy et al. [17], Kayid et al. [21], Mahmoud and Abdel-Alim [24] and Mahmoud and Diab [25].

The rest of this paper is organized as follows, In Section 2, testing exponentiality against $RNBU_{mgf}$ is proposed based on laplace transform order and another test based on goodness of fit approach follows as a special case. In Section 3, PAE is studied. Monte Carlo null distribution critical points are simulated for sample sizes $n = 5(5)50$ and the power estimates of the tests are also calculated for some common alternative distributions in Section 4. The case of right-censored data is considered, the critical values and the power estimates of the tests are tabulated in Section 5. Finally, in Section 6, we discuss some applications to elucidate the usefulness of the proposed tests in reliability analysis for complete and uncomplete data.

2 Testing Against $RNBU_{mgf}$ in laplace transform order

In this section, a test statistic based on laplace transform order is presented for testing $H_0 : F$ is exponential against the alternative $H_1 : F$ is belongs to $RNBU_{mgf}$ class but not exponential. Let $X_1, X_2, ..., X_n$ be a random sample from a population with distribution F, we use $\xi(\lambda, s)$ as a measure of departure from exponentiality. The following lemma is needed.

Lemma 2.1 If X is a random variable with distribution function F belonging to $RNBU_{mgf}$ class then,

$$\xi(\lambda,s) = \frac{1}{s^2\lambda^2}\phi(s)\phi(\lambda) + \left[\frac{1}{s\lambda(\lambda+s)}\mu - \frac{1}{s^2\lambda^2}\right]\phi(\lambda) - \left[\frac{1}{s\lambda(\lambda+s)}\mu + \frac{1}{s^2\lambda^2}\right]\phi(s) + \frac{1}{s^2\lambda^2}, \tag{2}$$

where,

$$\phi(s) = \int_0^\infty e^{-sx}dF(x), \quad \text{and, } \phi(\lambda) = \int_0^\infty e^{\lambda x}dF(x)$$

Proof: Since F is $RNBU_{mgf}$, multiplying Eq.(1) by e^{-st} and integrating both sides from 0 to ∞, then we have,

$$\int_0^\infty e^{-st}\int_0^\infty e^{\lambda x}\,\overline{W}(x+t)\,dxdt \le \int_0^\infty e^{-st}\,\overline{W}(t)\,dt\int_0^\infty e^{\lambda x}\,\overline{W}(x)dx$$

setting,

$$\begin{aligned}
L.H.S &= \int_0^\infty e^{-st}\int_0^\infty e^{\lambda x}\,\overline{W}(x+t)\,dxdt, \\
&= \int_0^\infty e^{-t(\lambda+s)}\left[\int_t^\infty e^{\lambda v}\,\overline{W}(v)\,dv\right]dt, \\
&= \frac{1}{(\lambda+s)}\left[\int_0^\infty \left(e^{\lambda v} - e^{-sv}\right)\overline{W}(v)\,dv\right], \\
&= \frac{1}{\mu}\left[\frac{1}{\lambda^2(\lambda+s)}\phi(\lambda) - \frac{1}{s^2(\lambda+s)}\phi(s) - \frac{1}{s\lambda}\mu + \frac{\lambda-s}{s^2\lambda^2}\right].
\end{aligned} \tag{3}$$

And,

$$R.H.S = \int_0^\infty e^{-st}\,\overline{W}(t)\,dt \int_0^\infty e^{\lambda x}\,\overline{W}(x)dx,$$
$$= \frac{1}{s\lambda\mu^2}\left(\frac{1}{s}\phi(s)+\mu-\frac{1}{s}\right)\left(\frac{1}{\lambda}\phi(\lambda)-\mu-\frac{1}{\lambda}\right),$$
$$= \frac{1}{\mu^2}\left[\frac{1}{s^2\lambda^2}\phi(s)\phi(\lambda)-\frac{1}{s^2\lambda^2}\phi(s)-\frac{1}{s^2\lambda^2}\phi(\lambda)-\frac{1}{s^2\lambda}\mu\phi(s)+\frac{1}{\lambda^2 s}\mu\phi(\lambda)-\frac{1}{s\lambda}\mu^2+\frac{\lambda-s}{s^2\lambda^2}\mu+\frac{1}{s^2\lambda^2}\right]. \tag{4}$$

Hence, from (3) and (4) the result follows.
Note that under $H_0 : \xi(\lambda,s)=0$, while under $H_1 : \xi(\lambda,s)>0$

Corollary 2.1 If we set $s=1$ in Eq. (2) then $\xi(\lambda,1)$ becomes a measure of departure from exponentiality based on goodness of fit approach

$$i.e \quad \xi(\lambda,1)=\frac{1}{\lambda^2}E\left(e^{-x}\right)\phi(\lambda)+\left[\frac{1}{\lambda(\lambda+1)}\mu-\frac{1}{\lambda^2}\right]\phi(\lambda)-\left[\frac{1}{\lambda(\lambda+1)}\mu+\frac{1}{\lambda^2}\right]E\left(e^{-x}\right)+\frac{1}{\lambda^2}. \tag{5}$$

and hence $H_0 : \xi(\lambda,1)=0$, while under $H_1 : \xi(\lambda,1)>0$.

2.1 Empirical test statistic for $RNBU_{mgf}$

Let $X_1, X_2, ..., X_n$ be a random sample from F. Let $\overline{F}_n(x)$ denote the empirical distribution of the survival function $\overline{F}(x)$ where,

$$\overline{F}_n(x)=\frac{1}{n}\sum_{i=1}^{n} I(X_i>x), \qquad dF_n(x)=\frac{1}{n}.$$

And let $\hat{\xi}(\lambda,s)$ be the empirical estimate of $\xi(\lambda,s)$ where,

$$\hat{\xi}(\lambda,s)=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{1}{s^2\lambda^2}e^{-sX_i}e^{\lambda X_j}+\left[\frac{1}{s\lambda(\lambda+s)}X_i-\frac{1}{s^2\lambda^2}\right]e^{\lambda X_j}-\left[\frac{1}{s\lambda(\lambda+s)}X_i+\frac{1}{s^2\lambda^2}\right]e^{-sX_j}+\frac{1}{s^2\lambda^2}\right\}. \tag{6}$$

To make the test statistic $\hat{\xi}(\lambda,s)$ scale invariant, we set $\widehat{\beta}(\lambda,s)=\frac{\hat{\xi}(\lambda,s)}{\overline{X}}$, then

$$\widehat{\beta}(\lambda,s)=\frac{1}{n^2\overline{X}}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(X_i,X_j). \tag{7}$$

where,

$$\phi(X_i,X_j)=\frac{1}{s^2\lambda^2}e^{-sX_i}e^{\lambda X_j}+\left[\frac{1}{s\lambda(\lambda+s)}X_i-\frac{1}{s^2\lambda^2}\right]e^{\lambda X_j}-\left[\frac{1}{s\lambda(\lambda+s)}X_i+\frac{1}{s^2\lambda^2}\right]e^{-sX_j}+\frac{1}{s^2\lambda^2}. \tag{8}$$

We define the symmetric kernel as,

$$\Psi(X_1,X_2)=\frac{1}{2!}\sum_R \phi(X_i,X_j).$$

Where the sum is over all arrangements of X_i and X_j, this leads to $\widehat{\beta}(\lambda,s)$ which is equivalent to U_n statistic given by,

$$U_n=\frac{1}{\binom{n}{2}}\sum_{i<j}\phi(X_i,X_j)$$

The following theorem summarizes the asymptotic normality of $\widehat{\beta}(\lambda,s)$.

Theorem 2.1 As $n\to\infty$, $\sqrt{n}\left(\widehat{\beta}(\lambda,s)-\xi(\lambda,s)\right)$ is asymptotically normal with mean 0 and variance

$$\sigma^2=Var\left[\frac{1}{s\lambda(1-\lambda)(1+s)}X-\frac{1-\lambda}{\lambda^2(\lambda+s)(1+s)}e^{\lambda X}+\frac{1+s}{s^2(\lambda+s)(1-\lambda)}e^{-sX}+\frac{s-\lambda-2s\lambda}{s^2\lambda^2(1+s)(1-\lambda)}\right]$$

Under H_0, the variance σ^2 reduces to

$$\sigma_0^2 = \frac{2}{(1-\lambda)^2(1+s)^2(1-2\lambda)(1+2s)(1+s-\lambda)}. \tag{9}$$

Proof : Let

$$\eta_1(X_1) = E[\phi(X_1,X_2)|X_1] = \frac{1}{s^2\lambda^2(1-\lambda)}e^{-sX} + \frac{1}{s\lambda(1-\lambda)(1+s)}X - \frac{1+s\lambda}{s^2\lambda^2(1+s)(1-\lambda)},$$

and,

$$\eta_2(X_2) = E[\phi(X_1,X_2)|X_2] = -\frac{(1-\lambda)}{\lambda^2(1+s)(\lambda+s)}e^{\lambda X} - \frac{s\lambda+s+\lambda}{s^2\lambda^2(1-\lambda)}e^{-sX} + \frac{1}{s^2\lambda^2}.$$

Consider,

$$\eta(X) = \frac{1}{s\lambda(1-\lambda)(1+s)}X - \frac{1-\lambda}{\lambda^2(\lambda+s)(1+s)}e^{\lambda X} + \frac{1+s}{s^2(\lambda+s)(1-\lambda)}e^{-sX} + \frac{s-\lambda-2s\lambda}{s^2\lambda^2(1+s)(1-\lambda)}.$$

Since,

$$E(\eta(X)) = 0, \qquad \sigma^2 = Var(\eta(X)).$$

Under H_0 the variance reduces to Eq. (9).

Remark: Notice that at $s=1$, we obtain the variance

$$\sigma^2 = Var\left[\frac{1}{2\lambda(1-\lambda)}X - \frac{1-\lambda}{2\lambda^2(1+\lambda)}e^{\lambda X} + \frac{2}{1-\lambda^2}e^{-X} + \frac{1-3\lambda}{2\lambda^2(1-\lambda)}\right].$$

Under H_0 we get the variance in the case of goodness of fit approach as,

$$\sigma_0^2(1) = \frac{1}{6(1-\lambda)^2(1-2\lambda)(2-\lambda)}.$$

3 The Pitman Asymptotic Efficiency

To judge on the quality of this procedure,we evaluate the Pitman's Asymptotic (PAE) for some commonly used distributions in reliability, which is defined as

$$PAE(\Delta(\theta)) = \frac{1}{\sigma_0}|\frac{d}{d\,\theta}\Delta(\theta)|_{\theta\to\theta_0}.$$

(i) Linear failure rate family, $\overline{F}_1(x) = \exp(-x-\theta x^2/2)$, $x \geq 0, \theta \geq 0$,
(ii) Makeham family, $\overline{F}_2(x) = \exp(-x-\theta(x+e^{-x}-1))$, $x \geq 0, \theta \geq 0$,
(iii) Weibull family, $\overline{F}_3(x) = \exp(-x^\theta)$, $x \geq 0, \theta \geq 1$,
(iv) Gamma family, $\overline{F}_4(x) = \int_x^\infty e^{-u}u^{\theta-1}du/\Gamma(\theta)$, $x > 0, \theta \geq 0$.
Note that the exponential distribution is attained at $\theta=\theta_0=0$ in (i), (ii) , and at $\theta=\theta_0=1$ in $(iii),(iv)$. Since

$$\xi_\theta(\lambda,s) = \frac{1}{s^2\lambda^2}\phi_\theta(s)\phi_\theta(\lambda) + \left[\frac{1}{s\lambda(\lambda+s)}\mu_\theta - \frac{1}{s^2\lambda^2}\right]\phi_\theta(\lambda) - \left[\frac{1}{s\lambda(\lambda+s)}\mu_\theta + \frac{1}{s^2\lambda^2}\right]\phi_\theta(s) + \frac{1}{s^2\lambda^2}.$$

The $PAE(\xi_\theta(\lambda,s))$ can be written as,

$$\begin{aligned} PAE(\xi_\theta(\lambda,s)) = \frac{1}{\sigma_0}\Bigg| & \frac{1}{s^2\lambda^2}\left[\phi'_\theta(\lambda)\left(\int_0^\infty e^{-sx}dF_\theta(x)\right) + \left(\int_0^\infty e^{\lambda x}dF_\theta(x)\right)\phi'_\theta(s)\right] \\ & + \left[\frac{1}{s\lambda(\lambda+s)}\mu_\theta - \frac{1}{s^2\lambda^2}\right]\left(\int_0^\infty e^{\lambda x}dF'_\theta(x)\right) + \left[\frac{1}{s\lambda(\lambda+s)}\mu'_\theta\right]\phi_\theta(\lambda) \\ & - \left[\frac{1}{s\lambda(\lambda+s)}\left(\int_0^\infty x dF_\theta(x)\right) + \frac{1}{s^2\lambda^2}\right]\phi'_\theta(s) - \left[\frac{1}{s\lambda(\lambda+s)}\mu'_\theta\right]\phi_\theta(s)\Bigg|_{\theta\to\theta_0}. \end{aligned}$$

In this case, we obtain,

$$PAE\left(\xi_\theta(\lambda,s),\overline{F}_1(x)\right) = \frac{1}{\sigma_0}\left|\frac{1}{(1+s)^2(1-\lambda)^2}\right|,$$

$$PAE\left(\xi_\theta(\lambda,s),\overline{F}_2(x)\right) = \frac{1}{\sigma_0}\left|\frac{1}{2(1+s)(2+s)(1-\lambda)(2-\lambda)}\right|,$$

$$PAE\left(\xi_\theta(\lambda,s),\overline{F}_3(x)\right) = \frac{1}{\sigma_0}\left|-\frac{\lambda\log[1+s]+s\log[1-\lambda]}{s\lambda(1+s)(1-\lambda)(s+\lambda)}\right|,$$

$$PAE\left(\xi_\theta(\lambda,s),\overline{F}_4(x)\right) = \frac{1}{\sigma_0}\left|\frac{\lambda^2(1+s)\log[1+s]+s(-\lambda(s+\lambda)+s(\lambda-1)\log[1-\lambda])}{s^2\lambda^2(1+s)(\lambda-1)(s+\lambda)}\right|.$$

The following table includes the asymptotic efficiencies of our proposed test $\xi_\theta(\lambda,s)$ at various values of s,λ.

Table 1. The asymptotic efficiencies of $\xi_\theta(\lambda,s)$.

Distribution	s	$\lambda=0.01$	$\lambda=0.02$	$\lambda=0.03$	$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$
LFR	1	0.86381	0.86151	0.85909	0.83887	0.79549	0.72139
	2	0.91130	0.90963	0.90786	0.89197	0.85391	0.78246
	3	0.93419	0.93288	0.93145	0.91793	0.88278	0.81284
	4	0.94769	0.94659	0.94538	0.93333	0.90000	0.83103
	5	0.95658	0.95563	0.95457	0.94354	0.91144	0.84314
Makeham	1	0.14325	0.14213	0.14100	0.13245	0.11785	0.09901
	2	0.17001	0.16883	0.16763	0.15844	0.14232	0.12082
	3	0.18590	0.18469	0.18345	0.17392	0.15694	0.13388
	4	0.19644	0.19521	0.19396	0.18421	0.16667	0.14258
	5	0.20395	0.20271	0.20144	0.19155	0.17361	0.14879
Weibull	1	0.52815	0.52476	0.52129	0.49480	0.44821	0.38516
	2	0.61366	0.61010	0.60646	0.57834	0.52764	0.45695
	3	0.66729	0.66361	0.65984	0.63054	0.57700	0.50128
	4	0.70503	0.70125	0.69738	0.66713	0.61145	0.53204
	5	0.73343	0.72957	0.72561	0.69459	0.63719	0.55494

4 Monte Carlo null distribution critical points

In practice, simulated percentiles for small samples are commonly used by statisticians and reliability analysts. We have simulated the upper percentile values for 90%, 95%, 98% and 99%. Tables 2, and 3 present these percentile values of the statistics $\widehat{\beta}(\lambda,s)$ in Eq. (7) and the calculations are based on 10000 simulated samples of sizes $n=5(5)50$.

Table 2. The upper percentile of $\widehat{\beta}(\lambda,s)$ at $s=1$

n	90%	95%	98%	99%
5	0.158141	0.214662	0.295260	0.362522
10	0.104143	0.134946	0.178586	0.216829
15	0.082785	0.105432	0.135249	0.156459
20	0.072365	0.089758	0.112053	0.131618
25	0.066243	0.080607	0.099079	0.111748
30	0.059578	0.071791	0.085999	0.098294
35	0.056283	0.067457	0.082178	0.093471
40	0.053088	0.064114	0.077583	0.087248
45	0.050666	0.060518	0.072671	0.081029
50	0.048221	0.057201	0.068969	0.076068

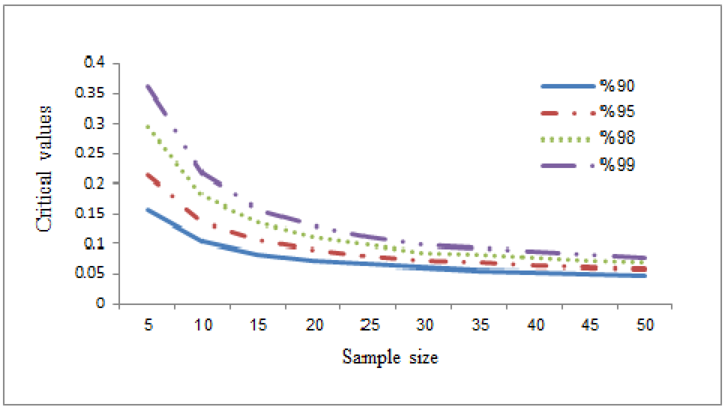

Fig. 1: Relation between critical values, sample size and confidence levels at $s = 1$, $\lambda = 0.1$

Table 3. The upper percentile of $\widehat{\beta}(\lambda, s)$ at $s = 5$

n	90%	95%	98%	99%
5	0.015482	0.020125	0.025696	0.030143
10	0.010910	0.013516	0.016381	0.018818
15	0.009146	0.011022	0.013339	0.014732
20	0.007969	0.009592	0.011306	0.012668
25	0.007226	0.008744	0.010393	0.011523
30	0.006741	0.008039	0.009696	0.010758
35	0.006266	0.007477	0.008848	0.009649
40	0.005940	0.007064	0.008259	0.009114
45	0.005691	0.006746	0.007887	0.008627
50	0.005427	0.006446	0.007504	0.008229

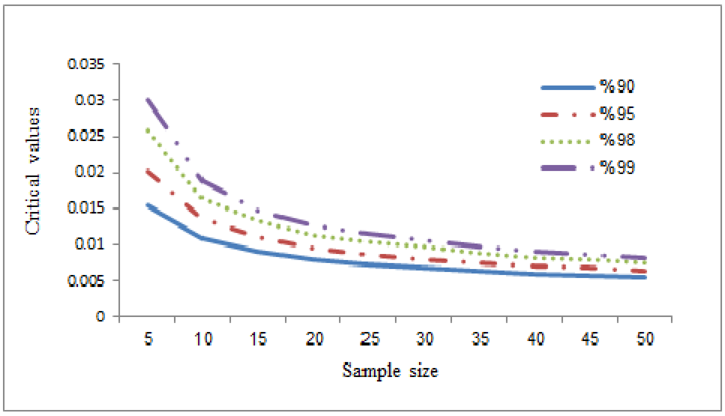

Fig. 2: Relation between critical values, sample size and confidence levels at $s = 5$, $\lambda = 0.1$

In view of Tables 2, 3, and its Figures $(1,2)$, it is noticed that the critical values increase as the confidence level increases and the values decrease as the sample size increases.

4.1 The Power Estimates

In this subsection, we present the power estimates of the test statistic $\widehat{\beta}(\lambda,s)$ at significance level $\alpha = 0.05$ using LFR, Weibull and Gamma distribution. The estimates are based on 10000 simulated samples of sizes $n = 10, 20$ and 30 with parameter $\theta = 2, 3$ and 4.

Table 4. Power estimates using $\alpha = 0.05$ at $s = 1$

Distribution	θ	$n=10$	$n=20$	$n=30$
LFR	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Weibull	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Gamma	2	0.9715	0.9788	0.9843
	3	0.9936	0.9955	0.9987
	4	0.9991	0.9996	1.0000

Table 5. Power estimates using $\alpha = 0.05$ at $s = 5$

Distribution	θ	$n=10$	$n=20$	$n=30$
LFR	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Weibull	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Gamma	2	0.9867	0.9937	0.9952
	3	0.9985	0.9995	0.9999
	4	0.9997	1.0000	0.9999

From Tables 4, 5, it is noted that the power of the test increases by increasing the value of the parameter θ and sample size n, and it is clear that our test has good powers.

5 Testing Against $RNBU_{mgf}$ class for censored data

The objective of this section is to propose a test statistic to test $H_0 : F$ is exponential versus $H_1 : F$ belongs to $RNBU_{mgf}$ with randomly right-censored data. Such censored data is usually the only information available in a life-testing model or in a clinical study where observations may be lost (censored) before the completion of this study. This experimental situation can formally be modeled as follows.

Suppose n objects are put on test, with true life times $X_1, X_2, ..., X_n$. Assume that $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d.) according to a continuous life distribution F. Let $Y_1, Y_2, ..., Y_n$ be (i.i.d.) according to a continuous life distribution G. and are independent of $X^{'}s$.

In the randomly right-censored model, we observe the pairs $(Z_j, \delta_j), j = 1, ..., n$ where $Z_j = \min(X_j, Y_j)$ and

$$\delta_j = \begin{cases} 1 & if \quad Z_j = X_j \ (\ j-\text{th observation is uncensored}) \\ 0 & if \quad Z_j = Y_j \ (\ j-\text{th observation is censored}) \end{cases}.$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < ... < Z_{(n)}$ denote the ordered $Z^{'}s$ and $\delta_{(j)}$ is the δ_j corresponding to $Z_{(j)}$ respectively.

Using the censored data $(Z_j, \delta_j), j = 1, ..., n$. Kaplan and Meier [20] proposed the product limit estimator.

$$\overline{F}_n(X) = 1 - F_n(X) = \prod_{[j: Z_{(j)} \le X]} \{(n-j)/(n-j+1)\}^{\delta_{(j)}}, X \in [0, Z_n].$$

Now, for testing $H_0 : \xi(\lambda,s) = 0$, against $H_1 : \xi(\lambda,s) > 0$, using the randomly right censored data, we propose the following test statistic:

$$\hat{\xi}^c(\lambda,s) = \frac{1}{\mu}\left[\frac{1}{s\lambda}\beta\eta + (\frac{1}{(\lambda+s)}\mu - \frac{1}{s\lambda})\beta - (\frac{1}{(\lambda+s)}\mu + \frac{1}{s\lambda})\eta + \frac{1}{s\lambda}\right] \tag{10}$$

Where,

$$\mu = \sum_{i=1}^{n} \prod_{m=1}^{i-1} C_m^{\delta_{(m)}} \left(Z_{(i)} - Z_{(i-1)}\right),$$

$$\beta = \sum_{j=1}^{n} e^{\lambda Z_{(j)}} \left(\prod_{p=1}^{j-2} C_p^{\delta_{(p)}} - \prod_{p=1}^{j-1} C_p^{\delta_{(p)}} \right),$$

$$\eta = \sum_{l=1}^{n} e^{-sZ_{(l)}} \left(\prod_{q=1}^{l-2} C_q^{\delta_{(q)}} - \prod_{q=1}^{l-1} C_q^{\delta_{(q)}} \right).$$

And,

$$C_k = \frac{n-k}{n-k+1}, \qquad dF_n(Z_{(j)}) = \overline{F}\left(Z_{j-1}\right) - \overline{F}\left(Z_j\right)$$

Tables 6, and 7. give the critical values for percentiles of $\hat{\xi}^c(\lambda,s)$ test for sample sizes $n = 20(5)30(10)81, 86$.based on 10000 replications.

Table 6. Critical values for percentiles of $\hat{\xi}^c(\lambda,s)$ test at $s = 1$

n	90%	95%	98%	99%
20	2.51899	3.02423	3.7453	4.20071
25	2.17860	2.64525	3.25387	3.69404
30	2.01479	2.42473	2.96989	3.31103
40	1.68255	2.00613	2.46125	2.74566
50	1.47031	1.76988	2.14072	2.38124
60	1.32854	1.60787	1.95139	2.20053
70	1.21383	1.45453	1.78388	1.98071
81	1.14788	1.36565	1.65734	1.87475
86	1.12950	1.32693	1.59007	1.76178

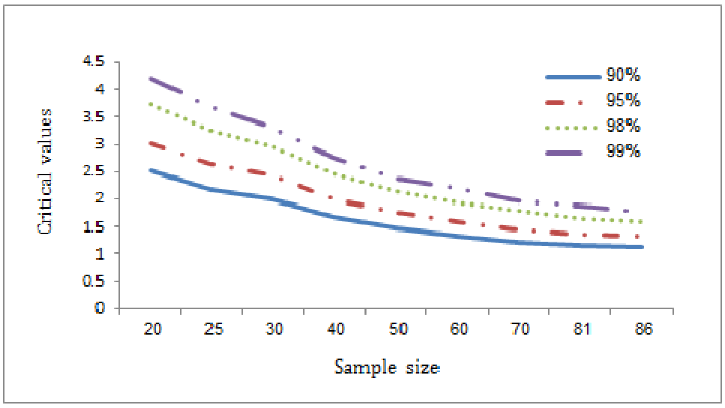

Fig. 3: Relation between critical values, sample size and confidence levels at $s = 1$, $\lambda = 0.1$

Table 7. Critical values for percentiles of $\hat{\xi}^c(\lambda, s)$ test at $s = 5$

n	90%	95%	98%	99%
20	0.72371	0.83678	0.96802	1.07043
25	0.64991	0.75973	0.88560	0.96595
30	0.59966	0.69154	0.80373	0.89796
40	0.52835	0.61313	0.69820	0.75932
50	0.48318	0.56127	0.64957	0.71423
60	0.43527	0.50692	0.58389	0.64405
70	0.40763	0.47746	0.56068	0.61346
81	0.38370	0.44472	0.52096	0.57518
86	0.37051	0.42810	0.49136	0.54651

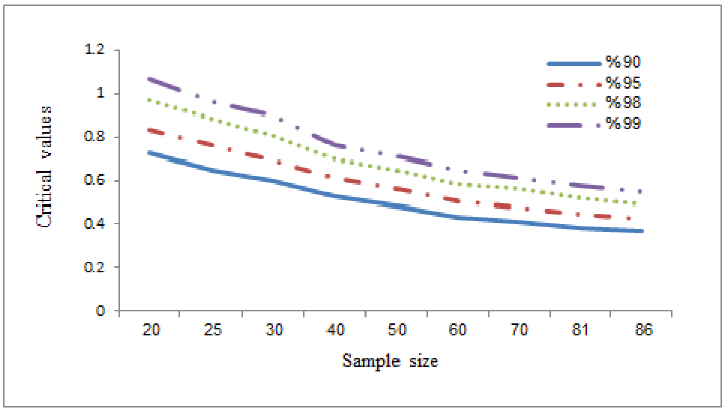

Fig. 4: Relation between critical values, sample size and confidence levels at $s = 5$, $\lambda = 0.1$

In view of Tables 6, 7, and its Fig. (3,4), it is noticed that the critical values increase as the confidence level increases and they decrease as the sample size increases.

5.1 The power estimates for $\hat{\xi}^c(\lambda, s)$

We present an estimation of the power for testing exponentiality Versus $RNBU_{mgf}$. Using significance level $\alpha = 0.05$ with suitable parameter values of θ at $n = 10, 20$ and 30, and for commonly used distributions in reliability such as LFR, Weibull and Gamma family alternatives at value of $s = 1, 5$ which is included in Tables 8, 9.

Table 8. Power estimates for $\hat{\xi}^c(\lambda, s)$ test at $s = 1$

Distribution	θ	$n = 10$	$n = 20$	$n = 30$
LFR	2	0.9945	0.9961	0.9943
	3	0.9927	0.9939	0.9922
	4	0.9896	0.9913	0.9866
Weibull	2	0.9998	0.9997	0.9996
	3	1.0000	0.9999	0.9996
	4	1.0000	1.0000	0.9997

Table 9. Power estimates for $\hat{\xi}^c(\lambda,s)$ test at $s=5$

Distribution	θ	$n=10$	$n=20$	$n=30$
LFR	2	0.9999	0.9999	0.9999
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Weibull	2	1.0000	1.0000	0.9998
	3	1.0000	1.0000	0.9992
	4	1.0000	1.0000	0.9990

6 Applications

In this section, we apply our test to some real data sets in the case of non censored and censored data at 95% confidence level.

6.1 Non censored data

Data-set #1. Consider the data set in Abouammoh et al. [3], these data represent a set of 40 patients suffering from blood cancer (leukemia) from one of ministry of health hospitals in Saudi Arabia. In this case, we get $\widehat{\beta}(\lambda,s)=0.746827$, at $s=1$, $\lambda=0.1$, $\widehat{\beta}(\lambda,s)=0.0567528$, at $s=5$, $\lambda=0.1$, and these values exceed the tabulated critical value in Tables 2, 3 in the two cases, it is evident that at the significant level 0.05 this data set has $RNBU_{mgf}$ property.

Data-set #2. Consider the data set given in Grubbs [19], This data set gives the times between arrivals of 25 customers at a facility. It is easy to show that $\widehat{\beta}(\lambda,s)=1.2696$, at $s=1$, $\lambda=0.1$, $\widehat{\beta}(\lambda,s)=0.0897996$, at $s=5$, $\lambda=0.1$, which are greater than the critical values of Tables 2, 3. Then we accept H_1 which states that the data set has $RNBU_{mgf}$ property.

Data-set #3. Consider the data set which represents failure times in hours, for a specific type of electrical insulation in an experiment in which the insulation has been subjected to a continuously increasing voltage stress (Lawless [23], p.138). We can see that the value of test statistic $\widehat{\beta}(\lambda,s)=0.0500617$, at $s=1$, $\lambda=0.1$, $\widehat{\beta}(\lambda,s)=0.0081738$, at $s=5$, $\lambda=0.1$, and these values are less than the tabulated critical values in Tables 2, 3. This means that the set of data has exponential property.

6.2 Censored data

Data-set #4. Consider the data from Susarla and Vanryzin [28], which represents 81 survival times (in months) of patients melanoma. Out of these 46 represents non-censored data. Now, taking into account the whole set of survival data (both censored and uncensored). It is found that the value of test statistic for the data set is given by $\hat{\xi}^c(\lambda,s)=128626.$, at $s=1$, $\lambda=0.1$, $\hat{\xi}^c(\lambda,s)=27742.8$, at $s=5$, $\lambda=0.1$, and these values are greater than the tabulated critical value in Tables 6, 7.in two cases This means that the data set has $RNBU_{mgf}$ property.

Data-set #5. On the basis of right censored data for lung cancer patients from Pena [27]. These data consist of 86 survival times (in month) with 22 right censored. Now account the whole set of survival data (both censored and uncensored), and computing the test statistic given by formula (10). It is found that at $s=1$, $\lambda=0.1$, $\hat{\xi}^c(\lambda,s)=3.19506$, at $s=5$, $\lambda=0.1$, $\hat{\xi}^c(\lambda,s)=0.692301$, which exceeds the tabulated values in Tables 6, 7 in two cases This means that the data set has $RNBU_{mgf}$ property.

References

[1] Abouammoh, A. M., Ahmed, A. N. (1988). “The New Better Than Used Class of Life Distribution.” Advances in Applied Probability, vol. 20, pp. 237-240.
[2] Abouammoh, A. M. and Ahmed, A. N. (1992). “On Renewal Failure Rate Classes of Life Distributions”, Statistics and probability Letters, vol. 14, pp. 211-217.
[3] Abouammoh, A. M., Abdulghani, S. A. and Qamber, I. S. (1994). “On partial orderings and testing of new better than renewal used classes”, Reliability Eng Syst Safety vol. 43, pp. 37-41.

[4] Abu-Youssef, S. E. (2004). "Moment inequality on new renewal better than used class of life distributions with hypothesis testing application", Applied mathematics and computation, vol. 149, pp. 651-659.
[5] Abu-Youssef, S. E. and Bakr, M. E. (2014). "A Goodness of Fit Approach to the Unknown Age (UBACT) Class of Life Distribution
[6] Abu-Youssef, S. E., El-Attar, H. E. and El-Toony, A. A. (2016). " Nonparametric Test for A Class of Life Time Distribution UBACT Based on The Laplace Transform", Global Journal of Mathematics, vol. 7, no. 2, pp. 741-747.
[7] Ahmad, I. A. (2001). "Moments inequalities of ageing families of distribution with hypothesis testing applications, J. Statist. Plan. Inf., vol. 92, pp. 121-132.
[8] Al-Gashgari, F. H., Shawky, A. I. and Mahmoud, M. A. W. (2016). "A Nonparametric Test for Testing Exponentiality Against NBUCA Class of Life Distributions Based on Laplace Transform", Quality and Reliability Engineering International, vol. 32, no. 1, pp. 29-36.
[9] Atallah, M. A., Mahmoud, M. A. W. and Al-Zahrani, B. M. (2014). "A New Test for Exponentiality Versus NBU mgf Life Distributions Based on Laplace Transform", Quality and Reliability Engineering International, vol. 30, no. 8, pp. 1353-1359.
[10] Barlow, R. E. and Proschan, F. (1981). "Statistical Theory of Reliability and Life Testing", To BeginWith, Silver Spring,Md, USA.
[11] Diab, L. S., EL-arishy, S. M. and Mahmoud, M. A.W. (2003). "Moment inequalities for testing new renewal better than used and renewal new better than used classes". I.J. Rel. Appl., vol. 4, pp. 97-123.
[12] Diab, L. S. and Mahmoud, M. A. W. (2007). "On testing exponentiality against HNRBUE based on a goodness of fit", International Journal of Reliability and Applications, vol. 8(1) , pp. 27-39.
[13] Diab, L.S. (2010). "Testing for NBUL using goodness of fit approach with application", Stat. papers., vol. 51, pp. 27-40.
[14] Diab, L.S. (2016). "Testing exponentiality against $RNBUE_{mgf}$ class of life distribution". Journal of Statistics: Advances in Theory and Applications. vol. 16, n. 2, pp. 163-183.
[15] EL-Arishy, S. M., Mahmoud, M. A. W. and Diab L. S. (2005). "Testing renewal new better than used life distributions based on u-test", Appl. Math. Model., vol. 29, pp. 784-796.
[16] EL-Arishy, S.M, Diab, L.S and Alim, A (2006). "Testing NRBU class of life distributions using a goodness of fit approach". Int J Reliability Appl. vol. 7(2), pp. 77-39.
[17] EL-Atfy, E. S., Diab, L. S. and EL-Arishy, S. M. (2016) "Testing Exponentiality Against NBRUrp Based on Goodness of Fit Approach", Journal of Statistics: Advances in Theory and Applications Volume 16, Number 2, Pages 255-273.
[18] Elbatal, I. (2009). "On Testing Statistics of Renewal New Better than Renewal Used Class of Life Distributions", Int. J. Contemp. Math. Sciences, vol. 4, no. 1, pp. 17 - 29.
[19] Grubbs, F. E. (1971). "Fiducial bounds on reliability for the two parameter negative exponential distribution". Technomet., vol. 13, pp. 873-876.
[20] Kaplan, E. L. and Meier, P. (1958). "Nonparametric estimation from incomplete observation". J. Amer. Statist.Assoc., vol. 53, pp.457-481.
[21] Kayid, M., Diab, L. S. and Alzughaibi, A. (2010). "Testing NBU (2) class of life distribution based on goodness of fit approach". Journal of King Saud University-Science, vol. 22 (4), pp. 241-245.
[22] Klar, B., and Muller, A.(2003). "Characterizations of classes of lifetime distributions generalizing the NBUE class", J.Appl.Prob., vol. 40, pp. 20-32.
[23] Lawless JF (1982). "Statistical models & methods for lifetime data", Wiley, New York.
[24] Mahmoud, M. A. W, Abdul Alim, N. A. (2006). "A goodness of fit approach to NBURFR and NBUARFR classes". Economic Quality Control , vol. 21, pp. 59-75.
[25] Mahmoud, M. A. W. and Diab, L. S. (2008). "A goodness of fit approach to decreasing variance residual life class of life distributions", JSTA ,vol. 7(1), pp. 119-136.
[26] Mahmoud, M. A. W, EL-Sagheer, R. M.and Etman,W. B. H.(2016). "Testing Exponentiality Against New Better Than Renwal Used in Laplace Transform Order", Journal of Statistics Applications & Probability, vol. 5, no. 2, pp. 279-285.
[27] Pena, A. E. (2002). "Goodness of fit tests with censored data". http://statman Stat.sc.edupenajtajkspresentedjtalk actronel.
[28] Susarla, V. and Vanryzin, J (1978). "Empirical bayes estimations of a survival function right censored observation". Ann. Statist., vol. 6, pp. 710-755.

S. M. El-Arisy is a professor of Mathematical statistics in Department of Mathematics, Faculty of Science (Girls Branch), Al-Azhar University, Cairo, Egypt. She received her PhD in Mathematical statistics in 1988 from Oregon State university Corvallis, U. S. A. Her research interests include: Characterization, statistical inference, and reliability.

L. S. Diab is a professor of Mathematical statistics in Department of Mathematics, Faculty of Science (Girls Branch), Al-Azhar University, Cairo, Egypt. She received her PhD in Mathematical statistics in 2004 from Al-Azhar University, Egypt. Her research interests include: Theory of reliability, ordered data, statistical inference, distribution theory, and Studied of classes of life distributions.

E. S. El-Atfy is an assistant lecturer of Mathematical statistics in Department of Mathematics, Faculty of Science (Girls Branch), Al-Azhar University, Cairo, Egypt. She received her M. Sc. in Mathematical statistics in 2017 from Al-Azhar University, Egypt. Her research interests include: Theory of reliability, Censored data, Life testing, Studied of classes of life distributions and characterization.
