Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/070301

Generalized Transmuted Power Function Distribution

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Received: 23 Apr. 2017, Revised: 28 Feb. 2018, Accepted: 4 Mar. 2018
Published online: 1 Nov. 2018

Abstract: In this paper, we introduce a new distribution called Generalized Transmuted Power Function Distribution (*GTPFD*). Some statistical properties are deduced. Finally, a real data application about the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli is used to illustrate. These real data show that the *GTPFD* can be considered as a good life time distribution comparing with other models.

Keywords: Transmuted Distribution, Generalized Transmuted Distribution, Power Function Distribution, Maximum likelihood

1 Introduction

Shaw and Buckley (2007) introduced a new class of distributions called transmuted distributions. If *G(x)* is the cumulative distribution function (*CDF*) of any random variable *X*, then the function

$$F(x) = G(x)\left[1+\lambda-\lambda G(x)\right],\ |\lambda| \le 1, \tag{1}$$

is called a transmuted distribution (TD). The probability density function (*pdf*) corresponding to (1) is given by

$$f(x) = \left[1+\lambda-2\lambda G(x)\right] g(x), \tag{2}$$

where $g(x)$ is the *pdf* of base distribution. Many transmuted distributions are proposed. Aryal and Tsokos (2011) presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Merovci (2013) proposed and studied the various structural properties of the transmuted Rayleigh distribution. Khan and King (2013) introduced the transmuted modified Weibull distribution. Transmuted Lomax distribution is presented by Ashour and Eltehiwy (2013). Elbatal et al. (2013) have presented transmuted generalized linear exponential distribution. Merovci and Puka (2014) introduced transmuted Pareto distribution. Abdul-Moniem (2015) proposed transmuted Burr type III distribution. Transmuted Gompertz distribution is presented by Abdul-Moniem and Seham (2015).

An extended of TD by adding two extra shape parameters called generalized transmuted distributions (*GTD*) have been introduced by Nofal et. al (2017). The cumulative distribution function (*CDF*) of *GTD* is

$$F(x) = [G(x)]^{a}\left\{1+\lambda-\lambda\,[G(x)]^{b}\right\}, \qquad a,b>0. \tag{3}$$

The probability density function (*pdf*) corresponding to (3) is

$$f(x) = g(x)\,[G(x)]^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\,[G(x)]^{b}\right\}. \tag{4}$$

A random variable X is said to have the three parameter power function distribution if its *pdf* is of the following form

$$g(x) = \frac{\alpha}{\theta}\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha-1}; \qquad \nu < x < \nu+\theta,\ (-\infty<\nu<\infty,\ \theta \text{ and } \alpha>0) \tag{5}$$

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This is a Pearson's Type-I distribution. If $\alpha = 1$ then the power function distribution coincides with the uniform distribution on the interval $(\nu, \nu+\theta)$.

The *CDF* corresponding to (5) is

$$G(x) = 1 - \left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}; \qquad \nu < x < \nu+\theta,\ (-\infty < \nu < \infty,\ \theta \text{ and } \alpha > 0) \tag{6}$$

Nofal et. al (2017), provided six special models of this family corresponding to the baseline Weibull, Lomax, Burr X, log-logistic, Lindley and Weibull geometric distributions. Here we introduce Generalized Transmuted Power Function Distribution (*GTPFD*) and its properties.

2 Generalized Transmuted Power Function Distribution (*GTPFD*)

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In this section, we introduce the *pdf* of *GTPFD* and its properties. Substituting (5) and (6) in (4), we get the *pdf* of *GTPFD* as follows

$$f(x) = \frac{\alpha}{\theta}\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha-1}\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}; \nu<x<\nu+\theta \tag{7}$$

The *pdf* for transmuted power function distribution (*TPFD*), generalized transmuted uniform distribution (*GTUD*), transmuted uniform distribution (*TUD*) and uniform distribution (*UD*) can be obtained by taking $a = b = 1$, $\alpha = 1$, $a = b = \alpha = 1$ and $(a = b = \alpha = 1\ \&\ \lambda = 0)$ respectively.

Table 1: Sub-models of the *GTPFD* distribution

No.	Distribution	α	a	b	λ	Author
1	*TPFD*	α	1	1	λ	Ul-Haq et al (2016)
2	*GTUD*	1	a	b	λ	New
3	*TUD*	1	1	1	λ	New
4	*UD*	1	1	1	0	

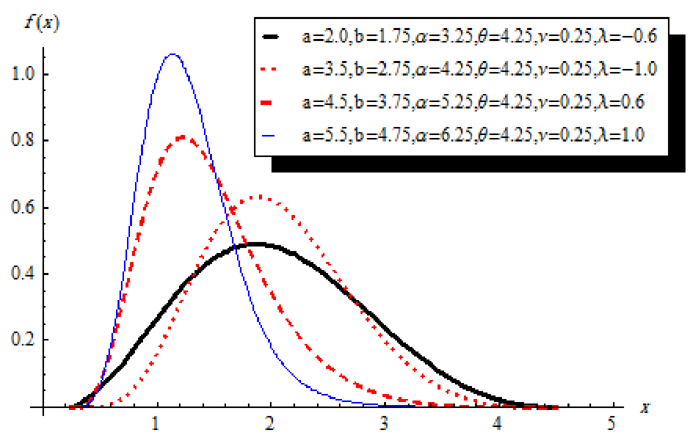

Fig. 1: pdf of GTPFD under different values of parameters

The *CDF*, survival function (*SF*), hazard rate function (*HR*) and reversed hazard rate function (*RHR*) corresponding (7) are

$$F(x) = \left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a}\left\{1+\lambda-\lambda\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}, \tag{8}$$

$$\bar{F}(x) = 1 - \left[1 - \left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a} \left\{1 + \lambda - \lambda \left[1 - \left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}, \tag{9}$$

$$h(x) = \frac{\frac{\alpha}{\theta}\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha-1}\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a-1}\left\{a(1+\lambda) - \lambda(a+b)\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}}{1-\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a}\left\{1+\lambda-\lambda\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}} \tag{10}$$

and

$$h^{*}(x) = \frac{\frac{\alpha}{\theta}\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha-1}\left\{a(1+\lambda) - \lambda(a+b)\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}}{\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]\left\{1+\lambda-\lambda\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}}. \tag{11}$$

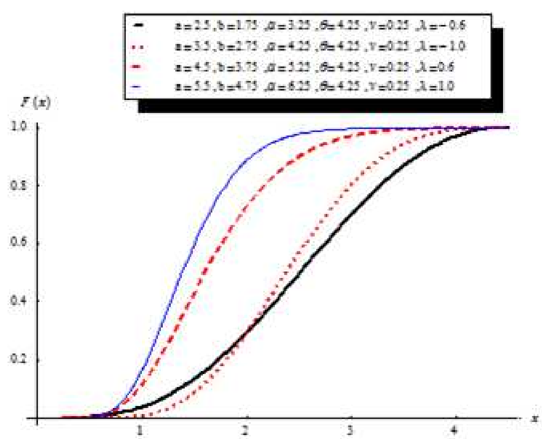

Fig. 2: CDF of GTPFD under different values of parameters

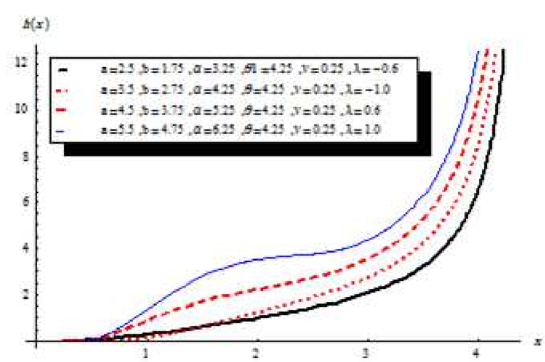

Fig. 3: HR of GTPFD under different values of parameters

The *HR* and *RHR* functions of *GTPFD* have the following properties

1. $\lim_{x \to \nu} h(x) = 0$
2. $\lim_{x \to \nu+\theta} h(x) = \infty$

3. $\lim_{x\to\nu} h^*(x) = \infty$
4. $\lim_{x\to\nu+\theta} h^*(x) = 0$

From these properties the *HR* is increasing and *RHR* is decreasing.

3 Statistical Properties

In this section some statistical properties of *GTPFD* are discussed.

3.1 Moments

The r^{th} traditional moments for *GTPFD* is

$$\mu_r^{/} = \frac{\alpha}{\theta}\int_{\nu}^{\nu+\theta} x^r \left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha-1}\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}dx$$

Using substitution

$$y = \left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}, \tag{12}$$

yields

$$\begin{aligned}\mu_r^{/} &= \alpha\int_0^1\left[\nu+\theta\left(1-y^{\frac{1}{\alpha}}\right)\right]^r(1-y)^{a-1}\left[a(1+\lambda)-\lambda(a+b)(1-y)^b\right]dy\\ &= \sum_{i=0}^{r}\sum_{j=0}^{i}\binom{r}{i}\binom{i}{j}\nu^{r-i}\theta^i(-1)^j\int_0^1 y^{\frac{j}{\alpha}}(1-y)^{a-1}\left[a(1+\lambda)-\lambda(a+b)(1-y)^b\right]dy\\ &= \sum_{i=0}^{r}\sum_{j=0}^{i}\binom{r}{i}\binom{i}{j}\nu^{r-i}\theta^i(-1)^j\beta_j\end{aligned} \tag{13}$$

where

$$\beta_j = \left[a(1+\lambda)\beta\left(\frac{j}{\alpha}+1,a\right)-\lambda(a+b)\beta\left(\frac{j}{\alpha}+1,a+b\right)\right].$$

The first two moments can be obtained by taking r =1 and 2 in (13) as follows:

$$\mu_1^{/} = \nu+\theta(1-\beta_1)$$

and

$$\mu_2^{/} = (\nu+\theta)(\nu+\theta-2\theta\beta_1)+\theta^2\beta_2$$

The variance (σ^2), standard deviation (σ) and coefficient of variation (CV) for *GTPFD* are

$$\sigma^2 = \theta^2\left[\beta_2-(\beta_1)^2\right],$$

$$\sigma = \theta\sqrt{\beta_2-(\beta_1)^2},$$

and

$$CV = \frac{\theta\sqrt{\beta_2-(\beta_1)^2}}{\nu+\theta(1-\beta_1)}.$$

3.2 Quantile and Median

The quantile x_q of the *GTPFD* is the real solution of the following equation

$$q^{\frac{1}{a}} = (1+\lambda)\left[1-\left(\frac{\nu+\theta-x_q}{\theta}\right)^{\alpha}\right] - \lambda\left[1-\left(\frac{\nu+\theta-x_q}{\theta}\right)^{\alpha}\right]^{b+1} \tag{14}$$

The equation (14) has no closed-form solution in x_q, so we have different cases by substituting the parametric values in the above quantile equation. So the derived special cases are

1.The q^{th} quantile of the *GTPFD* by substituting $b = 1$.

$$x_q = \nu+\theta-\theta\left(\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\lambda q^{\frac{1}{a}}}}{2\lambda}\right)^{\frac{1}{\alpha}}$$

2.The q^{th} quantile of the *GTPFD* by substituting $\lambda = -1$.

$$x_q = \nu+\theta-\theta\left(1-q^{\frac{1}{a(b+1)}}\right)^{\frac{1}{\alpha}}$$

3.The q^{th} quantile of the *GTPFD* by substituting $b = 1$ and $\lambda = -1$.

$$x_q = \nu+\theta-\theta\left(1-q^{\frac{1}{2a}}\right)^{\frac{1}{\alpha}}$$

By putting q = 0.5 in equation (14) we can get the median of *GTPFD*.

3.3 Mode

The mode of *GTPFD* is the solve the following equation with respect to x

$$\begin{aligned}&\left[(1-\alpha)+(a\alpha-1)\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]\left\{a(1+\lambda)-\lambda(a+b)\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}\\&-\lambda\alpha b(a+b)\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}=0\end{aligned} \tag{15}$$

The equation (15) has no closed-form solution in x, so we have different cases by substituting the parametric values in the above equation. So the derived special cases are

1.The mode of the *GTPFD* by substituting $b = 1$.

$$x = \nu+\theta-\theta\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right)^{\frac{1}{\alpha}},$$

where $A = \lambda(a+1)(\alpha a+\alpha-1)$, $B = a(\alpha a-1)+\lambda(a-3\alpha a+2-2\alpha)$ and $C = (a-\lambda)(1-\alpha)$

1.The mode of the *GTPFD* by substituting $\lambda = -1$.

$$x = \nu+\theta-\theta\left[\frac{\alpha-1}{\alpha(a+b)-1}\right]^{\frac{1}{\alpha}};\ \alpha>1,\ \alpha(a+b)>1.$$

3.4 Information entropies

The Shannon and Reny entropy for *GTPFD* have been obtained in this section.

3.4.1 Shannon entropy

The Shannon entropy for any distribution can be defined as $E\left[-\ln f\left(x\right)\right]$.

For *GTPFD* the Shannon entropy is

$$\begin{aligned} E\left[-\ln f\left(x\right)\right] &= -\ln\left(\frac{\alpha}{\theta}\right)-\left(\alpha-1\right)E\left[\ln\left(\frac{\nu+\theta-X}{\theta}\right)\right]-\left(a-1\right)E\left\{\ln\left[1-\left(\frac{\nu+\theta-X}{\theta}\right)^{\alpha}\right]\right\} \\ &-E\left[\ln\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\left(\frac{\nu+\theta-X}{\theta}\right)^{\alpha}\right]^{b}\right\}\right] \\ &= -\ln\left(\alpha\right)-\alpha E\left[\ln\left(\nu+\theta-X\right)\right]-\left(a-1\right)E\left\{\ln\left[1-\left(\frac{\nu+\theta-X}{\theta}\right)^{\alpha}\right]\right\} \\ &-E\left[\ln\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\left(\frac{\nu+\theta-X}{\theta}\right)^{\alpha}\right]^{b}\right\}\right] \\ &= -\ln\left(\alpha\right)-\alpha I_1-\left(a-1\right)I_2-I_3 \end{aligned} \tag{16}$$

Where

$$\begin{aligned} I_1 = E\left[\ln\left(\nu+\theta-X\right)\right] &= \tfrac{\alpha}{\theta}\int_{\nu}^{\nu+\theta}\ln\left(\nu+\theta-x\right)\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha-1}\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a-1} \\ &\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}dx \end{aligned}$$

Using substitution (12), we get

$$I_1=\ln\left(\theta\right)+\frac{1}{\alpha}\left[-C-\left(1+\lambda\right)\psi\left(a+1\right)+\lambda\psi\left(a+b+1\right)\right]. \tag{17}$$

$$\begin{aligned} I_2 &= E\left\{\ln\left[1-\left(\tfrac{\nu+\theta-X}{\theta}\right)^{\alpha}\right]\right\} \\ &= \tfrac{\alpha}{\theta}\int_{\nu}^{\nu+\theta}\ln\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha-1}\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a-1} \\ &\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}dx \end{aligned}$$

Using substitution (12), we get

$$I_2=\frac{-\left(1+\lambda\right)}{a}+\frac{\lambda}{a+b}=\frac{-a-b\left(1+\lambda\right)}{a\left(a+b\right)}. \tag{18}$$

and

$$\begin{aligned} I_3 &= E\left[\ln\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\left(\tfrac{\nu+\theta-X}{\theta}\right)^{\alpha}\right]^{b}\right\}\right] \\ &= \tfrac{\alpha}{\theta}\int_{\nu}^{\nu+\theta}\ln\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha-1} \\ &\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{a-1}\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left[1-\left(\tfrac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}dx \end{aligned}$$

Using substitution (12), we get

$$\begin{aligned} I_3 &= \int_0^1\ln\left[a\left(1+\lambda\right)-\lambda\left(a+b\right)\left(1-y\right)^{b}\right]\left(1-y\right)^{a-1}\left[a\left(1+\lambda\right)-\lambda\left(a+b\right)\left(1-y\right)^{b}\right]dy \\ &= a\left(1+\lambda\right)\int_0^1\ln\left[a\left(1+\lambda\right)-\lambda\left(a+b\right)\left(1-y\right)^{b}\right]\left(1-y\right)^{a-1}dy \\ &-\lambda\left(a+b\right)\int_0^1\ln\left[a\left(1+\lambda\right)-\lambda\left(a+b\right)\left(1-y\right)^{b}\right]\left(1-y\right)^{a+b-1}dy \end{aligned}$$

There is no solution for this integration, we can get it for $\lambda=-1$

$$I_3=\left(a+b\right)\left[\ln\left(a+b\right)\int_0^1\left(1-y\right)^{a+b-1}dy+b\int_0^1\left(1-y\right)^{a+b-1}\ln\left(1-y\right)dy\right]$$

$$= (a+b)\left[\frac{\ln(a+b)}{a+b} - \frac{b}{(a+b)^2}\right] = \frac{(a+b)\ln(a+b)-b}{a+b} \tag{19}$$

Where $\Psi(x) = \frac{d}{dx}\ln(\Gamma(x))$ and C is Eular constant.

Using the results (17), (18) and (19) in (16) and simplifying, we get the shannon entropy for $\lambda = -1$ as:

$$E[-\ln f(x)] = -\ln(\alpha) - \alpha\ln(\theta) + C + \psi(a+b+1) + 1 + \ln(a+b) - \frac{1}{a+b} \tag{20}$$

3.4.2 Renyi entropy

Renyi entropy is defined as

$$I_R(\gamma) = \frac{1}{\gamma-1}\log\int_R f^{\gamma}(x)\,dx;\quad \gamma > 0 \text{ and } \gamma \neq 1.$$

Now using the density function of *GTPFD*, we get

$$\int_R f^{\gamma}(x)\,dx = \left(\frac{\alpha}{\theta}\right)^{\gamma}\int_{\nu}^{\nu+\theta}\left(\frac{\nu+\theta-x}{\theta}\right)^{\gamma(\alpha-1)}\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{\gamma(a-1)} \left\{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x}{\theta}\right)^{\alpha}\right]^{b}\right\}^{\gamma}dx$$

Using substitution (12), we get

$$\begin{aligned}
&\int_R f^{\gamma}(x)\,dx \\
&= \left(\frac{\alpha}{\theta}\right)^{\gamma-1}\int_0^1 y^{(\gamma-1)\left(1-\frac{1}{\alpha}\right)}(1-y)^{\gamma(a-1)}\left[a(1+\lambda)-\lambda(a+b)(1-y)^b\right]^{\gamma}dy \\
&= \left(\frac{\alpha}{\theta}\right)^{\gamma-1}\sum_{i=0}^{\infty}\binom{\gamma}{i}(-1)^i[a(1+\lambda)]^{\gamma-i}[\lambda(a+b)]^i\int_0^1 y^{(\gamma-1)\left(1-\frac{1}{\alpha}\right)}(1-y)^{\gamma(a-1)+bi}dy \\
&= \left(\frac{\alpha}{\theta}\right)^{\gamma-1}\sum_{i=0}^{\infty}\binom{\gamma}{i}(-1)^i[a(1+\lambda)]^{\gamma-i}[\lambda(a+b)]^i\beta\left[(\gamma-1)\left(1-\frac{1}{\alpha}\right)+1,\gamma(a-1)+bi+1\right]
\end{aligned}$$

Then, we get the Renyi entropy as:

$$\begin{aligned}
I_R(\gamma) = &\frac{1}{\gamma-1}\left\{(\gamma-1)[\log(\alpha)-\log(\theta)] + \log\left(\sum_{i=0}^{\infty}\binom{\gamma}{i}(-1)^i[a(1+\lambda)]^{\gamma-i}[\lambda(a+b)]^i\right)\right. \\
&\left.+\log\left(\beta\left[(\gamma-1)\left(1-\frac{1}{\alpha}\right)+1,\gamma(a-1)+bi+1\right]\right)\right\}.
\end{aligned} \tag{21}$$

4 Maximum Likelihood Estimators (MLE)

In this section, we consider maximum likelihood estimators (MLE) of *GTPFD*. Let $x_1, x_2, \ldots, x_n$be a random sample of size n from *GTPFD*, then the log-likelihood function $L(\nu,\theta,\alpha,a,b,\lambda)$ can be written as

$$\begin{aligned}
L(\nu,\theta,\alpha,a,b,\lambda) &= n[\ln(\alpha)-\ln(\theta)] + (\alpha-1)\sum_{i=1}^n[\ln(\nu+\theta-x_i)-\ln(\theta)] \\
&+ (a-1)\sum_{i=1}^n\ln\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right] + \sum_{i=1}^n\ln\left\{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^b\right\} \\
&= n[\ln(\alpha)-\alpha\ln(\theta)] + (\alpha-1)\sum_{i=1}^n\ln(\nu+\theta-x_i) + (a-1)\sum_{i=1}^n\ln\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right] \\
&+ \sum_{i=1}^n\ln\left\{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^b\right\}.
\end{aligned}$$

Then the normal equations are

$$\begin{aligned}\frac{\partial \mathrm{L}}{\partial \alpha} &= -n\ln(\theta)+\sum_{i=1}^{n}\ln(\nu+\theta-x_i)-(a-1)\sum_{i=1}^{n}\frac{\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\ln\left(\frac{\nu+\theta-x_i}{\theta}\right)}{1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}} \\ &\quad -\sum_{i=1}^{n}\frac{\lambda b(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b-1}\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\ln\left(\frac{\nu+\theta-x_i}{\theta}\right)}{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}}=0,\end{aligned} \tag{22}$$

$$\frac{\partial \mathrm{L}}{\partial a}=\sum_{i=1}^{n}\ln\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]+\sum_{i=1}^{n}\frac{(1+\lambda)-\lambda\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}}{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}}=0, \tag{23}$$

$$\frac{\partial \mathrm{L}}{\partial b}=\sum_{i=1}^{n}\frac{-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}\ln\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]-\lambda\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}}{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}}=0, \tag{24}$$

$$\frac{\partial \mathrm{L}}{\partial \lambda}=\sum_{i=1}^{n}\frac{a-(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}}{a(1+\lambda)-\lambda(a+b)\left[1-\left(\frac{\nu+\theta-x_i}{\theta}\right)^{\alpha}\right]^{b}}=0. \tag{25}$$

The MLE of α, a, b and λ can be obtain by solving the equations (22), (23), (25) and (25) with $\nu=\min(x)$and $\theta=\max(x)-\min(x)$.

5 Applications

In this Section we fit *GTPFD* to real data sets and compare the fitness with the *TPFD*, *PFD*, *GTUD* and *TUD*. The set of data represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). In order to compare distributions, we consider the *K-S* (Kolmogorov-Smirnov) statistic, -2logL, *AIC* (Akaike Information Criterion), *AICC* (Akaike Information Criterion Corrected), *BIC* (Bayesian Information Criterion). The best distribution corresponds to lower *K-S*, *-2logL*, *AIC*, *BIC*, *AICC* statistics value.

Where, $AIC=2m-2\ln L$, $\quad AICC=AIC+\frac{2m(m+1)}{n-m-1}$, $BIC=m\ln(n)-2\ln L$

and $K-S=\max\limits_{1\le i\le n}\left[F(x_i)-\frac{i-1}{n},\frac{i}{n}-F(x_i)\right]$ where $F_n(x)=\frac{1}{n}\sum_{i=1}^{n}I_{x_i\le x}$ is empirical distribution function, $F(x)$ is comulative distribution function, *m* is the number of parameters in the statistical model and *n* the sample size.

Table 2: Maximum-likelihood estimates, *AIC*, *BIC* and *AICC* values, and *K-S* statistics for the 72 guinea pigs infected with virulent tubercle bacilli with $\hat{\nu}=0.1$ and $\hat{\theta}=5.45$.

Model	MLEs				Measures				
	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	*K-S*	*-2logL*	*AIC*	*BIC*	*AICC*
GTPFD	3.164	1.409	0.267	-0.289	0. 149	182.265	194.265	195.557	207.925
TPFD	2.007	-	-	0.312	0.231	198.947	206.947	207.544	216.054
PFD	2.576	-	-	-	0.185	196.013	202.013	202.366	208.843
GTUD	-	0.7	0.199	-0.205	0.309	237.784	247.784	248.693	259.168
TUD	-	-	-	1	0.176	198.225	204.225	204.578	211.055

Table 2 shows parameter MLEs, the values of K-S, -2logL, AIC, BIC, AICC statistics for the data set. From these results, it is evident that the *GTPFD* distribution is the best distribution for fitting these data set compared to other distributions considered here. And is a strong competitor to other distributions commonly used in literature for fitting lifetime data.

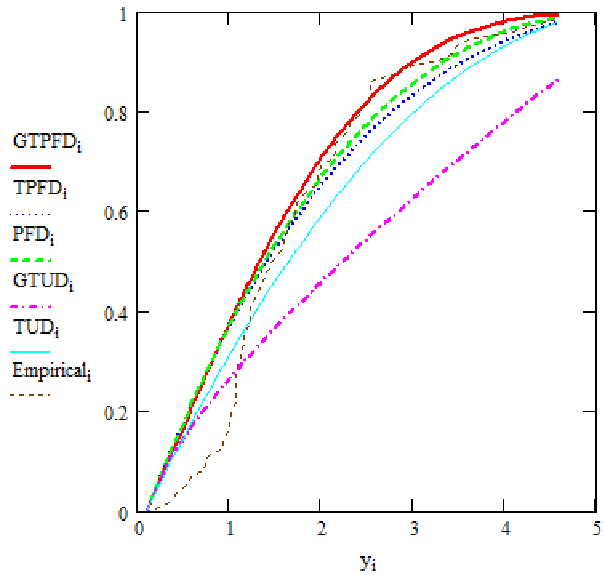

Fig. 4: Empirical, fitted *GTPFD*, *TPFD*, *PFD*, *GTUD* and *TUD CDF* of 72 guinea pigs infected with virulent tubercle bacilli

Table 3: Empirical means and mean squared errors

n	MLE	MSE
10	α =1.547	0.021
	λ =- 0.506	3.014×10^{-4}
	a =2.083	0.066
	v =0.406	6.437×10^{-4}
	θ =1.916	0.023
20	α =1.523	0.012
	λ =- 0.5	1.531×10^{-4}
	a =2.027	0.040
	v =0.4	3.444×10^{-4}
	θ =1.997	1.492×10^{-3}
30	α =1.487	1.551×10^{-3}
	λ =- 0.498	7.048×10^{-5}
	a =2.011	0.011
	v =0.399	1.923×10^{-5}
	θ =1.96	3.405×10^{-3}
40	α =1.51	1.544×10^{-3}
	λ =- 0.5	9.373×10^{-6}
	a =2.029	3.410×10^{-3}
	v =0.398	1.794×10^{-5}
	θ =1.986	1.139×10^{-3}
50	α =1.505	6.416×10^{-4}
	λ =- 0.5	6.274×10^{-7}
	a =2.01	1.326×10^{-3}
	v =0.401	1.199×10^{-5}
	θ =1.994	5.802×10^{-4}

6 Simulation

In this section, we conduct simulation studies to assess on the finite sample behavior of the MLEs of α, λ, a, v and θ. All results were obtained from 1000 replications. In each replication, a random sample of size n is drawn from the *GTPFD*. The true parameter values used in the data generating processes are $\alpha = 1.5$, $\lambda = -0.5$, $a = b = 2$, $v = 0.4$ and $\theta = 2$. The Table 3 reports the empirical means and the mean squared errors (MSE) of the corresponding estimators for sample sizes n = 10; 20; 30; 40 and 50.

7 Conclusion

In this paper, we introduce a new distribution called generalized transmuted power function distribution and presented its theoretical properties. The estimation of parameters is approached by the method of maximum likelihood. We compare the new distribution with its baseline distributions. An application of the generalized transmuted power function distribution to real data show that the new distribution can be used quite effectively to provide better than the transmuted power function, power function, generalized transmuted uniform and transmuted uniform distributions.

Acknowledgment

The authors would like to express their thanks to the editor for important remarks, careful reading and support.

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