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Shifted Polynomials in a Convection Problem

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The onset of convection in a horizontal layer of fluid heated from below in the presence of a gravity field varying across the layer is investigated. The eigenvalue problem governing the linear stability of the mechanical equilibria of the fluid layer in the case of free boundaries is solved using a Galerkin method based on shifted polynomials (Legendre and Chebyshev polynomials).

Keywords: Convection, variable gravity field.

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1 Problem Setting

Physical problems concerning the motion of fluids in the presence of a variable gravity field can be encountered in many practical applications, e.g. convection problems in porous media (in [1] the effect of a nonconstant gravity field on the stability of a horizontal fluid saturated isotropic porous layer was investigated; in the presence of a magnetic field, rotation and a variable gravity field, in [14], it is shown that for a rotating fluid in a porous medium, the stability is assured when gravity is upwards decreasing and in [13], that the presence of a finite Larmor radius is a necessary condition of stability), crystal growth problems (it is known that a microgravity environment allows a better understanding of the processes involving protein crystal growth [20]) or problems concerning the mass transport in the Earth's system [10] to quote but a few.

In this paper we analyze the influence of such gravity fields varying across the layer on the stability bounds in a convection problem that arises in a horizontal layer of fluid heated from below. The gravity field acting in the z-direction is orthogonal to the fluid layer and is assumed to depend on the vertical coordinate z only [19]. For such a variable gravity field different points of the fluid experience different buoyancy forces. As a consequence, part of a fluid layer tends to become unstable while the other tends to remain stable, the mechanical equilibrium turning into a convective motion. Pradhan and Samal [15] studied this problem estimating the growth rate in time dependence of the temperature fields in the framework of inviscid theory.

Experimental measurements of the Earth's upper atmosphere show that the atmospheric density decreases as the altitude increases as an approximately exponentially function of the vertical height [11]. Taking these into account, consider a layer of heat-conducting viscous fluid contained between the planes $z = 0$ and $z = h$ [19]. The equations governing the convective motion and the conducting state are [8]

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \text{grad})\mathbf{v} = -\frac{1}{\rho}\text{grad}p + \nu\Delta\mathbf{v} + \mathbf{g}(z)\alpha T, \\ \text{div}\mathbf{v} = 0, & t > 0 \\ \dfrac{\partial T}{\partial t} + (\mathbf{v} \cdot \text{grad})T = k\Delta T, \end{cases} \tag{1.1}$$

where ν is the coefficient of kinematic viscosity, ρ the density, α the thermal expansion coefficient, k the thermal diffusitivity, p is the pressure, T the temperature, $\mathbf{v}$ the velocity and the gravity $g(z)$ is defined by $\mathbf{g}(z) = gH(z)\mathbf{k}$, where g is a constant.

The linear stability of the conduction stationary solution characterized by $\mathbf{v} = 0$ of equations (1.1), written in the nondimensional form, against normal mode perturbations, is governed by a two-point problem for the ordinary differential equation [2]

$$\begin{cases} (D^2 - a^2)^2 W = RH(z)a^2\Theta, \\ (D^2 - a^2)\Theta = -RW, \end{cases} \tag{1.2}$$

where $D = d/dz$, R^2 is the Rayleigh number, a is the wavenumber and W and Θ are the amplitudes of the velocity and the temperature field perturbations.

Consider $H(z) = 1 + \varepsilon h(z)$, $z \in (0, 1)$. The parameter ε represents a scale for $h(z)$. In this case, the two-point problem for (1.2) consists of the ordinary differential equations [19]

$$\begin{cases} (D^2 - a^2)^2 W = R[1 + \varepsilon h(z)]a^2\Theta, \\ (D^2 - a^2)\Theta = -RW \end{cases} \tag{1.3}$$

and the usual boundary conditions for free boundaries read

$$W = D^2 W = \Theta = 0 \quad \text{at } z = 0, 1. \tag{1.4}$$

We look for the smallest eigenvalue R^2 (the Rayleigh number) in (1.3)-(1.4) defining the neutral manifold.

In the case of rigid boundaries, i.e. the boundary conditions are

$$W = DW = \Theta = 0 \text{ at } z = 0, 1 \tag{1.5}$$

the eigenvalue problem has been studied by us in some previous papers, e.g. [4], [5] and by Straughan in [19]. He performed numerical evaluations of the Rayleigh number by using the energy method for some various gravity fields and our numerical evaluations obtained in [4], [5] were similar to those obtained in [19]. Herein, for the same varying gravity fields used in the case of rigid boundaries we will obtain numerical results in the case of free boundaries.

In [7], [8] it is proved that in this situation, when $H(z) \geq 0$ across the layer, the principle of exchange of stabilities holds. In our case ε is a small parameter and for the chosen functions $h(z)$ the inequality $H(z) \geq 0$ is valid, therefore the principle of exchange of stabilities holds no matter what the boundary conditions. We can mention that there are cases in which this condition is not satisfied and the principle of exchange of stabilities holds.

Although the analytical and numerical investigation of the eigenvalue problems governed by systems of ordinary differential equations with variable coefficients and depending on many physical parameters, using spectral methods, started in the 60's, their study has not become a simple one. In [3] characteristic value problems represented by systems of ordinary differential equations with variable coefficients, that are not self-adjoint in the usual sense, were solved by methods patterned after those applicable in the cases where the eigenvalue problems were defined by self-adjoint operators. In [16] Roberts found a variational basis for these methods, he showed that finding the solution of a characteristic equations of the considered type is equivalent to a variational procedure.

The expansion sets of functions used for the various fields encountered in the convection problems from hydrodynamic stability theory (e.g. the velocity, temperature, concentration field) must have a basic property: they must be easy to evaluate. That is why, most of the times, the trigonometric and the polynomial functions that are easy to evaluate, are used. A second property is the completeness of the expansion sets of functions. This assures that each function of the given space can be written as a linear combination of functions from the considered set (or, more likely, as a limit of such a linear combination). The Chebyshev polynomials, the Legendre polynomials, the Hermite functions, the sine and cosine functions, satisfy this condition.

In [4] the two-point problem (1.3)-(1.4) was investigated using methods based on Fourier series of trigonometric functions for the particular variable gravity field given by $h(z) = -z$. Herein, the trial and the test sets of functions will consists in Chebyshev and Legendre polynomials. However, when the boundary conditions are very complicated the Galerkin approach is not easy to apply. This is one of the major problems in applying spectral methods. It is difficult to find sets of functions that satisfy all boundary conditions moreover if they involve high order derivatives or physical parameters. Here we overcome this problem by defining a new function. In order to write the boundary conditions in a simpler form, the function $\Psi = (D^2 - a^2)W$ was introduced and taking into account that $W = D^2W = 0$ at $z = 0, 1$ we get $\Psi = 0$ at $z = 0, 1$.

By denoting $\mathbf{U} = (W, \Psi, \Theta)$ the eigenvector in (1.3)-(1.4), the two-point problem can be rewritten

$$\begin{cases} L_1\mathbf{U} = 0 \Leftrightarrow (D^2 - a^2)W - \Psi = 0, \\ L_2\mathbf{U} = 0 \Leftrightarrow (D^2 - a^2)\Psi - R(1 + \epsilon h(z))a^2\Theta = 0, \\ L_3\mathbf{U} = 0 \Leftrightarrow (D^2 - a^2)\Theta + RW = 0, \end{cases} \tag{1.6}$$

with the boundary conditions

$$W = \Psi = \Theta = 0 \text{ at } z = 0, 1. \tag{1.7}$$

2 Methods Based on Shifted Polynomials

In order to perform not only an analytical study, but also to obtain numerical evaluations for the Rayleigh number R^2, each of the functions from the unknown eigenvector $\mathbf{U}$ is approximated by a truncated series of orthogonal polynomials, in our case Chebyshev and Legendre polynomials. These polynomials are orthogonal on $[-1, 1]$. Since in this problem the range is $[0, 1]$, we will use shifted polynomials, orthogonal on $[0, 1]$, obtained from the original polynomials by a variable transformation.

The Chebyshev polynomials were widely used in spectral methods for ordinary differential equations, e.g. [2], [6], [12]. Here we present only some basic properties of these polynomials necessary for our study.

The Chebyshev polynomials (of the first kind) of degree n, $T_n(z)$, are orthogonal on $[-1, 1]$ with respect to the weight function $w(z) = 1/\sqrt{1 - z^2}$, i.e.

$$\int_{-1}^{1} T_n(z)T_m(z)w(z) = \frac{\pi}{2}c_n\delta_{mn}, \quad c_n = \begin{cases} 2, \text{ if } n = 0, \\ 1, \text{ if } n \geq 1 \end{cases} .$$

The shifted Chebyshev polynomials (of the first kind) (SCP) of degree n on $(0, 1)$, $T_n^*(z)$, are defined by the relation $T_n^*(z) = T_n(2z - 1)$. The following orthogonality relation holds

$$\int_{0}^{1} T_n^*(z)T_m^*(z)w^*(z)dz = \begin{cases} \frac{\pi}{2}c_n\delta_{nm}, \text{ if } i = j, \\ 0, \text{ if } i \neq j, \end{cases} \tag{2.1}$$

with respect to the weight function $w^*(z) = 1/\sqrt{z(1 - z)}$. The recurrence relation between T_n^* has the form

$$T_n^*(z) = 2(2z - 1)T_{n-1}^*(z) - T_{n-2}^*(z).$$

Similarly to Shen [18], let us introduce $M_1 = \{\Phi_k^*(z)\}_{k\in\mathbb{Z}}$, a the complete set of orthogonal functions in $L^2(0, 1)$, $\Phi_k^*(z)$ defined by

$$\Phi_k^*(z) = T_k^*(z) - T_{k+2}^*(z) \tag{2.2}$$

and satisfying boundary conditions of the type $\Phi_k^*(0) = \Phi_k^*(1) = 0$. Then the unknown functions Ψ, W, Θ can be expanded upon the complete set M_1 and they satisfy automatically all the boundary conditions. We have

$$W = \sum_{k=0}^{n} W_k \Phi_k^*(z), \quad \Theta = \sum_{k=0}^{n} \Theta_k \Phi_k^*(z), \quad \Psi = \sum_{k=0}^{n} \Psi_k \Phi_k^*(z). \tag{2.3}$$

The system (1.3) can then be written in terms of the expansion functions only. Imposing the condition that left-hand sides of equations of the system are orthogonal on Φ_i^*, $i = 0, 1, ..., n$, we get the algebraic system

$$\begin{cases} \sum\limits_{k=0}^{n} \Big\{ \Big((D^2 - a^2)\Phi_k^*(z), \Phi_i^*(z)\Big) W_k - (\Phi_k^*(z), \Phi_i^*(z))\Psi_k \Big\} = 0, \\ \sum\limits_{k=0}^{n} \Big\{ \Big((D^2 - a^2)\Phi_k^*(z), \Phi_i^*(z)\Big) \Psi_k - Ra^2((1 + \epsilon h(z))\Phi_k^*(z), \Phi_i^*(z))\Theta_k \Big\} = 0, \\ \sum\limits_{k=0}^{n} \Big\{ \Big((D^2 - a^2)\Phi_k^*(z), \Phi_i^*(z)\Big) \Theta_k + R(\Phi_k^*(z), \Phi_i^*(z)) W_k \Big\} = 0 \end{cases} \tag{2.4}$$

in the unknown coefficients W_k, Ψ_k, Θ_k. Since not all these coefficients are null, the condition that the determinant of the system vanish is imposed leading to the secular equation.

Following [12] it is easy to deduce the derivation formulae

$$(\Phi_k^*(z))' = 2\Big\{ 2k \sum_{\substack{r=0 \\ k-r \text{ odd}}}^{k-1} T_r^*(z) - 2(k+2) \sum_{\substack{r=0 \\ k+2-r \text{ odd}}}^{k+1} T_r^*(z) \Big\} \tag{2.5}$$

for the first derivative of the function Φ_k^* and

$$(\Phi_k^*(z))'' = 4\Big\{ \sum_{\substack{r=0 \\ k-r \text{ even}}}^{k-2} (k-r)k(k+r)T_r^*(z) - \sum_{\substack{r=0 \\ k+2-r \text{ even}}}^{k} (k+2-r)(k+2)(k+2+r)T_r^*(z) \Big\} \tag{2.6}$$

for the second one. In the numerical evaluations we will take into account that the first term in each of the involved sums is halved.

The presence of the varying gravity field leads to nonconstant coefficients, such that the analytical expression of the scalar product $(h(z)\Phi_k^*(z), \Phi_i^*(z))$ is based on the relation [6]

$$z^r T_s(z) = \frac{1}{2^r} \sum_{i=0}^{r} C_r^i T_{s-r+2i}(z). \tag{2.7}$$

The analytical expressions of all the other scalar products from (2.4) are taking with respect to the weight function $w^*(z)$ and deduced by taking into account the orthogonality relation (2.1).

The secular equation leading to the neutral values of the Rayleigh number can be deduced in a similar way by using shifted Legendre polynomials.

Let

$$H_0^1(0,1) = \{f | f, f' \in L^2(0,1), f(0) = f(1) = 0\},$$

be a Hilbert space and denote by L_k the Legendre polynomials defined on $(-1,1)$. Then the shifted Legendre polynomials Q (SLP) on $(0,1)$ are defined by the relation $Q_k(x) = L_k(2x-1)$ and they are orthogonal on the interval $(0,1)$, i.e.

$$\int_0^1 Q_i Q_j dz = \frac{1}{2i+1}\delta_{ij}.$$

Using the identity [9]

$$2(2i+1)Q_i(z) = Q'_{i+1}(z) - Q'_{i-1}(z), \tag{2.8}$$

we define the set M_2 of orthogonal functions ϕ_i, $M_2 = \{\phi_i\}_{i=\overline{1,n}}$.

$$\phi_i(z) = \int_0^z Q_i(t)dt = \frac{Q_{i+1} - Q_{i-1}}{2(2i+1)}, i = 1, 2, ...$$

that satisfy boundary conditions of the type $\phi_i(0) = \phi_i(1) = 0$ at $z = 0$ and 1 such that the set M_2 is complete in $H_0^1(0,1)$.

Therefore we can write the unknown functions W, Ψ, Θ as series in the form

$$W = \sum_{i=1}^n W_i \phi_i(z), \;\; \Psi = \sum_{i=1}^n \Psi_i \phi_i(z), \;\; \Theta = \sum_{i=1}^n \Theta_i \phi_i(z). \tag{2.9}$$

The secular equation is obtained following the same steps in the analytical study as before, i.e.

$$\begin{vmatrix} ((D^2-a^2)\phi_i,\phi_k) & -1 & 0 \\ 0 & ((D^2-a^2)\phi_i,\phi_k) & -Ra^2((1+\epsilon h(z))\phi_i,\phi_k) \\ ((D^2-a^2)\phi_i,\phi_k) & 0 & ((D^2-a^2)\phi_i,\phi_k) \end{vmatrix} = 0. \tag{2.10}$$

3 Numerical Evaluations

Numerical evaluations for the studied eigenvalue problem were obtained mostly for rigid boundary cases for different significant values of the scale parameter ϵ and the wavenumber a. In [4] we obtained approximative numerical evaluations of the critical Rayleigh number for both free and rigid boundaries using spectral methods based on Fourier trigonometric series (TFS), for the particular variable gravity field $h(z) = -z$.

We regain the critical values of the Rayleigh number in the classical case, i.e. $\varepsilon = 0$ $a^2 = 2.2214$, $Ra = 657.511$ for both free boundaries. The number of functions in the expansion sets is small ($n = 4$), for $n > 4$ the improvement of the numerical values of the Rayleigh number is insignificant and would not justify more computational time. Three variable gravity fields from [19], i.e. $h(z) = -z$, $h(z) = -z^2$ and $h(z) = z^2 - 2z$ were taken into consideration.

In Table 1 the evaluations for the particular case $h(z) = -z$ are presented in comparison with the ones obtained before in [4]. Taking this into consideration the method can be considered suitable for other varying gravity fields as well. The numerical results presented in Tables 1,2,3 show that a decreasing gravity field enlarges the domain of stability.

ϵ	a^2	$R^2 - SCP$	$R^2 - SLP$	$R^2 - TFS$
0.0	4.92	657.512	675.05	657.5133416
0.01	4.92	660.747	678.45	660.8174287
0.03	4.92	667.653	685.33	667.5262351
0.33	4.92	787.363	808.303	787.4411276
0.2	5.00	730.459	749.95	730.6101972
0.2	9.00	829.44	846.70	829.4751258
0.5	7.5	930.982	952.07	931.6381300
0.5	9.00	994.393	1015.27	995.3701503
0.75	10.0	1251.178	1276.05	1255.126920

Table 1. Numerical values of the Rayleigh number for various values of the parameters for $h(z) = -z$.

ϵ	a^2	$R^2 - SCP$	$R^2 - SLP$
0.0	4.92	657.512	675.05
0.01	4.92	659.41	676.99
0.03	4.92	663.17	680.90
0.33	4.92	725.06	745.21
0.2	5.00	696.80	715.87
0.2	9.00	791.24	808.22
0.5	7.5	813.28	833.21
0.5	9.00	868.72	888.54
0.75	10.0	993.51	1016.20

Table 2. Numerical values of the Rayleigh number for various values of the parameters for $h(z) = -z^2$.

ϵ	a^2	$R^2 - SCP$	$R^2 - SLP$
0.0	4.92	657.512	675.05
0.01	4.92	662.29	679.91
0.03	4.92	671.95	689.83
0.33	4.92	861.25	882.05
0.2	5.00	767.40	787.44
0.2	9.00	871.37	889.03
0.5	7.5	1088.2	1110.46
0.5	9.00	1162.4	1184.11
0.75	10.0	1687.8	1713.45

Table 3. Numerical values of the Rayleigh number for various values of the parameters for $h(z) = z^2 - 2z$.

4 Conclusions

In this paper we presented two shifted polynomials-based methods for the study of the linear stability of the mechanical equilibrium of a horizontal layer of a viscous incompressible fluid heated from below in the case of a variable gravity field.

Since it has the advantage of optimal analysis, i.e. the analytical study is not a tedious one and the obtained numerical results are very good compared to other methods, the Galerkin method was chosen for the study of the eigenvalue problem (1.3)-(1.4).

We provided numerical results for various gravity fields, decreasing but not all linear. These results proved to agree quite well with the results obtained by us with the Galerkin method based on trigonometric functions [4] for a particular gravity field and also with the classical one existing in the literature. The numerical evaluations of the Rayleigh number were obtained for different values of the physical parameters, allowing a conclusion on the effects of these parameters on the stability domain. It was seen that the domain of stability decreases as the gravity field is increasing.

Although both expansion sets of functions lead to good numerical evaluations of the Rayleigh number, it seems that the method based on the expansion upon shifted Chebyshev polynomials is more effective. However, in most cases, the Legendre polynomials are preferred in the Galerkin approach and the Chebyshev polynomials are considered suitable for the collocation methods.

References

[1] S. M. Alex and P. R. Patil, Effect of variable gravity field on thermal instability in a porous medium with inclined temperature gradient and vertical troughflow, *J of*

Porous Media **5**(2002), 137–147.

[2] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, 2nd. ed., Dover, New York, 2000.

[3] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic stability*, Oxford Univ. Press, 1961.

[4] I. Dragomirescu, Approximate neutral surface of a convection problem for variable gravity field, *Rend. Sem. Mat. Univ. Pol. Torino* **64**, (2006), 331–342.

[5] I. Dragomirescu, A SLP-based method for a convection problem for a variable gravity field , *Proceedings of Aplimat 2007*, Bratislava, 149–154.

[6] C. I. Gheorghiu, *Spectral Methods for Differential Problems*, Casa Cartii de Stiinta, Cluj-Napoca, 2007.

[7] I. Herron, On the principle of exchange of stabilities in Rayleigh-Bénard convection, *SIAM J. Appl. Math.* **61** (2000), 1362–1368.

[8] I. Herron, Onset of convection in a porous medium with internal heat source and variable gravity, *IJES* **39** (2001), 201–208.

[9] A. A. Hill and B. Straughan, A Legendre spectral element method for eigenvalues in hydromagnetic stability, *J of Computational and Applied Mathematics* **193** (2003), 363–381.

[10] K. H. Ilk, *Introduction to the Gravity Field Theory, Lecture Notes*, part 3, 2nd Tropical School of Geodesy, Indonesia, Bandung 4-16 November, ITB Press, Bandung, 1996.

[11] D. King Hale, Upper atmosphere studies by ranging to satellites, *Phil. Trans. Roy. Soc. London A.* **284** (1977), 555–563.

[12] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman&Hall, 2003.

[13] K. Prakash and V. Kapoor, Effects of finite larmor radius and variable gravitational field on thermal instability of fluid layer under rotation in porous medium , *Czech. J of Physics* **48** (1998), 599–608.

[14] K. Prakash and H. Raj, Effect of variable gravitational field on thermal instability of a rotating fluid layer with magnetic field in porous medium, *Czech. J of Physics* **47**(1997), 793–800.

[15] G. Pradhan and P. Samal, *Thermal instability of a fluid layer under variable body forces*, *J. Math. Anal. Appl.* **122** (1987), 1167–1181.

[16] P. H. Roberts, Characteristic value problems posed by differential equations arising in hydrodynamic and hydromagnetics, *J. Math. Analysis and Applications* **1** (1960), 195–214.

[17] J. Shen, Efficient spectral - Galerkin method I, Direct solvers of second and fourth order equations by using Legendre polynomials, *SIAM J. Sci Stat. Comput* **15** (1994), 1489–1505.

[18] J. Shen, Efficient spectral-Galerkin method II, Direct solvers of second and fourth order equations by using Chebyshev polynomials, *SIAM J. Sci. Stat. Comput.* **16** (1995), 74–87.

[19] B. Straughan, *The Energy Method, Stability, and Nonlinear Convection*, Springer, Berlin, 2003.

[20] P. G. Vekilov, Protein crystal growth – microgravity aspects, *Advances in Space Research* **24** (1999), 1231–1240.