Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/120125

Estimation of Finite Population Mean Through a Two-Parameter Ratio- Product-Ratio-Type Exponential Estimator in Systematic Sampling

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Received: 19 Oct. 2018, Revised: 11 Jul. 2019, Accepted: 17 Jul. 2019.
Published online: 1 Jan. 2023.

Abstract: In this Paper, we suggest a two parameter ratio-product-ratio-type exponential estimator for estimating the finite population mean in systematic sampling. The bias and mean squared error of the suggested estimator are obtained to the first degree of approximation. It has been shown that the proposed estimator is better than the usual unbiased estimator, Swain's (1964) ratio estimator, Shukla's (1971) product estimator and Singh et al's (2011) estimators under some realistic conditions. An empirical study has been under taken to evaluate the performance of the suggested estimator over other existing estimators.

Keywords: Study variate, Auxiliary variate, Bias, Mean Squared Error, Empirical study.
AMS Classification: 62D05.

1 Introduction

It iswell known that the use of auxiliary information at the estimation stage improves the efficiency of an estimate. Ratio, product and regression methods of estimation are good examples in this context. The method of systematic sampling first studied by Madow and Madow (1944) and is widely used in survey of finite populations. Apart from it systematic sampling provides estimators which are more efficient than simple random sampling or stratified random sampling for certain types of populations; see Cochran (1946), Gautschi (1957) and Hajeck (1959). Further few authors have paid their attention towards the estimation of population mean using auxiliary information in systematic sampling, for instance, see Swain (1964), Shukla (1971), Singh (1967), Kushwaha and Singh (1989), Banarasi et al (1993), Singh and Singh (1998), Singh et al (2011), Singh and Jatwa (2012) ,Tailor et al (2013) ,Singh et al (2015), Tarry and Singh (2015), Singh and Pal (2017a, b) and Pal et al (2018).

Let us consider a finite population $\Omega=\left(\Omega_1,\Omega_2,\ldots,\Omega_N\right)$ of *N* units. The *N* units of the population *U* are numbered from 1 to *N* in some order. Let *N*=*nk*, where n and k are positive integers. Thus there will be *k* samples (clusters) each of size n. Then observe the study variate y and the auxiliary variate *x* for each and every unit selected in the sample. Let $\left(y_{ij}, x_{ij}\right)$ for $i=1,2,\ldots,k$ and $j=1,2,\ldots,n$; denote the value of j^{th} unit in the i^{th} sample. The systematic sample means $\bar{y}_{sy}=(1/n)\sum_{j=1}^{n} y_{ij}$ and $\bar{x}_{sy}=(1/n)\sum_{j=1}^{n} x_{ij}$ are the unbiased estimators of population means

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$\bar{Y}=(1/N)\sum_{i=1}^{k}\sum_{j=1}^{n}y_{ij}$ and $\bar{X}=(1/N)\sum_{i=1}^{k}\sum_{j=1}^{n}x_{ij}$ of y and x respectively.

Let $S_y^2=\frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij}-\bar{Y}\right)^2$, $S_x^2=\frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{X}\right)^2$ and

$S_{xy}=\frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij}-\bar{Y}\right)\left(x_{ij}-\bar{X}\right)$ be population mean squares/variances of y and x respectively; and population covariance between y and x. Also $C_y^2=S_y^2/\bar{Y}^2$ and $C_x^2=S_x^2/\bar{X}^2$ are the square of coefficients of variation of y and x respectively.

To obtain the bias and *MSE* of the suggested estimator up to first order of approximation, we write,

$\bar{y}_{sy}=\bar{Y}(1+e_0)$, $\bar{x}_{sy}=\bar{X}(1+e_1)$

such that

$E(e_i)=0 \quad \forall i=0,1,$

and

$E\left(e_0^2\right)=\theta\rho_y^*C_y^2,\ E\left(e_1^2\right)=\theta\rho_x^*C_x^2,\ E(e_0e_1)=\theta\, k\, C_x^2\sqrt{\rho_y^*\rho_x^*}$,

where $\theta=\frac{(N-1)}{nN}$, $\rho=\frac{S_{yx}}{S_yS_x}$, $k=\rho\frac{C_y}{C_x}$, $\rho_y^*=\left[1+(n-1)\rho_y\right]$, $\rho_x^*=\left[1+(n-1)\rho_x\right]$,

$\rho^*=\left(\rho_y^*/\rho_x^*\right)^{1/2}$; ρ_y and ρ_x are the intraclass correlation coefficients among the pair of units for the variables y and x.

The variance/ mean squared error (*MSE*) of the usual unbiased estimator $\bar{y}_{sy}$ for the population mean $\bar{Y}$ is given by

$$Var\left(\bar{y}_{sy}\right)=MSE\left(\bar{y}_{sy}\right)=\theta\bar{Y}^2\rho_y^*C_y^2 \quad (1.1)$$

Swain (1964) and Shukla (1971) defined ratio and product estimators for population mean $\bar{Y}$ respectively as

$$\bar{y}_{Rsy}=\bar{y}_{sy}\left(\frac{\bar{X}}{\bar{x}_{sy}}\right) \quad (1.2)$$

and

$$\bar{y}_{Psy}=\bar{y}_{sy}\left(\frac{\bar{x}_{sy}}{\bar{X}}\right) \quad (1.3)$$

To the first degree of approximation, the mean squared errors (*MSEs*) of the estimators $\bar{y}_{Rsy}$ and $\bar{y}_{Psy}$ are respectively given by

$$MSE\left(\bar{y}_{Rsy}\right)=\theta\bar{Y}^2\left[\rho_y^*C_y^2+\rho_x^*C_x^2\left(1-2k^*\right)\right] \quad (1.4)$$

$$MSE\left(\bar{y}_{Psy}\right)=\theta\bar{Y}^2\left[\rho_y^*C_y^2+\rho_x^*C_x^2\left(1+2k^*\right)\right] \quad (1.5)$$

where $k^*=\rho\left(\frac{C_y}{C_x}\right)\sqrt{\frac{\rho_y^*}{\rho_x^*}}=k\left(\rho_y^*/\rho_x^*\right)^{1/2}$.

Ratio-type and product-type exponential estimators in systematic sampling for population mean $\bar{Y}$ due to Singh et al

(2011) are respectively given by

$$\bar{y}_{\mathrm{Re}\,sy} = \bar{y}_{sy}\exp\left(\frac{\bar{X}-\bar{x}_{sy}}{\bar{X}+\bar{x}_{sy}}\right), \quad (1.6)$$

$$\bar{y}_{Pesy} = \bar{y}_{sy}\exp\left(\frac{\bar{x}_{sy}-\bar{X}}{\bar{x}_{sy}+\bar{X}}\right). \quad (1.7)$$

To the first degree of approximation, the mean squared errors of the estimators $\bar{y}_{\mathrm{Re}\,sy}$ and $\bar{y}_{Pesy}$ are respectively

$$MSE\left(\bar{y}_{Re\,sy}\right)=\theta\bar{Y}^2\left[\rho_y^* C_y^2+\frac{\rho_x^* C_x^2}{4}\left(1-4k^*\right)\right], \quad (1.8)$$

$$MSE\left(\bar{y}_{Pesy}\right)=\theta\bar{Y}^2\left[\rho_y^* C_y^2+\frac{\rho_x^* C_x^2}{4}\left(1+4k^*\right)\right]. (1.9)$$

In this paper we have suggested a two-parameter-ratio-product-ratio-type exponential estimator for population mean $\bar{Y}$ in systematic sampling on the line of Singh and Yadav (2018) and its properties are studied under large sample approximation. An empirical study is carried out.

2 The Proposed Two Parameter Ratio-Product-Ratio-Type Exponential Estimator in Systematic Sampling

For estimating the population mean $\bar{Y}$ of the study variable *y* in systematic sampling, we define the following ratio-product-ratio-type exponential estimator

$$d_{e(\eta,\delta)} = \bar{y}_{sy}\left[\eta\, exp\left\{\frac{(1-2\delta)\left(\bar{x}_{sy}-\bar{X}\right)}{\left(\bar{x}_{sy}+\bar{X}\right)}\right\}+(1-\eta)exp\left\{\frac{(1-2\delta)\left(\bar{X}-\bar{x}_{sy}\right)}{\left(\bar{x}_{sy}+\bar{X}\right)}\right\}\right], \quad (2.1)$$

(η,δ) being constants. The goal of this paper is to derive values for these constants (η,δ) such that the bias and /or the mean squared error (*MSE*) of $d_{e(\eta,\delta)}$ is minimal.

We note that $d_{e(\eta,\delta)} = d_{e(1-\eta,\,1-\delta)}$, that is the estimator $d_{e(\eta,\delta)}$ is invariant under point reflection through the point $(\eta,\delta)=\left(\frac{1}{2},\frac{1}{2}\right)$. For $\delta=\frac{1}{2}$, $d_{e(\eta,\delta)}$ reduces to systematic sample mean $d_{e(\eta,1/2)}=\bar{y}_{sy}$. For $(\eta,\delta)=(1,1)$, $d_{e(\eta,\delta)}$ reduces to $\bar{y}_{Resy}$ while for $(\eta,\delta)=(1,0)$ it boils down to $\bar{y}_{Pesy}$.

2.1 Bias and Mean Squared Error (MSE) of the SuggestedEstimator

To obtain the bias and *MSE* of the suggested estimator $d_{e(\eta,\delta)}$, we write

$\bar{y}_{sy} = \bar{Y}\left(1+e_y\right)$, $\bar{x}_{sy} = \bar{X}\left(1+e_x\right)$

such that

$E\left(e_y\right)=E\left(e_x\right)=0$

and

$E\left(e_y^2\right)=\theta\rho_y^* C_y^2,$

$$E\left(e_x^2\right)=\theta\rho_x^* C_x^2,$$

$$E\left(e_y e_x\right)=\theta\sqrt{\rho_y^*\rho_x^*}\quad k\,C_x^2,$$

$$=\theta\,k^*\rho_x^* C_x^2\ ,$$

Expressing (2.1) in terms of e_y and e_x, we have

$$d_{e(\eta,\delta)}=\bar{Y}\left(1+e_y\right)\left[\eta\,exp\left\{\frac{(1-2\delta)\,e_x}{(2+e_x)}\right\}+(1-\eta)exp\left\{\frac{-(1-2\delta)e_x}{(2+e_x)}\right\}\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[\eta\,exp\left\{\frac{(1-2\delta)\,e_x}{2}\left(1+\frac{e_x}{2}\right)^{-1}\right\}+(1-\eta)exp\left\{\frac{-(1-2\delta)e_x}{2}\left(1+\frac{e_x}{2}\right)^{-1}\right\}\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[\begin{array}{l}\eta\left\{1+\frac{(1-2\delta)\,e_x}{2}\left(1+\frac{e_x}{2}\right)^{-1}+\frac{(1-2\delta)^2\,e_x^2}{8}\left(1+\frac{e_x}{2}\right)^{-2}+...\right\}\\+(1-\eta)\left\{1-\frac{(1-2\delta)e_x}{2}\left(1+\frac{e_x}{2}\right)^{-1}+\frac{(1-2\delta)e_x^2}{8}\left(1+\frac{e_x}{2}\right)^{-2}+...\right\}\end{array}\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[\begin{array}{l}\eta\left\{1+\frac{(1-2\delta)\,e_x}{2}\left(1-\frac{e_x}{2}+\frac{e_x^2}{8}-...\right)+\frac{(1-2\delta)^2\,e_x^2}{8}+...\right\}\\+(1-\eta)\left\{1-\frac{(1-2\delta)e_x}{2}\left(1-\frac{e_x}{2}+\frac{e_x^2}{8}-...\right)+\frac{(1-2\delta)^2\,e_x^2}{8}+...\right\}\end{array}\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[1+\frac{(1-2\delta)\,e_x}{2}\left\{\eta-\frac{\eta e_x}{2}-(1-\eta)+\frac{(1-\eta)e_x}{2}+...\right\}+\frac{(1-2\delta)^2\,e_x^2}{8}+...\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[1+\frac{(1-2\delta)\,e_x}{2}\left\{2\eta-1+\frac{e_x}{2}(1-2\eta)+...\right\}+\frac{(1-2\delta)^2\,e_x^2}{8}+...\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[1-\frac{(1-2\delta)(1-2\eta)e_x}{2}\left\{1-\frac{e_x}{2}+...\right\}+\frac{(1-2\delta)^2\,e_x^2}{8}+...\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[1-\frac{(1-2\delta)(1-2\eta)e_x}{2}\left\{1-\frac{e_x}{2}+...\right\}+\frac{(1-2\delta)^2\,e_x^2}{8}+...\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[1-\frac{(1-2\delta)(1-2\eta)}{2}e_x+\frac{(1-2\delta)(1-2\eta)}{4}e_x^2+\frac{(1-2\delta)^2}{8}e_x^2+...\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[1-\frac{(1-2\delta)(1-2\eta)}{2}e_x+\frac{(1-2\delta)}{8}e_x^2\left\{2\,(1-2\eta)+(1-2\delta)\right\}+...\right]$$

$$=\bar{Y}\left(1+e_y\right)\left[1-\frac{(1-2\delta)(1-2\eta)}{2}e_x+\frac{(1-2\delta)(3-4\eta-2\delta)}{8}e_x^2+...\right]$$

$$= \bar{Y}\left[\begin{array}{l} 1 + e_y - \frac{(1-2\delta)(1-2\eta)}{2} e_x - \frac{(1-2\delta)(1-2\eta)}{2} e_x e_y \\ + \frac{(1-2\delta)(3-4\eta-2\delta)}{8} e_x^2 + \frac{(1-2\delta)(3-4\eta-2\delta)}{8} e_y e_x^2 \ldots \end{array}\right]$$

Neglecting terms of $e_x's$ having power greater than two we have

$$d_{e(\eta,\delta)} = \bar{Y}\left[1 + e_y - \frac{(1-2\delta)(1-2\eta)}{2} e_x + \frac{(1-2\delta)(3-4\eta-2\delta)}{8} e_x^2 - \frac{(1-2\delta)(1-2\eta)}{2} e_y e_x\right]$$

or

$$\left(d_{e(\eta,\delta)} - \bar{Y}\right) = \bar{Y}\left[e_y - \frac{(1-2\delta)(1-2\eta)}{2}\left(e_x + e_y e_x\right) + \frac{(1-2\delta)(3-4\eta-2\delta)}{8} e_x^2\right] \quad (2.2)$$

Taking expectation of both sides of (2.2) we get the bias of $d_{e(\eta,\delta)}$ to the first degree of approximation as

$$B\left(d_{e(\eta,\delta)}\right) = \frac{\theta\bar{Y}}{8}\rho_x^* C_x^2 (1-2\delta)\left[(3-4\eta-2\delta) - 4(1-2\eta)k^*\right] \quad (2.3)$$

Squaring both sides of (2.2) and neglecting terms of *e's* having power greater than two we have

$$\left(d_{e(\eta,\delta)} - \bar{Y}\right)^2 = \bar{Y}^2\left[e_y^2 - (1-2\eta)(1-2\delta) e_y e_x + \frac{(1-2\eta)^2 (1-2\delta)^2}{4} e_x^2\right] \quad (2.4)$$

Taking expectation of both sides of (2.4) we get the *MSE* of $d_{e(\eta,\delta)}$ to the first degree of approximation as

$$MSE\left(d_{e(\eta,\delta)}\right) = \theta\bar{Y}^2\left[\rho_y^* C_y^2 + \frac{(1-2\eta)(1-2\delta)}{4}\rho_x^* C_x^2\left\{(1-2\eta)(1-2\delta) - 4k^*\right\}\right] \quad (2.5)$$

which is minimized for

$$\frac{(1-2\eta)\ (1-2\delta)}{2} = k^*. \quad (2.6)$$

Thus the resulting minimum *MSE* of $d_{e(\eta,\delta)}$ is given by

$$MSE_{min}\left(d_{e(\eta,\delta)}\right) = \theta\rho_y^*\ \bar{Y}^2 C_y^2\left(1-\rho^2\right). \quad (2.7)$$

So we established the following theorem.

Theorem 2.1 : To the first degree of approximation

$$MSE\left(d_{e(\eta,\delta)}\right) \geq \theta\bar{Y}^2\rho_y^* C_y^2\left(1-\rho^2\right)$$

with equality holding if

$$(1-2\eta)(1-2\delta) = 2k^*.$$

3 Comparison of Mean Squared Errors and Choice of Parameters

3.1 Comparing the MSE of the Systematic Sample Mean $\bar{y}_{sy}$ to the Proposed Estimator $d_{e(\eta,\delta)}$

From (1.1) and (2.5) we have

$$MSE\left(d_{e(\eta,\delta)}\right) - MSE\left(\bar{y}_{sy}\right) = \theta\bar{Y}^2\frac{(1-2\eta)(1-2\delta)}{2}\rho_x^* C_x^2\left[\frac{(1-2\eta)(1-2\delta)}{2} - 2k^*\right]$$

which is less than zero if

$$(1-2\eta)(1-2\delta)\left[(1-2\eta)(1-2\delta) - 4k^*\right] < 0. \quad (3.1)$$

Therefore, either

(i) $\eta > \frac{1}{2}, \delta > \frac{1}{2}$ and $k^* > \frac{(1-2\eta)(1-2\delta)}{4}$,

(ii) $\eta < \frac{1}{2}, \delta > \frac{1}{2}$ and $k^* < \frac{(1-2\eta)(1-2\delta)}{4}$,

(iii) $\eta > \frac{1}{2}, \delta < \frac{1}{2}$ and $k^* < \frac{(1-2\eta)(1-2\delta)}{4}$,

(iv) $\eta < \frac{1}{2}, \delta < \frac{1}{2}$ and $k^* > \frac{(1-2\eta)(1-2\delta)}{4}$.

For further details see Singh and Yadav (2018, p.233).

Thus the envisaged estimator $d_{e(\eta,\delta)}$ is superior to the systematic sample mean $\bar{y}_{sy}$ as long as the conditions (i) to (iv) are satisfied.

3.2 Comparing the MSE of the Ratio-Type Exponential estimator $\bar{y}_{Re\,sy}$ *in systematic sampling to the proposed Estimator* $d_{e(\eta,\delta)}$

In this section it is desired to obtain the range of plausible values for (η,δ) where the envisaged estimator $d_{e(\eta,\delta)}$ is more efficient than the ratio-type exponential estimator $\bar{y}_{Resy}$ in systematic sampling.

Inserting $(\eta,\delta)=(1,1)$ in (2.5) we get the *MSE* of the ratio-type exponential estimator $\bar{y}_{Resy}$ as

$$MSE(\bar{y}_{Re\,sy}) = \theta\bar{Y}^2\left[\rho_y^* C_y^2 + \frac{\rho_x^*}{4}(1-4k^*)\right] \quad (3.2)$$

From (2.5) and (3.2) we have

$$MSE(\bar{y}_{Resy}) - MSE(d_{e(\eta,\delta)}) = \frac{\theta\bar{Y}^2\rho_x^* C_x^2}{4}\{1-(1-2\eta)(1-2\delta)\}\{1+(1-2\eta)(1-2\delta)-4k^*\}$$

which is non-negative if

$$(2\eta\delta-\eta-\delta)\left[2k^*-1-(2\eta\delta-\alpha-\beta)\right] > 0 \quad (3.3)$$

that is if

$$\left.\begin{array}{ll} either & (2k^*-1) > (2\eta\delta-\eta-\delta) > 0 \\ or & (2k^*-1) < (2\eta\delta-\eta-\delta) < 0 \end{array}\right\} (3.4)$$

So the proposed estimator $d_{e(\eta,\delta)}$ will dominate over ratio type exponential estimator $\bar{y}_{Resy}$ as long as the condition (3.4) is satisfied.

3.3 Comparing the MSE of the Product-Type Exponential Estimator $\bar{y}_{Pesy}$ *to proposed Estimator* $d_{e(\eta,\delta)}$

In this Section we will obtain the range of (η,δ) in which the proposed estimator $d_{e(\eta,\delta)}$ is better than the product-type exponential estimator $\bar{y}_{Pesy}$.

From (1.9) and (2.5) we have

$$MSE(\bar{y}_{Pesy}) - MSE(d_{e(\eta,\delta)}) = \theta\bar{Y}^2\rho_x^* C_x^2\{1+2\eta\delta-\eta-\delta\}\{2k^*-(2\eta\delta-\eta-\delta)\}$$

which is non- negative if

$$\{1+2\eta\delta-\eta-\delta\}\{2k^{*}-(2\eta\delta-\eta-\delta)\}>0 \quad (3.5)$$

i.e. if

$$\left.\begin{array}{ll} either & 2k^{*}>(2\eta\delta-\eta-\delta)>-1 \\ or & 2k^{*}<(2\eta\delta-\eta-\delta)<-1 \end{array}\right\} \quad (3.6)$$

Thus the suggested estimator $d_{e(\eta,\delta)}$ is more efficient than the product-type exponential estimator $\bar{y}_{Pesy}$ as long as the condition (3.6) is satisfied.

3.4 Comparing the MSE of the Swain's Ratio Estimator $\bar{y}_{Rsy}$ *to the Proposed Estimator* $\boldsymbol{d_{e(\eta,\delta)}}$

In this section we will derive the range of constants (η,δ)in which the suggested estimator $d_{e(\eta,\delta)}$ is more efficient than the ratio-type exponential estimator $\bar{y}_{Resy}$ in systematic sampling.

From (1.4) and (2.5) we have

$$MSE(\bar{y}_{Rsy})-MSE(d_{e(\eta,\delta)})=\theta\bar{Y}^{2}\rho_{x}^{*}C_{x}^{2}\left[1-\frac{(1-2\eta)(1-2\delta)}{2}\right]\left[1+\frac{(1-2\eta)(1-2\delta)}{2}-2k^{*}\right]$$

which is positive if

$$\left[1-\frac{(1-2\eta)(1-2\delta)}{2}\right]\left[1+\frac{(1-2\eta)(1-2\delta)}{2}\right]>0$$

$$\left.\begin{array}{ll} i.e.\ if\ either & (2k^{*}-1)>\dfrac{(1-2\eta)(1-2\delta)}{2}>1 \\ or & (2k^{*}-1)>\dfrac{(1-2\eta)(1-2\delta)}{2}<1 \end{array}\right\} \quad (3.7)$$

So the proposed estimator $d_{e(\eta,\delta)}$ is better than the ratio estimator $\bar{y}_{Rsy}$ if the condition (3.7) holds good.

3.5 Comparing the MSE of the Shukla's Product-Estimator $\boldsymbol{\bar{y}_{Psy}}$ *to the envisaged estimator* $\boldsymbol{d_{e(\eta,\delta)}}$

Here we will obtain the range of (η,δ) in which the suggested estimator $d_{e(\eta,\delta)}$ is more efficient than the Shukla's product estimator $\bar{y}_{Psy}$.

From (1.5) and (2.5) we have

$$MSE(d_{e(\eta,\delta)})-MSE(\bar{y}_{Psy})=\theta\bar{Y}^{2}\rho_{x}^{*}C_{x}^{2}\left\{1-\frac{(1-2\eta)(1-2\delta)}{2}\right\}\left[1-\frac{(1-2\eta)(1-2\delta)}{2}+2k^{*}\right]$$

which is less than zero if

$$\left\{1-\frac{(1-2\eta)(1-2\delta)}{2}\right\}\left\{1-\frac{(1-2\eta)(1-2\delta)}{2}+2k^{*}\right\}<0$$

$$\left.\begin{array}{ll} i.e.\ if\ either & (1+2k^{*})<\dfrac{(1-2\eta)(1-2\delta)}{2}<1 \\ or & 1<\dfrac{(1-2\eta)(1-2\delta)}{2}<(1+2k^{*}) \end{array}\right\} (3.8)$$

Thus the suggested estimator $d_{e(\eta,\delta)}$is more efficient than the product estimator $\bar{y}_{Psy}$ in systematic sampling as long as the condition (3.8) is satisfied.

3.6 Comparing the MSE of the proposed estimator $\boldsymbol{d_{e(\eta,\delta)}}$ *with* $\bar{y}_{sy}$, $\bar{y}_{Re\,sy}$, $\bar{y}_{Pesy}$, $\bar{y}_{Rsy}$ *and* $\bar{y}_{Psy}$ *when one of the two constants has pre-assigned value.*

Let η_0 be the preassigned value of the constant η. Then the proposed estimator $d_{e(\eta,\delta)}$ takes the form :

$$d_{e(\eta,\delta)} = \bar{y}_{sy}\left[\eta_0\, exp\left\{\frac{(1-2\delta)\left(\bar{x}_{sy}-\bar{X}\right)}{\left(\bar{x}_{sy}+\bar{X}\right)}\right\}+(1-\eta_0)exp\left\{\frac{(1-2\delta)\left(\bar{X}-\bar{x}_{sy}\right)}{\left(\bar{x}_{sy}+\bar{X}\right)}\right\}\right] \quad (3.9)$$

Further, let δ_0 be the preassigned value of the constant δ. Then the suggested estimator $d_{e(\eta,\delta)}$ turns out to be the estimator.

$$d_{e(\eta,\delta)} = \bar{y}_{sy}\left[\eta\, exp\left\{\frac{(1-2\delta_0)\left(\bar{x}_{sy}-\bar{X}\right)}{\left(\bar{x}_{sy}+\bar{X}\right)}\right\}+(1-\eta)exp\left\{\frac{(1-2\delta_0)\left(\bar{X}-\bar{x}_{sy}\right)}{\left(\bar{x}_{sy}+\bar{X}\right)}\right\}\right] \quad (3.10)$$

Putting the value of $\eta=\eta_0$ in (2.5) we get the MSE of the estimator $d_{e(\eta,\delta)}$ to the first degree of approximation as

$$MSE\left(d_{e(\eta,\delta)}\right)=\theta\bar{Y}^2\left[\rho_y^*C_y^2+\frac{(1-2\eta_0)(1-2\delta)}{4}\rho_x^*C_x^2\left\{(1-2\eta_0)(1-2\delta)-4k^*\right\}\right] (3.11)$$

From (1.1) and (3.11) we have

$$MSE\left(\bar{y}_{sy}\right)-MSE\left(d_{e(\eta,\delta)}\right)=\theta\bar{Y}^2\,\frac{(1-2\eta_0)(1-2\delta)}{4}\rho_x^*C_x^2\left[4k^*-(1-2\eta_0)(1-2\delta)\right]$$

which is positive if

$$(1-2\eta_0)(1-2\delta)\left[4k^*-(1-2\eta_0)(1-2\delta)\right]>0$$

i.e. if

$$\left.\begin{array}{lll} either & \eta_0<\frac{1}{2}, & \frac{1}{2}\left[1-\frac{4k^*}{(1-2\eta_0)}\right]<\delta<\frac{1}{2} \\ or & \eta_0>\frac{1}{2}, & \left[\delta^2-\left\{1-\frac{2k^*}{(1-2\eta_0)}\right\}\delta+\left\{\frac{1}{4}-\frac{k^*}{(1-2\eta_0)}\right\}\right]>0 \end{array}\right\} (3.12)$$

Thus the proposed estimator $d_{e(\eta,\delta)}$ is more efficient than usual unbiased estimator $\bar{y}_{sy}$ as long as the condition (3.12) is satisfied.

From (1.4) and (3.11) we have

$$MSE\left(\bar{y}_{Rsy}\right)-MSE\left(d_{e(\eta,\delta)}\right)=\theta\,\bar{Y}^2\rho_x^*C_x^2\left\{1-\frac{(1-2\eta_0)(1-2\delta)}{2}\right\}\left[1+\frac{(1-2\eta_0)(1-2\delta)}{2}-2k^*\right]$$

which is greater than zero if

$$\left\{1-\frac{(1-2\eta_0)(1-2\delta)}{2}\right\}\left\{1+\frac{(1-2\eta_0)(1-2\delta)}{2}-2k^*\right\}>0$$

i.e. if

$$\left.\begin{array}{ll} either & \frac{1}{2}\left[1-\frac{2}{(1-2\eta_0)}\right]<\delta<\frac{\left(3-2\eta_0-4k^*\right)}{2(1-2\eta_0)} \\ or & \frac{\left(3-2\eta_0-4k^*\right)}{2(1-2\eta_0)}<\delta<\frac{1}{2}\left[1-\frac{2}{(1-2\eta_0)}\right] \end{array}\right\} (3.13)$$

So the proposed estimator $d_{e(\eta,\delta)}$ will dominate over the ratio estimator $\bar{y}_{Rsy}$ as long as the condition (3.13) is satisfied.

From (1.5) and (3.11) we have

$$MSE(\bar{y}_{Psy})-MSE(d_{e(\eta,\delta)})=\theta\,\bar{Y}^2\rho_x^*C_x^2\left\{1-\frac{(1-2\eta_0)(1-2\delta)}{2}\right\}\left[1+\frac{(1-2\eta_0)(1-2\delta)}{2}+2k^*\right]$$

which is positive if

$$\left.\begin{array}{ll}\text{either} & \frac{1}{2}\left[1-\frac{2}{(1-2\eta_0)}\right]<\delta<\frac{(3-2\eta_0+4k^*)}{2(1-2\eta_0)}\\ \text{or} & \frac{(3-2\eta_0+4k^*)}{2(1-2\eta_0)}<\delta<\frac{1}{2}\left[1-\frac{2}{(1-2\eta_0)}\right]\end{array}\right\}\quad(3.14)$$

Thus it follows that the proposed estimator $d_{e(\eta,\delta)}$ is better than the product estimator $\bar{y}_{Psy}$ if the condition (3.14) holds good.

From (1.8) and (2.11) we have

$$MSE(\bar{y}_{Re\,sy})-MSE(d_{e(\eta,\delta)})=\theta\,\bar{Y}^2\rho_x^*C_x^2\,\{1-(1-2\eta_0)(1-2\delta)\}\left[1+(1-2\eta_0)(1-2\delta)-4k^*\right]$$

which is non- negative if

$$[1-(1-2\eta_0)(1-2\delta)]\left[1+(1-2\eta)(1-2\delta)-4k^*\right]>0$$

i.e. if

$$\left.\begin{array}{ll}\text{either} & \frac{\eta_0}{(2\eta_0-1)}<\delta<\frac{(\eta_0+2k^*)}{(2\eta_0-1)}\\ \text{or} & \frac{(\eta_0+2k^*)}{(2\eta_0-1)}<\delta<\frac{\eta_0}{(2\eta_0-1)}\end{array}\right\}\quad(3.15)$$

So the proposed estimator $d_{e(\eta,\delta)}$ is more efficient than the ratio-type exponential estimator $\bar{y}_{Resy}$ as long as the condition (3.15) is satisfied.

Further, from (1.9) and (3.11) we have

$$MSE(\bar{y}_{Pesy})-MSE(d_{e(\eta,\delta)})=\theta\,\bar{Y}^2\rho_x^*C_x^2\left\{1-\frac{(1-2\eta_0)(1-2\delta)}{2}\right\}\left[1-\frac{(1-2\eta_0)(1-2\delta)}{2}+2k^*\right]$$

which is greater than zero if

$$\left\{1-\frac{(1-2\eta_0)(1-2\delta)}{2}\right\}\left\{1-\frac{(1-2\eta_0)(1-2\delta)}{2}+2k^*\right\}>0$$

i.e. if

$$\left.\begin{array}{ll}\text{either} & \delta>\left[\frac{(2\eta_0+1)}{2(2\eta_0-1)}+\frac{2k^*}{(2\eta_0-1)}\right]\\ \text{or} & \delta<\frac{(2\eta_0+1)}{2(2\eta_0-1)}\end{array}\right\}\quad(3.16)$$

Thus the proposed estimator $d_{e(\eta,\delta)}$ will dominate over the product-type exponential estimator $\bar{y}_{Pesy}$ if the condition (3.16) is satisfied.

In similar fashion we can easily derive the conditions under which the envisaged estimator $d_{e(\eta,\delta)}$ is better than the estimators $\bar{y}_{sy}$, $\bar{y}_{Rsy}$, $\bar{y}_{Re\,sy}$, $\bar{y}_{Psy}$ and $\bar{y}_{Pesy}$.

4 Unbiased Asymptotically Optimum Estimator (*AOE*)

The bias of $d_{e(\eta,\delta)}$ at (2.3) would be zero if

$$\delta=\frac{1}{2} \text{ or } \delta=1.5-2\eta-2k^{*}+4\eta k^{*} \quad (4.1)$$

The proposed ratio-product-ratio-type exponential estimator $d_{e(\eta,\delta)}$, inserted with the values of η from (4.1), becomes an approximately unbiased estimator for the population mean $\bar{Y}$. In the three dimensional parameter space $(\eta,\delta,k^{*})\in R^{3}$, these unbiased estimators lie on a plane (in the case $\delta=\frac{1}{2}$).

From (2.6) and (4.1), the values of the parameters (η,δ) can be derived for the suggested estimator $d_{e(\eta,\delta)}$ becomes at least up to first order approximation an unbiased AOE. We derive a line with $\delta=\frac{1}{2}$ (recall that on this line the proposed estimator $d_{e(\eta,\delta)}$ always turns out to the systematic sample mean $\bar{y}_{sy}$)

$$\delta=\frac{1}{2},\ k^{*}=0 \quad (4.2)$$

or a "curve'' $\left(\eta^{*}(k^{*}),\delta^{*}(k^{*}),\delta^{*}(k^{*})\right)\in R^{3}$ in the parameter space

$$\eta^{*}(k^{*})=\frac{1}{2}\left\{1\pm\sqrt{\frac{k^{*}}{(2k^{*}-1)}}\right\},\qquad \delta^{*}(k^{*})=\frac{1}{2}\left\{1\pm\sqrt{k^{*}(2k^{*}-1)}\right\}. \qquad (4.3)$$

Putting the values of $\eta^{*}(k^{*})$ and $\delta^{*}(k^{*})$ from (4.3) in (2.1), the suggested estimator $d_{e(\eta,\delta)}$ takes the form:

$$\bar{y}^{*}_{sye}(k^{*})=d_{e(\eta^{*}(k^{*}),\,\delta^{*}(k^{*}))}$$

$$=\frac{\bar{y}_{sy}}{2}\left[\left\{1+\sqrt{\frac{k^{*}}{(2k^{*}-1)}}\right\}exp\left\{\frac{2\sqrt{k^{*}(2k^{*}-1)}(\bar{X}-\bar{x}_{sy})}{(\bar{X}+\bar{x}_{sy})}\right\}+\left\{1-\sqrt{\frac{k^{*}}{(2k^{*}-1)}}\right\}exp\left\{\frac{2\sqrt{k^{*}(2k^{*}-1)}(\bar{X}-\bar{x}_{sy})}{(\bar{X}+\bar{x}_{sy})}\right\}\right] \qquad (4.4)$$

It can be easily derived to the first degree of approximation that

$$B\left(\bar{y}^{*}_{sye}(k^{*})\right)=0, \qquad (4.5)$$

$$MSE\left(\bar{y}^{*}_{sye}(k^{*})\right)=\theta\rho^{*}_{y}S^{2}_{y}(1-\rho^{2}). \qquad (4.6)$$

Thus the estimator $\bar{y}^{*}_{sye}(k^{*})$ is an unbiased AOE to the first degree of approximation,

Putting (2.6) in (2.3) we obtained the bias of an AOE to the first degree of approximation,

$$B\left(d_{e(\eta,\,\delta)}\right)=\frac{\theta\bar{Y}\rho^{*}_{x}C^{2}_{x}}{8}\left[4k^{*}(1-2k^{*})+(1-2\delta)^{2}\right] \quad (4.7)$$

It follows from (2.6) and (4.7) that the bias can only be made zero if

$k^* \le 0$ or $k^* \ge \frac{1}{2}$.

Otherwise, there is always a non-negative contribution coming from the term $4k^*\left(1-2k^*\right)$ that does not vanish regardless the value of $\delta \in R$. For $\delta = \frac{1}{2}$ in (4.7) , we get least possible bias in $d_{e(\eta,\delta)}$.

Remark 4.1- The optimum value of $(1-2\eta)(1-2\delta)$ depends upon the value of k^* which can be obtained either from the past data or a pilot survey or experienced gathered in due course of time, for instance, see Reddy (1978) and Srivenkatramna and Tracy (1981).

5 Empirical Study

To examine the merits of the proposed estimator over the other existing estimators, we have considered a natural population data earlier used by Tailor et al (2013) and Khan and Singh(2015).
The parametric values available are:

$\bar{X} = 44.47, S_x^2 = 149.55, C_x = 0.28, S_{xy} = 538.57, \bar{Y} = 80, S_y^2 = 2000, C_y = 0.56, S_{yz} = -902.86,$

$\bar{Z} = 48.40, S_z^2 = 427.83, C_z = 0.43, S_{xz} = -241.06, \rho_{xy} = 0.9848, \rho_{yz} = -0.9760, \rho_{xz} = -0.9530$

$\rho_x = 0.707, \rho_y = 0.6652, \rho_z = 0.5487, N = 15, n = 3.$

For given value of δ, we have computed the optimum value of η space using the following formula:

$$\eta_{opt} = \frac{1}{2}\left[1 - \frac{2k^*}{(1-2\delta)}\right] \quad (5.1)$$

For given value of η, we have computed the optimum value of δ using the following formula:

$$\delta_{opt} = \frac{1}{2}\left[1 - \frac{2k^*}{(1-2\eta)}\right] \quad (5.2)$$

For the given data set we have computed the optimum values of $(\eta_{opt}, \delta_{opt})$ for the given values of (δ, η) respectively and findings are shown in Tables 5.1 and 5.2 .

Table 5.1: Optimum values of δ_{opt} for given values of η .

η	δ_{opt}
-3	0.223553
-2.75	0.202287
-2.5	0.177478
-2.25	0.148158
-2	0.112974
-1.75	0.069971
-1.5	0.016217
-1.25	-0.05289
-1	-0.14504
-0.75	-0.27405
-0.5	-0.46757
-0.25	-0.79009

0	-1.43513
0.25	-3.37026
0.75	4.370264
1	2.435132
1.25	1.790088
1.5	1.467566
1.75	1.274053
2	1.145044
2.25	1.052895
2.5	0.983783
2.75	0.930029
3	0.887026

It is observed from Table 5.1 that the optimum value of δ $\left(i.e.\ \delta_{opt}\right)$ increases as the value of η increases.

Table 5.2: Optimum values of η_{opt} for given values of δ

δ	η_{opt}
-3	0.223553
-2.75	0.202287
-2.5	0.177478
-2.25	0.148158
-2	0.112974
-1.75	0.069971
-1.5	0.016217
-1.25	-0.05289
-1	-0.14504
-0.75	-0.27405
-0.5	-0.46757
-0.25	-0.79009
0	-1.43513
0.25	-3.37026
0.75	4.370264
1	2.435132
1.25	1.790088
1.5	1.467566
1.75	1.274053
2	1.145044
2.25	1.052895
2.5	0.983783
2.75	0.930029
3	0.887026

It is observed from Table 5.2 that the optimum value of η $\left(i.e.\ \eta_{opt}\right)$ increases as the value of δ increases.

We have further computed the percent relative efficiency (PRE) of the proposed estimator $d_{e(\eta,\delta)}$ with respect to $\bar{y}_{sy}$, $\bar{y}_{Rsy}$ and $\bar{y}_{Resy}$ using the following formula:

$$PRE\left(d_{e(\eta,\delta)},\bar{y}_{sy}\right)=\frac{\rho_y^* C_y^2}{\left[\rho_y^* C_y^2+\frac{\rho_x^* C_x^2}{4}\left\{(1-2\eta)^2(1-2\delta)^2-4k^*(1-2\eta)(1-2\delta)\right\}\right]}\times 100 \qquad (5.3)$$

$$PRE\left(d_{e(\eta,\delta)},\bar{y}_{Rsy}\right)=\frac{\left[\rho_y^* C_y^2+\rho_x^* C_x^2\left(1-4k^*\right)\right]}{\left[\rho_y^* C_y^2+\frac{\rho_x^* C_x^2}{4}\left\{(1-2\eta)^2(1-2\delta)^2-4k^*(1-2\eta)(1-2\delta)\right\}\right]}\times 100 \qquad (5.4)$$

$$PRE\left(d_{e(\eta,\delta)},\bar{y}_{Re\,sy}\right)=\frac{\left[\rho_y^* C_y^2+\frac{\rho_x^* C_x^2}{4}\left(1-4k^*\right)\right]}{\left[\rho_y^* C_y^2+\frac{\rho_x^* C_x^2}{4}\left\{(1-2\eta)^2(1-2\delta)^2-4k^*(1-2\eta)(1-2\delta)\right\}\right]}\times 100 \qquad (5.5)$$

Findings are shown in Table 5.3 .

Table 5.3: PREs of the suggested class of estimators $d_{e(\eta,\delta)}$ with respect to the systematic sample mean $\bar{y}_{sy}$, ratio estimator $\bar{y}_{Rsy}$ in systematic sampling and ratio-type exponential estimator $\bar{y}_{Re\,sy}$ in systematic sampling.

$\left(d_{e(\eta,\delta)}\right)$	$PRE\left(d_{e(\eta,\delta)},\bar{y}_{sy}\right)$	$PRE\left(d_{e(\eta,\delta)},\bar{y}_{Rsy}\right)$	$PRE\left(d_{e(\eta,\delta)},\bar{y}_{Re\,sy}\right)$
(-3,0.35)	428.958	110.110	241.763
(-3, 0.30)	957.931	245.894	539.894
(-3, 0.25)	2557.041	656.373	1441.159
(-3,0.223553)	3307.767	849.078	1864.271
(-3,0.20)	2683.049	688.718	1512.177
(-3, 0.15)	1011.310	259.596	569.979
(-3,0.10)	446.549	114.626	251.677
(-2, 0.25)	389.571	100.000	219.564
(-2, 0.30)	658.791	169.107	371.298
(-2, 0.112974)	3307.767	849.078	1864.271
(-2, 0.15)	2557.041	656.373	1441.159
(-2, 0.10)	3192.686	819.538	1799.411
(-2, 0.05)	1788.700	459.146	1008.119
(-2.75, 0.30)	742.403	190.569	418.421
(-2.75,0.25)	1813.568	465.529	1022.135
(-2.75,0.202287)	3307.767	849.078	1864.271
(-2.75,0.15)	1662.637	426.786	937.069
(-2.75,0.10)	691.043	177.385	389.475
(-2.5, 0.30)	587.602	150.833	331.175
(-2.5, 0.25)	1261.593	323.841	711.039
(-2.5,0.177478)	3307.767	849.078	1864.271
(-2.5, 0.15)	2683.049	688.718	1512.177
(-2.5,0.10)	1160.152	297.802	653.867
(-2.5, 0.05)	550.253	141.246	310.125
(-2.25, 0.30)	474.275	121.743	267.303
(-2.25,0.25)	897.001	230.253	505.553

(-2.25,0.148158)	3307.767	849.078	1864.271

Table 5.3 continued….

$\left(d_{e(\eta,\delta)}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{sy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Rsy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Re\,sy}\right)$
(-2.25,0.15)	3304.860	848.332	1862.633
(-2.25,0.10)	2066.122	530.358	1164.475
(-2.25,0.05)	945.982	242.826	533.160
(-1.75, 0.25)	499.510	128.220	281.526
(-1.75, 0.15)	1566.937	402.221	883.132
(-1.75,0.069971)	3307.767	849.078	1864.271
(-1.75,0.10)	2860.347	734.229	1612.103
(-1.75,0.05)	3093.735	794.138	1743.642
(-1.5, 0.25)	389.571	100.000	219.564
(-1.5, 0.15)	957.931	245.894	539.894
(-1.5, 0.016217)	3307.767	849.078	1864.271
(-1.5, 0.10)	1685.844	432.743	950.148
(-1.5, 0.05)	2860.347	734.229	1612.103
(-1.5, 0.00)	3192.686	819.538	1799.411
(-1.25,0)	2557.041	656.373	1441.159
(-1.25,0.10)	957.931	245.894	539.894
(-1.25,0.15)	621.789	159.608	350.443
(-1.25,-0.05289)	3307.767	849.078	1864.271
(-1.25,0.05)	1566.937	402.221	883.132
(-1.25,-0.10)	2683.049	688.718	1512.177
(-1.25,-0.15)	1662.637	426.786	937.069
(-1.25,-0.25)	651.552	167.248	367.217
(-1, 0)	1261.593	323.841	711.039
(-1, 0.10)	587.602	150.833	331.175
(-1, 0.15)	428.958	110.110	241.763
(-1, -0.14504)	3307.767	849.078	1864.271
(-1, 0.05)	841.062	215.894	474.026
(-1, -0.10)	2860.347	734.229	1612.103
(-1, -0.15)	3301.515	847.474	1860.747
(-1, -0.25)	1788.700	459.146	1008.119
(-0.75,0)	658.791	169.107	371.298
(-0.75,0.10)	389.571	100.000	219.564
(-0.75,-0.27405)	3307.767	849.078	1864.271
(-0.75,0.05)	499.510	128.220	281.526
(-0.75,-0.10)	1261.593	323.841	711.039
(-0.75,-0.15)	1813.568	465.529	1022.135
(-0.75,-0.15)	3208.392	823.570	1808.263
(-0.5, 0.00)	389.571	100.000	219.564
(-0.5, -0.75)	886.034	227.438	499.372
(-0.5, -0.50)	3192.686	819.538	1799.411
(-0.5, -0.45)	3273.161	840.195	1844.767

Table 5.3 continued….

$\left(d_{e(\eta,\delta)}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{sy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Rsy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Re\,sy}\right)$
(-0.5, -0.46757)	3307.767	849.078	1864.271
(-0.5, -0.35)	2244.696	576.196	1265.120
(-0.5, -0.25)	1261.593	323.841	711.039

(-0.5, -0.15)	742.403	190.569	418.421
(-0.25, -0.5)	1261.593	323.841	711.039
(-0.25, -1.0)	1788.700	459.146	1008.119
(-0.25, -0.79009)	3307.767	849.078	1864.271
(-0.25, -1.25)	651.552	167.248	367.217
(-0.25, -1.35)	469.698	120.568	264.724
(0.00, -0.50)	389.571	100.000	219.564
(0.00, -1.00)	1261.593	323.841	711.039
(0.00, -1.25)	2557.041	656.373	1441.159
(0.00, -1.43513)	3307.767	849.078	1864.271
(0.00, -1.50)	3192.686	819.538	1799.411
(0.00, -1.75)	1788.700	459.146	1008.119
(0.00, -2.00)	886.034	227.438	499.372
(0.00, -2.25)	494.585	126.956	278.750
(0.25, -1.50)	389.571	100.000	219.564
(0.25, -1.75)	499.510	128.220	281.526
(0.25, -3.37026)	3307.767	849.078	1864.271
(0.25, -2.00)	658.791	169.107	371.298
(0.25, -2.25)	897.001	230.253	505.553
(0.75, 2.50)	389.571	100.000	219.564
(0.75, 2.75)	499.510	128.220	281.526
(0.75, 3.00)	658.791	169.107	371.298
(0.75, 3.15)	789.667	202.701	445.059
(0.75, 3.25)	897.001	230.253	505.553
(0.75, 3.50)	1261.593	323.841	711.039
(0.75, 3.75)	1813.568	465.529	1022.135
(0.75, 4.00)	2557.041	656.373	1441.159
(0.75, 4.15)	2996.442	769.164	1688.807
(0.75, 4.25)	3208.392	823.570	1808.263
(0.75, 4.370264)	3307.767	849.078	1864.271
(0.75, 4.50)	3192.686	819.538	1799.411
(0.75, 4.75)	2527.316	648.743	1424.405
(0.75, 5.00)	1788.700	459.146	1008.119
(0.75, 5.15)	1436.898	368.841	809.842
(0.75, 5.25)	1244.739	319.515	701.540
(0.75, 5.50)	886.034	227.438	499.372
(0.75, 5.75)	651.552	167.248	367.217
(0.75, 6.00)	494.585	126.956	278.750

Table 5.3 continued…

$\left(d_{e(\eta,\delta)}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{sy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Rsy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Re\,sy}\right)$
(0.75, 6.15)	424.989	109.092	239.526
(1.00, 1.50)	389.571	100.000	219.564
(1.00, 1.75)	658.791	169.107	371.298
(1.00, 2.00)	1261.593	323.841	711.039
(1.00, 2.15)	1949.848	500.511	1098.942
(1.00, 2.25)	2557.041	656.373	1441.159
(1.00, 2.50)	3192.686	819.538	1799.411
(1.00, 2.435132)	3307.767	849.078	1864.271
(1.00, 2.75)	1788.700	459.146	1008.119
(1.00, 3.00)	886.034	227.438	499.372
(1.00, 3.15)	615.104	157.892	346.675
(1.00, 3.25)	494.585	126.956	278.750
(1.00, 3.30)	446.549	114.626	251.677

(1.00, 3.35)	404.884	103.931	228.194
(1.25, 1.50)	1261.593	323.841	711.039
(1.25, 1.75)	3208.392	823.570	1808.263
(1.25, 2.00)	1788.700	459.146	1008.119
(1.25, 1.790088)	3307.767	849.078	1864.271
(1.25, 2.15)	945.982	242.826	533.160
(1.25, 2.25)	651.552	167.248	367.217
(1.25, 2.30)	550.253	141.246	310.125
(1.25, 2.40)	404.884	103.931	228.194
(1.5, 1.00)	389.571	100.000	219.564
(1.5, 1.15)	742.403	190.569	418.421
(1.5, 1.25)	1261.593	323.841	711.039
(1.5, 1.467566)	3307.767	849.078	1864.271
(1.5, 1.50)	3192.686	819.538	1799.411
(1.5, 1.75)	886.034	227.438	499.372
(1.75, 1.00)	658.791	169.107	371.298
(1.75, 1.15)	1813.568	465.529	1022.135
(1.75, 1.25)	3208.392	823.570	1808.263
(1.75, 1.30)	3192.686	819.538	1799.411
(1.75, 1.274053)	3307.767	849.078	1864.271
(1.75, 1.35)	2527.316	648.743	1424.405
(1.75, 1.45)	1244.739	319.515	701.540
(1.75, 1.50)	886.034	227.438	499.372
(2.00, 1.00)	1261.593	323.841	711.039
(2.00, 1.15)	3301.515	847.474	1860.747
(2.00, 1.25)	1788.700	459.146	1008.119
(2.00, 1.30)	1160.152	297.802	653.867
(2.00, 1.145044)	3307.767	849.078	1864.271

Table 5.3 continued…

$\left(d_{e(\eta,\delta)}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{sy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Rsy}\right)$	$PRE\left(d_{e(\eta,\delta)}, \bar{y}_{Re\,sy}\right)$
(2.00, 1.35)	780.407	200.325	439.841
(2.00, 1.40)	550.253	141.246	310.125
(2.00, 1.45)	404.884	103.931	228.194
(2.25, 1.00)	2557.041	656.373	1441.159
(2.25, 1.15)	1662.637	426.786	937.069
(2.25, 1.052895)	3307.767	849.078	1864.271
(2.25, 1.25)	651.552	167.248	367.217
(2.25, 1.30)	446.549	114.626	251.677
(2.5, 0.75)	389.571	100.000	219.564
(2.5, 1.00)	3192.686	819.538	1799.411
(2.5, 0.983783)	3307.767	849.078	1864.271
(2.5, 1.15)	691.043	177.385	389.475
(2.5, 1.20)	446.549	114.626	251.677
(2.75, 0.75)	499.510	128.220	281.526
(2.75, 0.930029)	3307.767	849.078	1864.271
(2.75, 1.00)	1788.700	459.146	1008.119
(3.00, 0.75)	658.791	169.107	371.298
(3.00, 0.887026)	3307.767	849.078	1864.271
(3.00, 1.00)	886.034	227.438	499.372

Table 5.3 exhibits that the envisaged estimator $d_{e(\eta,\delta)}$ has the largest efficiency at the optimum value of δ for given value of η. It is also observed from Table 5.3 that there is considerable gain in efficiency by using the proposed estimator

$d_{e(\eta,\delta)}$ even if δ departs from its exact optimum value for given η. So there is enough scope of selecting the values of (η, δ) for getting estimators more efficient than the estimators $\bar{y}_{sy}$, $\bar{y}_{Rsy}$ and $\bar{y}_{Resy}$. Thus our recommendation is in the favor of the proposed estimator $d_{e(\eta,\delta)}$ for its use in practice.

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