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Modeling with Fractional Laplace Transform by Difference Operator

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Abstract: In this research article, we develop and analyse the discrete fractional Laplace transform and obtain the properties of discrete fractional Laplace transform. Also, the discrete fractional Laplace transforms of certain functions like polynomial factorial, exponential functions, trigonometric functions, etc are derived. Moreover, we equate two types of solutions namely closed and summation forms and verify its efficacy with numerical results by MATLAB.

Keywords: Difference operator, Laplace transform, gamma function, fractional operator.

1 Introduction

Fractional calculus which deals with integration and differentiation with respect to arbitrary order has gained lot of interest during the last three decades. Since L'Hopital's letter to Leibniz in 1695 has raised a relevant question namely, "What does $\frac{\partial^m f(x)}{\partial x^m}$ mean if $m = \frac{1}{2}$?". This is considered to be the initiation of the idea of fractional calculus (Diethelm, 2010, Hilfer, 2000, Lazarevic, et al., 2014, Millar & Ross, 1993, Kumar & Saxena, 2016). These debates ended with several types fractional operators in which the fractionalizing process mainly depends on iterating the integral or the derivative. For example the Riemann-Liouville derivatives were defined after setting the Riemann-Liouville integrals through the Leibniz-Cauchy formula. For the applications one can refer, [1–8]. In the last three years some new types of fractional operators with nonsingular kernels have appeared where the fractionalizing manner depends on a limiting process using delta dirac functions. For such fractional operators and their discrete versions we may refer to [9–14].

In 1989, Miller and Ross [15] initiated the process to develop the theory for fractional finite differences. Further developments took place in 2007 and 2012 when the authors [16, 17] put forth several results and a discrete transform method for fractional order difference equation. In 2009, the authors [18,19] have been founded the significance of Laplace transform technique. One can refer [20–25] and [26]- [37].

In this paper, we continue to build on their work [16, 17] to develop properties of fractional h-differences, define new type fractional Laplace and extorial transforms, and develop a method to apply the above transforms to polynomial

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factorial, geometric and logarithmic functions. Therefore, we review some notations and from the fractional calculus in Section 2 and in Section 3, we present some relevant results on fractional $\ell-$difference operators and its properties. This results helps to solve problems on fractional Laplace transform in Section 4 and Section 5. Numerical examples are provided to illustrate of our findings.Conclusions are depicted in Section 6.

2 Preliminaries

Definition 1. *The ℓ-difference operator for the real valued function $\psi(\lambda)$, is defined by*

$$\Delta_\ell \psi(\lambda) = \frac{\psi(\lambda+\ell)-\psi(\lambda)}{\ell}, \tag{1}$$

and the infinite sum is defined by

$$\Delta_\ell^{-1}\psi(\lambda)|_\theta^\infty = \Delta_\ell^{-1}\psi(\infty) - \Delta_\ell^{-1}\psi(\theta) = \ell\sum_{i=0}^{\infty}\psi(\theta+i\ell). \tag{2}$$

Note that when $\Delta_\ell^{-1}\psi(\lambda)$ at ∞ is 0, it is obvious to denote ${}_\infty\Delta_\ell^{-1}\psi(\theta) = -\ell\sum\limits_{i=0}^{\infty}\psi(\theta+i\ell)$ and hence ${}_\infty\Delta_\ell^{-1}\psi(\lambda) = -\ell\sum\limits_{i=0}^{\infty}\psi(\lambda+i\ell)$ is obtained by $\Delta_\ell({}_\infty\Delta_\ell^{-1}\psi(\lambda)) = \psi(\lambda)$.

Example 1. Since $\Delta_\ell\frac{1}{2^\lambda} = \frac{1}{\ell}\left[\frac{1}{2^{\lambda+\ell}} - \frac{1}{2^\lambda}\right] = \frac{1}{\ell 2^\lambda}\left(\frac{1}{2^\ell}-1\right)$, it is clear that $\Delta_\ell^{-1}\frac{1}{2^\lambda} = \left(\frac{2^\ell}{1-2^\ell}\right)\frac{\ell}{2^\lambda}$ and $\Delta_\ell^{-1}\frac{1}{2^\infty} = 0$. By considering $\psi(\lambda) = \frac{1}{2^\lambda}$ and $\theta = 0$ in (2) we get

$$\Delta_\ell^{-1}\frac{1}{2^\lambda}\Big|_0^\infty = \ell\sum_{i=0}^{\infty}\frac{1}{2^{i\ell}} \Rightarrow \Delta_\ell^{-1}\frac{1}{2^\infty} - \Delta_\ell^{-1}\frac{1}{2^0} = \ell\sum_{i=0}^{\infty}\frac{1}{2^{i\ell}} \Rightarrow \frac{\ell 2^\ell}{1-2^\ell} = \ell\sum_{i=0}^{\infty}\frac{1}{2^{i\ell}}. \tag{3}$$

Here one can apply any real $h>0$.

Even though the Example 1 is simple, arriving the relations (1) and (2) for fractional order difference operator ${}_\infty\Delta_\ell^{-\nu}$, ν is fraction, is complex but we have tried and achieved it. When $\ell \to 1$, operator Δ_ℓ^ν coincides with the operators used in [31,33,35]. As we are trying to find fractional order difference for polynomial factorial we give the following definitions and lemmas.

Definition 2. *[32] The polynomial factorial function, for $\ell>0, \nu\in R$ is defined by*

$$\lambda_\ell^{[\nu]} = \ell^\nu\frac{\Gamma(\frac{\lambda}{\ell}+\nu)}{\Gamma(\frac{\lambda}{\ell})}, \tag{4}$$

where $\lambda_\ell^{[0]} = 1$ and $\frac{\lambda}{\ell}+\nu, \frac{\lambda}{\ell}, \notin \{0,-1,-2,-3,...\}$.

Remark. [34] The Euler Gamma function of an infinite product as defined by
$\Gamma(\lambda) = \frac{1}{\lambda}\prod\limits_{j=1}^{\infty}\frac{(1+\frac{1}{j})^\lambda}{(1+\frac{\lambda}{j})}, j\notin\{0,-1,-2,-3,...\}$.

The properties of the polynomial factorials are given in the following lemma.

Lemma 1. *The results are satisfied the polynomial factorials;*
(i) $\ell(\frac{\lambda}{\ell}+\nu)\lambda_\ell^{[\nu]} = \lambda_\ell^{[\nu+1]}$, *(ii)* $\ell(\frac{\lambda}{\ell}-\nu)\lambda_\ell^{(\nu)} = \lambda_\ell^{(\nu+1)}$,
(iii) $\lambda_\ell^{[m+n]} = \lambda_\ell^{[m]}(\lambda+m\ell)_\ell^{[n]}$,
(iv) $\Delta_\ell\lambda_\ell^{[\nu]} = \nu(\lambda+\ell)_\ell^{[\nu-1]}$ *and (v)* $\Delta_\ell\lambda_\ell^{(\nu)} = \nu\lambda_\ell^{(\nu-1)}$.

Proof. The proof of (i) and (ii) are simple, hence omitted.

To prove (iii), taking $\nu = m+n$ in (4) gives $\lambda_\ell^{[m+n]} = \ell^{m+n}\dfrac{\Gamma(\frac{\lambda}{\ell}+m+n)}{\Gamma(\frac{\lambda}{\ell})}$

$$= \ell^m \ell^n \left[\frac{(\frac{\lambda}{\ell}+m+n-1)(\frac{\lambda}{\ell}+m+n-2)\cdots(\frac{\lambda}{\ell}+m)\Gamma(\frac{\lambda}{\ell}+m)}{\Gamma(\frac{\lambda}{\ell})}\right]$$

$$= \lambda_\ell^{[m]} \ell^n \left[\frac{(\frac{\lambda+m\ell}{\ell}+n-1)(\frac{\lambda+m\ell}{\ell}+n-2)\cdots(\frac{\lambda+m\ell}{\ell}+m)\Gamma(\frac{\lambda+m\ell}{\ell})}{\Gamma(\frac{\lambda+m\ell}{\ell})}\right],$$

which yields $\quad \lambda_\ell^{[m+n]} = \lambda_\ell^{[m]} \ell^n \dfrac{\Gamma(\frac{\lambda+m\ell}{\ell}+n)}{\Gamma(\frac{\lambda+m\ell}{\ell})} = \lambda_\ell^{[m]}(\lambda+m\ell)_\ell^{[n]}.$

The results of (iv) and (v) are derived by taking ${}_a\Delta_\ell$ on polynomial factorials.

3 Fractional ℓ-difference operator and its properties

This section presents infinite sum representations of functions in $h^\nu(\infty)$-space. $h^\nu(\infty)$-space is the collection of all real valued functions $\psi(\lambda)$ such that $\sum_{r=0}^{\infty}(-1)^r \dfrac{\ell^{-\nu}\Gamma(\nu+1)}{\Gamma(r+1)\Gamma(\nu-r+1)}\psi(\lambda+r\ell)$ is finite. In the continuous case, we have definite integrals as well as indefinite integrals. As an analogous to continuous case, we define indefinite fractional difference using Gamma functions.

Definition 3. *For $\nu > 0$ and the real valued function ψ, the fractional ℓ-difference operator Δ_ℓ^ν is defined by*

$$\Delta_\ell^\nu \psi(\lambda) = \ell^{-\nu}\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\nu+1)}{\Gamma(i+1)\Gamma(\nu-i+1)}\psi(\lambda+i\ell). \tag{5}$$

The following example is result of (5).

Example 2. The following identities are arrived from (5):

(i)
$\Delta_\ell^{\frac{1}{2}}\psi(\lambda) = \ell^{\frac{1}{2}}\left[\psi(\lambda) - \frac{1}{2}\psi(\lambda+\ell) - \frac{1}{8}\psi(\lambda+2\ell) - \frac{1}{16}\psi(\lambda+3\ell) - \frac{5}{128}\psi(\lambda+4\ell) - \frac{7}{256}\psi(\lambda+5\ell) - \frac{21}{1024}\psi(\lambda+6\ell) - \ldots\right]$

(ii) $\Delta_\ell^{-\frac{1}{2}}\psi(\lambda) =$
$\ell^{-\frac{1}{2}}\left[\psi(\lambda) + \frac{1}{2}\psi(\lambda+\ell) + \frac{3}{8}\psi(\lambda+2\ell) + \frac{5}{16}\psi(\lambda+3\ell) + \frac{35}{128}\psi(\lambda+4\ell) + \frac{63}{256}\psi(\lambda+5\ell) + \frac{231}{1024}\psi(\lambda+6\ell) + \ldots\right]$.

We can easily verified that $\Delta_\ell^{\frac{1}{2}}({}_\infty\Delta_\ell^{-\frac{1}{2}}\psi(\lambda)) = \psi(\lambda)$.

Remark.(i) If $\nu = m$ is a positive integer, then the infinite sum (5) becomes finite sum,
(i.e) $\Delta_\ell^m \psi(\lambda) = \ell^{-m}\sum_{i=0}^{m}(-1)^i \dfrac{\Gamma(m+1)}{\Gamma(i+1)\Gamma(m-i+1)}\psi(\lambda+i\ell)$, since the remaining terms are zero.
(ii) Taking $\psi(\lambda) = \psi_1(\lambda) + \psi_2(\lambda)$ in (5) gives the linear property of Δ_ℓ^ν.

Theorem 1. *For two fractions ν and $\mu > 0$, the following identities hold*
(i) $\Delta_\ell^\nu(\Delta_\ell^\mu \psi(\lambda)) = \Delta_\ell^{\nu+\mu}\psi(\lambda)$ and (ii) $\Delta_\ell^\nu(\Delta_\ell^{-\nu}\psi(\lambda)) = \psi(\lambda)$.

Proof. (i) From the Definition 3, we have

$$\Delta_\ell^\nu(\Delta_\ell^\mu \psi(\lambda)) = \Delta_\ell^\nu\left(\sum_{i=0}^{\infty}(-1)^i \frac{\ell^{-\mu}\Gamma(\mu+1)}{\Gamma(i+1)\Gamma(\mu-i+1)}\psi(\lambda+i\ell)\right)$$

$$= \Delta_\ell^\nu\left[\psi(\lambda) - \mu\psi(\lambda+\ell) + \frac{\mu(\mu-1)}{2!}\psi(\lambda+2\ell) - \frac{\mu(\mu-1)(\mu-2)}{3!}\psi(\lambda+3\ell) + \cdots\right]$$

$$= \left[\ell^{-\nu}\sum_{i=0}^{\infty}(-1)^i \tfrac{\Gamma(\nu+1)}{\Gamma(i+1)\Gamma(\nu-i+1)}\psi(\lambda+i\ell) - \mu\sum_{i=0}^{\infty}(-1)^i \tfrac{\ell^{-\nu}\Gamma(\nu+1)}{\Gamma(i+1)\Gamma(\nu-i+1)}\psi(\lambda+(i+1)\ell)\right.$$
$$\left. + \frac{\mu(\mu-1)}{2!}\sum_{i=0}^{\infty}(-1)^i \tfrac{\ell^{-\nu}\Gamma(\nu+1)}{\Gamma(i+1)\Gamma(\nu-i+1)}\psi(\lambda+(i+2)\ell) + \cdots\right].$$

By expanding the summation and rearranging the terms, we get

$$\Delta_\ell^\nu(\Delta_\ell^\mu\psi(\lambda) = \Big[\psi(\lambda)-(\nu+\mu)\psi(\lambda+\ell)+\frac{(\nu+\mu)(\nu+\mu-1)}{2!}\psi(\lambda+2\ell)$$
$$-\frac{(\nu+\mu)(\nu+\mu-1)(\nu+\mu-2)}{3!}\psi(\lambda+2=3\ell)+\cdots\Big]$$
$$=\ell^{-\mu-\nu}\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(\nu+\mu)}{\Gamma(i+1)\Gamma(\nu+\mu-i+1)}\psi(\lambda+i\ell)=\Delta_\ell^{\nu+\mu}\psi(\lambda).$$

(ii) The proof follows by taking $\mu=-\nu$ in (i), since $\Delta_\ell^{-\nu}$ is inverse of Δ_ℓ^ν for $\nu>0$.

Definition 4. *For $\nu>0$ and the real valued function $\psi(\lambda)$. The inverse fractional ℓ-difference operator is defined by*

$$\Delta_\ell^{-\nu}\psi(\lambda)\Big|_\theta^\infty=-\ell^\nu\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\psi(\theta+i\ell). \quad (6)$$

Remark. $\Delta_\ell^{-\nu}\psi(\infty)-\Delta_\ell^{-\nu}u(\theta)=-\sum_{i=0}^{\infty}(-1)^i\frac{\ell^\nu\Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\psi(\theta+i\ell)$ yields $\Delta_\ell^{-\nu}\psi(\theta)=\sum_{i=0}^{\infty}(-1)^i\frac{\ell^\nu\Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\psi(\theta+i\ell)$ and it is denoted as ${}_\infty\Delta_\ell^{-\nu}\psi(\theta)$.

Definition 5. *The Caputo ℓ-difference operator for $0<\nu<1$, is defined by*

$${}_a\Delta_\ell^\nu\psi(\lambda)=\Delta_\ell({}_a\Delta_\ell^{-(1-\nu)}\psi(\lambda)). \quad (7)$$

Theorem 2. *Let $\nu>0$ be a fraction, $\psi\in[0,\infty)$ and $c\neq 0$. Then, we have*

$${}_a\Delta_\ell^{-\nu}\frac{1}{c^\lambda}\Big|_a^\infty=\frac{\ell^\nu}{c^\lambda}\left(1-\frac{1}{c^\ell}\right)^{-\nu}\Big|_a^\infty=-\sum_{i=0}^{\infty}(-1)^i\frac{\ell^\nu\Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\frac{1}{c^{a+i\ell}}. \quad (8)$$

Proof. The expression $\Delta_\ell^\nu\frac{1}{c^\lambda}=\sum_{i=0}^{\infty}(-1)^i\frac{\ell^{-\nu}\Gamma(\nu+1)}{\Gamma(i+1)\Gamma(\nu-i+1)}\frac{1}{c^{\lambda+\ell}}$ is obtained by taking $\psi(\lambda)=\frac{1}{c^\lambda}$ in (5). The binomial expansion for rational index gives ${}_a\Delta_\ell^\nu\frac{1}{c^\lambda}=\frac{\ell^{-\nu}}{c^\lambda}\left(1-\frac{1}{c^\ell}\right)^\nu$.
Now (8) follows by taking ${}_a\Delta_\ell^{-\nu}$ on both sides, linear property and replacing ψ by a.

Example 3. For the particular values of $c=2$, $h=3, a=3$ and $\nu=0.5$ in (8) gives

$${}_3\Delta_3^{-0.5}\frac{1}{2^\lambda}\Big|_3^\infty=\frac{3^{0.5}}{2^\lambda}\left(1-\frac{1}{2^3}\right)^{-0.5}\Big|_3^\infty=\sum_{i=0}^{\infty}(-1)^i\frac{3^{0.5}\Gamma(0.5)}{\Gamma(i+1)\Gamma(0.5-i)}\frac{1}{2^{3+3i}}.$$

Which are verified by MATLAB with the coding as given below:

$3.\wedge(0.5)(1./8).\times((7./8).\wedge(-0.5))=symsum(((-1).\wedge i.\times 3.\wedge(0.5).\times gamma(0.5))./(gamma(i+1).\times gamma(0.5-i).\times(2.\wedge(3+3.\times i))),i,0,inf)$.

Corollary 1. *Let ψ be the real valued function and $\nu,\ell>0$, then we have*

$${}_a\Delta_\ell^{-\nu}\frac{1}{e^{s\lambda}}\Big|_a^\infty=\frac{\ell^\nu}{e^{s\lambda}}\left(1-\frac{1}{e^{s\ell}}\right)^{-\nu}\Big|_a^\infty=-\sum_{i=0}^{\infty}(-1)^i\frac{\ell^\nu\Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\frac{1}{e^{sa+i\ell}}. \quad (9)$$

Proof. Replacing $\frac{1}{c^\lambda}$ by $\frac{1}{e^{s\lambda}}$ in Theorem 2, we get the proof.

4 Fractional Laplace transform by ℓ-difference operator

Here, we define and develop a new type fractional Laplace transform and it properties. The fractional Laplace transform of certain functions are derived and presented with numerical examples.

Definition 6. *For the real valued function $\psi(\lambda)$. Then the Fractional Laplace Transform(FLT) is defined as*

$$L_\ell^\nu[\psi(\lambda)] = \Delta_\ell^{-\nu}\psi(\lambda)e^{-s\lambda}\Big|_0^\infty = -\sum_{i=0}^{\infty}(-1)^i \frac{\ell^\nu \Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\psi(i\ell)e^{-si\ell}. \tag{10}$$

Theorem 3. *Let $\nu,\ell > 0$ and $\lambda \in [0,\infty)$, we have*

$$_\theta\Delta_\ell^{-\nu}\lambda_\ell^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\lambda_\ell^{(\mu+\nu)}. \tag{11}$$

Proof. Since for $\nu > 0$, the proof follows by proving $\Delta_\ell[f(\lambda)] = \Delta_\ell[g(\lambda)]$,
where $g(\lambda) = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\lambda_\ell^{(\mu+\nu)}$ and $f(\lambda) = {}_\theta\Delta_\ell^{-\nu}\lambda_\ell^{(\mu)}$.
From the Lemma 1, we get

$$\Delta g(\lambda) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\Delta\lambda_\ell^{(\mu+\nu)} = \frac{\Gamma(\mu+1)(\mu+\nu)\ell}{\Gamma(\mu+\nu+1)}\lambda_\ell^{(\mu+\nu-1)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu)}\lambda_\ell^{(\mu+\nu-1)}. \tag{12}$$

By repeating from Lemma 1, which gives

$$\Delta_\ell[f(\lambda)] = \Delta_\ell({}_\theta\Delta_\ell^{-\nu})\lambda_\ell^{(\mu)} = {}_\theta\Delta_\ell^{-(\nu-1)}\lambda_\ell^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu)}\lambda_\ell^{(\mu+\nu-1)}. \tag{13}$$

Equating (12) and (13), we get the proof.

Theorem 4. *Let $\nu,\ell > 0$, and $\lambda \in [0,\infty)$, then we have*

$$\Delta_\ell^{-\nu}[\lambda_\ell^{(\mu)}e^{-s\lambda}] = \sum_{i=1}^{\mu+1}\frac{(-1)^{i-1}\ell^\nu\mu^{(i-1)}}{(i-1)!}\frac{(\nu+i-2)^{(i-1)}\ell^{i-1}}{(1-e^{-s\ell})^{i+\nu-1}}\frac{\lambda_\ell^{(\mu+1-i)}}{e^{s(\lambda+(i-1)\ell)}}. \tag{14}$$

Proof. The proof follows by proceeding in the same way of Theorem 3.

Corollary 2. *Let $\nu,\ell > 0$, and $\lambda \in [0,\infty)$, then we have*

$$\Delta_\ell^{-\nu}[\lambda_\ell^{(\mu)}e^{-s\lambda}]\Big|_0^\infty = \frac{(-1)^\mu\ell^\nu\mu^{(\mu)}}{\mu!}\frac{(\nu+\mu-1)^{(\mu)}\ell^\mu}{(1-e^{-s\ell})^{\mu+\nu}}\frac{1}{e^{s\mu\ell}}. \tag{15}$$

Proof. Taking the limits 0 to ∞ in (14), we get the proof.

Corollary 3. *By applying the fractional Laplace transform, we get*

$$L_\ell^\nu[\lambda_\ell^{(\mu)}] = \frac{(-1)^\mu\ell^\nu\mu^{(\mu)}}{\mu!}\frac{(\nu+\mu-1)^{(\mu)}\ell^{\mu+1}}{(1-e^{-s\ell})^{\mu+\nu}}\frac{1}{e^{s\mu\ell}} = \sum_{i=0}^{\infty}(-1)^i\frac{\ell^\nu\Gamma(1-\nu)(i\ell)_\ell^{(\mu)}e^{-si\ell}}{\Gamma(i+1)\Gamma(1-\nu-i)}. \tag{16}$$

Proof. Product on both sides of (15) by ℓ, which gives

$$L_\ell^\nu[\lambda_\ell^{(\mu)}] = \Delta_\ell^{-\nu}[\lambda_\ell^{(\mu)}e^{-s\lambda}]\Big|_0^\infty = \frac{(-1)^\mu\ell^\nu\mu^{(\mu)}}{\mu!}\frac{(\nu+\mu-1)^{(\mu)}\ell^{\mu+1}}{(1-e^{-s\ell})^{\mu+\nu}}\frac{1}{e^{s\mu\ell}}. \tag{17}$$

Now the proof of (16) is arrived from (10).

Example 4. For the particular values of $\mu = 2$ in (16), we have

$$L_\ell^\nu[\lambda_\ell^{(2)}] = \frac{(\nu)(\nu+1)\ell^{\nu+3}}{(1-e^{-s\ell})^{2+\nu}e^{2s\ell}} = \sum_{i=0}^{\infty}(-1)^i \frac{\ell^\nu \Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}(i\ell)_\ell^{(2)} e^{-si\ell}. \tag{18}$$

For $\nu = 0.6, s = 4$ and $h = 3$ in (18), which is verified and the coding as: $(0.6.*(1.6).*27).*3.\wedge(0.6)./((1-exp(-12)).\wedge(2.6).*exp(24)) = 3.*symsum(((-1).\wedge i.*3.\wedge(0.6).*gamma(0.4).*(3.*i).*(3.*i-3).*exp(-12.*i))./(gamma(i+1).*gamma(0.4-i)),i,0,inf)$.

The diagrams of outcomes of FLT for polynomial factorial generated by MATLAB are shown below. Figure 1. tells that the input function(signal) as polynomial factorial and Figure 2. tells that the output signal in the frequency domain by varying the values of ν.

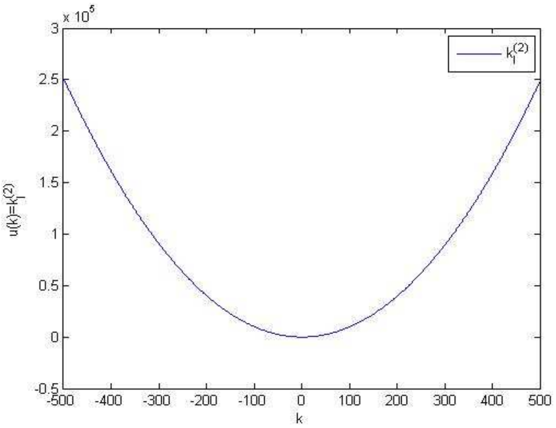

Fig. 1: Time Domain Signal for Polynomial Factorial

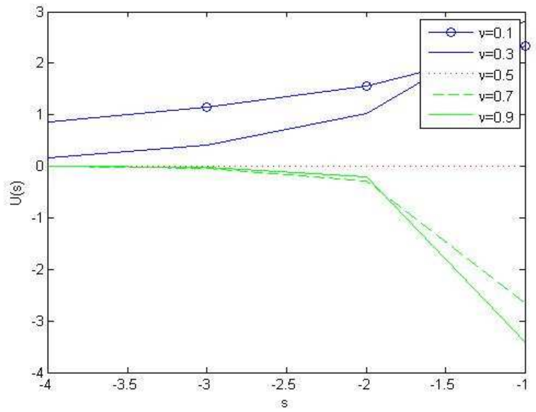

Fig. 2: Frequency Signal for Fraction

5 Fractional extorial transform by ℓ-difference operator

Definition 7. *Let $\nu, h > 0$, we define the extorial function as*

$$e^{\lambda_\ell^{[\nu]}} = 1 + \frac{\lambda_\ell^{[\nu]}}{1!} + \frac{\lambda_\ell^{[2\nu]}}{2!} + \frac{\lambda_\ell^{[3\nu]}}{3!} + \cdots \tag{19}$$

Remark. (i) In particular value of $\nu = 1$, (19) gives

$$e^{\lambda_\ell^{[1]}} = 1 + \frac{\lambda_\ell^{[1]}}{1!} + \frac{\lambda_\ell^{[2]}}{2!} + \frac{\lambda_\ell^{[3]}}{3!} + \cdots \tag{20}$$

(ii) Since $e^{\lambda} \neq e^{\lambda_\ell^{[1]}}$.

Definition 8. *For the real valued function ψ, then the Fractional extorial transform is defined by*

$$\mathscr{E}_\ell^\nu[\psi(\lambda)] = \Delta_\ell^{-\nu}\psi(\lambda)e^{-s\lambda_\ell^{[\nu]}}\Big|_0^\infty = -\sum_{i=0}^{\infty}(-1)^i \frac{\ell^\nu \Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\psi(i\ell)e^{-s(i\ell)_\ell^{[\nu]}}. \tag{21}$$

The fractional extorial transform of certain functions like polynomial factorial and logarithmic functions are obtained.

Theorem 5. *Let $\lambda \in [0,\infty)$, then*

$$\mathscr{E}_\ell^\nu[\lambda_\ell^{(\mu)}] = -\sum_{i=0}^{\infty}(-1)^i \frac{\ell^\nu \Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}(i\ell)_\ell^{(\mu)}e^{-s(i\ell)_\ell^{[\nu]}}. \tag{22}$$

Proof. Taking $\psi(\lambda) = \lambda_\ell^{(\mu)}$ in (21).

Theorem 6. *Let $\lambda \in [0,\infty)$, then*

$$\mathscr{E}_\ell^\nu[\log a\lambda] = -\sum_{i=0}^{\infty}(-1)^i \frac{\ell^\nu \Gamma(1-\nu)}{\Gamma(i+1)\Gamma(1-\nu-i)}\log a(i\ell)e^{-s(i\ell)_\ell^{[\nu]}}, \tag{23}$$

Proof. Applying the fractional extorial transform for the function $\psi(\lambda) = \log at$ we get (23).

Remark. To get the classical Laplace transform one can apply $\ell \to 0$ and $\nu = 1$.

6 Conclusion

In this research, we have defined the discrete fractional Laplace transform using inverse fractional difference operators and obtained several related results. Also, we have proposed a new type of fractional extorial transform and eventually discussed its sums. The properties and several results have been obtained by its transform for certain functions. A verification of our findings by MATLAB has been provided as well.

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