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Liouville-Weyl Fractional Hamiltonian Systems: Existence Result

César Torres[1,*] *and Willy Zubiaga*[2]

[1] Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n. Trujillo, Perú
[2] Escuela de Post Grado, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n. Trujillo, Perú

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Abstract: In this work we investigate the following fractional Hamiltonian systems ${}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t,u(t))$, where $\alpha \in (1/2,1)$, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a positive definite symmetric matrix, $W(t,u) = a(t)V(t)$ with $a \in C(\mathbb{R},\mathbb{R}^+)$ and $V \in C^1(\mathbb{R}^n,\mathbb{R})$. By using the Mountain pass theorem and assuming that there exist $M > 0$ such that $(L(t)u,u) \geq M|u|^2$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$ and V satisfies the global Ambrosetti-Rabinowitz condition and other suitable conditions, we prove that the above mentioned equation at least has one nontrivial weak solution.

Keywords: Fractional hamiltonian systems, fractional spaces, variational methods, Mountain pass theorem.

1 Introduction

In this manuscript we study the existence of weak solution for the fractional systems

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t,u(t)), \tag{1}$$

where $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ be a positive definite symmetric matrix which satisfy the following condition

(V_1)There is $M > 0$ such that

$$(L(t)x,x) \geq M|x|^2 \text{ for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n,$$

$W(t,x) = a(t)V(x)$ with $a \in C(\mathbb{R},\mathbb{R}^+)$ and $V \in C^1(\mathbb{R}^n,\mathbb{R})$.

Fractional calculus methods and techniques are playing and important role in science and engineering [1–8]. Since fractional equations are important in theory and application, in the last years, researches on fractional differential equations have achieved significant development. Recently, the study of fractional differential equations through variational methods become an important filed of study for several researches [9–14]. A new research trend started to be developed, especially for fractional differential equations having both the left and right fractional derivatives, because, in general, the fixed-point theory is not a suitable tool to show existence results to this type of equations. Namely, it is not obvious to convert this type of fractional equation into an equivalent integral equation and then transformed into some fixed-point problem.

Jiao and Zhou [10], have used critical point theory and variational methods to show the existence of at least one nontrivial weak solution for

$$\begin{aligned} &{}_tD^{\alpha}_{T}({}_{0}D^{\alpha}_t u(t)) = \nabla F(t,u(t)), \text{ a.e. } t \in [0,T], \\ &\quad u(0) = u(T) = 0. \end{aligned} \tag{2}$$

Following this result, Torres [11], has considered a kind of fractional Hamiltonian systems with the left and right fractional derivatives on $\mathbb{R}$, namely

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t,u(t)), \quad t \in \mathbb{R} \tag{3}$$

such that $\alpha \in (1/2,1)$, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ denotes a symmetric matrix-valued function, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. Under the following hypotheses,

* Corresponding author e-mail: ctl_576@yahoo.es

$(L)\forall\, t\in\mathbb{R}$, the matrix $L(t)$ is positive definite and symmetric. Moreover, there is $l\in C(\mathbb{R},(0,\infty))$ such that $\lim_{|t|\to\infty} l(t)=\infty$ and

$$(L(t)x,x)\geq l(t)|x|^2,\ \ \forall\,(t,x)\in\mathbb{R}\times\mathbb{R}^n. \tag{4}$$

$(W_1) W\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and there exist $\mu>2$ obeying

$$0<\mu W(t,x)\leq(x,\nabla W(t,x)),\ \text{ for all } t\in\mathbb{R}\ \text{ and } x\in\mathbb{R}^n\setminus\{0\}.$$

$(W_2)|\nabla W(t,x)|=o(|x|)$ as $x\to 0$ uniformly with respect to $t\in\mathbb{R}$.
(W_3)There exists $\overline{W}\in C(\mathbb{R}^n,\mathbb{R})$ obeying

$$|W(t,x)|+|\nabla W(t,x)|\leq|\overline{W(x)}|\ \text{ for every } x\in\mathbb{R}^n \text{ and } t\in\mathbb{R}.$$

The author showed that (3) possesses at least one nontrivial weak solution via Mountain Pass Theorem.

For the case $\alpha=1$, (3) becomes

$$u''-L(t)u+\nabla W(t,u)=0, \tag{5}$$

such that $W:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ denotes a given function. It was firstly recognized from Poincare [15] the existence of homoclinic solutions of Hamiltonian systems (5) and its importance on the study of the behaving on dynamical systems. During the last decades the variational methods and critical point theory were utilised to investigate the existence and multiplicity of homoclinic solutions [16, 17].

Recently, by utilising the genus properties of critical point theory, in [14] it was proved some new criterion to guarantee the existence of infinitely many solutions of (3) for the case that $W(t,u)$ is subquadratic as $|u|\to+\infty$ and L fulfills (L).

Note that, condition (L) is the so-called coercive condition and is a little demanding. Thus, for a simple choice like $L(t)=sI_n$, where $s>0$ and I_n is the $n\times n$ identity matrix, the condition (4) is not satisfied, so we can not get any existence result for problem (1). Motivated by this dificulty, in the present work we focus our attention on the case that $L(t)$ is uniformly bounded from below, namely $L(t)$ satisfies (V_1).

We suppose that $W(t,u)=a(t)V(u)$ fulfills the followings

$(V_2) a:\mathbb{R}\to\mathbb{R}^+$ which is a continuous function obeying

$$\lim_{|t|\to+\infty} a(t)=0;$$

$(V_3) V\in C^1(\mathbb{R}^n,\mathbb{R})$ and there is a constant $\mu>2$ fulfilling

$$0<\mu V(x)\leq(\nabla V(x),x)\ \text{ for all }\ x\in\mathbb{R}^n\setminus\{0\};$$

$(V_4)\nabla V(x)=o(|x|)$ as $|x|\to 0$.

Thus, we can present the main result, namely

Theorem 1. *Let $\frac{1}{2}<\alpha<1$. If $(V_1)-(V_4)$ are satisfy, problem (3) possesses at least one nontrivial solution.*

In [11], assuming that (L) holds, the author introduced some compact embedding theorem (see Lemma 2.2 in [11]). This compact result plays a crucial role in order to prove Palais-Smale condition. Since we are assuming (V_1), we can not get a similar compact embedding, hence the main difficulty of this work is the lack of compactness. In order to show Palais-Smale compactness result, we get a new compact embedding under condition (V_1), for more details see Lemma 2 below.

The rest of the manuscript has the following organisation: in Section 2, some preliminary results are given. The proof of Theorem 1 is presented in Section 3.

2 Preliminaries

Below the variational framework for the problem (1) is considered, for more details see [18].

Let $\alpha\in(0,1)$. For a suitable function u, the Liouville-Weyl fractional derivatives are defined as

$$_{-\infty}D_x^{\alpha}u(x)=\frac{d}{dx}{}_{-\infty}I_x^{1-\alpha}u(x)\quad\text{ and }\quad {}_xD_{\infty}^{\alpha}u(x)=-\frac{d}{dx}{}_xI_{\infty}^{1-\alpha}u(x), \tag{6}$$

where ${}_{-\infty}I_x^\alpha$ and ${}_xI_\infty^\alpha$ are the left and right Liouville-Weyl fractional integrals given by

$$ {}_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}u(\xi)d\xi \quad \text{and} \quad {}_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}u(\xi)d\xi. $$

Let $2 \le p < +\infty$ and denote by $L^p(\mathbb{R},\mathbb{R}^n)$ the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under

$$ \|u\|_{L^p}^p := \int_{\mathbb{R}} |u(t)|^p dt, $$

and $L^\infty(\mathbb{R},\mathbb{R}^n)$ is the Banach space of essentially-bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ such that

$$ \|u\|_\infty := esssup\{|u(t)| : t \in \mathbb{R}\}. $$

Note that, for $u \in L^p(\mathbb{R},\mathbb{R}^n)$, we have

$$ \mathscr{F}({}_{-\infty}I_x^\alpha u(x)) = (i\omega)^{-\alpha}\widehat{u}(\omega) \qquad \text{and} \qquad \mathscr{F}({}_xI_\infty^\alpha u(x)) = (-i\omega)^{-\alpha}\widehat{u}(\omega), $$

and for $u \in C_0^\infty(\mathbb{R},\mathbb{R}^n)$, we have

$$ \mathscr{F}({}_{-\infty}D_x^\alpha u(x)) = (i\omega)^{\alpha}\widehat{u}(\omega) \qquad \text{and} \qquad \mathscr{F}({}_xD_\infty^\alpha u(x)) = (-i\omega)^{\alpha}\widehat{u}(\omega), $$

where $C_0^\infty(\mathbb{R},\mathbb{R}^n)$ denotes the space of infinitely differentiable functions with vanishing at infinity.
For $\alpha \in (0,1)$, let us define the fractional space

$$ I_{-\infty}^\alpha(\mathbb{R},\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|.\|_{I_{-\infty}^\alpha}}, $$

where

$$ \|u\|_{I_{-\infty}^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2\right)^{1/2}, \tag{7} $$

and

$$ |u|_{I_{-\infty}^\alpha} = \|{}_{-\infty}D_x^\alpha u\|_{L^2} $$

Moreover, we consider the fractional Sobolev space

$$ H^\alpha(\mathbb{R},\mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|.\|_\alpha}. $$

where

$$ \|u\|_\alpha = \left(\|u\|_{L^2}^2 + |u|_\alpha^2\right)^{1/2}, $$

and

$$ |u|_\alpha = \||w|^\alpha \widehat{u}\|_{L^2}. \tag{8} $$

Since

$$ |w|^\alpha \widehat{u} \in L^2(\mathbb{R},\mathbb{R}^n), \;\; \text{for all} \;\; u \in L^2(\mathbb{R},\mathbb{R}^n). \tag{9} $$

Then

$$ |u|_{I_{-\infty}^\alpha} = \||w|^\alpha \widehat{u}\|_{L^2}. \tag{10} $$

Therefore, $I_{-\infty}^\alpha(\mathbb{R},\mathbb{R}^n)$ and $H^\alpha(\mathbb{R},\mathbb{R}^n)$ are equivalent with equivalent semi-norm and norm.

Theorem 2.[11] *If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R},\mathbb{R}^n) \subset C(\mathbb{R},\mathbb{R}^n)$ and there is a constant $C = C_\alpha$ fulfilling*

$$ \|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)| \le C\|u\|_\alpha, \tag{11} $$

such that $C(\mathbb{R},\mathbb{R}^n)$ is the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$.

*Remark.*From Theorem 2, the embedding $H^\alpha(\mathbb{R},\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R},\mathbb{R}^n$ is continuous for all $q \in [2,\infty]$ and $1/2 < \alpha < 1$, since

$$ \int_{\mathbb{R}} |u(x)|^q dx \le \|u\|_\infty^{q-2}\|u\|_{L^2}^2. $$

Below we discuss the fractional space used for the variational framework of (3). We consider

$$X^{\alpha} = \left\{ u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^n) | \int_{\mathbb{R}} |_{-\infty}D_t^{\alpha} u(t)|^2 + L(t)u(t).u(t)dt < \infty \right\}.$$

X^{α} be a Hilbert space such that

$$\langle u, v \rangle_{X^{\alpha}} = \int_{\mathbb{R}} ({}_{-\infty}D_t^{\alpha} u(t), \ {}_{-\infty}D_t^{\alpha} v(t)) + L(t)u(t).v(t)dt.$$

Besides the related norm is given by

$$\|u\|^2_{X^{\alpha}} = \langle u, u \rangle_{X^{\alpha}}.$$

Lemma 1.[11] *Assume L fulfills* (V_1)*. Then* X^{α} *is continuously embedded in* $H^{\alpha}(\mathbb{R}, \mathbb{R}^n)$.

In order to recover the Palais-Smale condition, we are going to show a new compact embedding result under the condition (V_1). Thus, if we denote by $L_a^p(\mathbb{R}, \mathbb{R}^n)$ the weighted space of measurable functions $u : \mathbb{R} \to \mathbb{R}^n$ such that

$$\|u\|^p_{p,a} = \int_{\mathbb{R}} a(t)|u(t)|^p dt < \infty,$$

then we have the following important result.

Lemma 2. *Suppose that* (V_1), (V_2) *hold. Then, the imbedding* $X^{\alpha} \hookrightarrow L_a^2(\mathbb{R}, \mathbb{R}^n)$ *is continuous and compact.*

Proof. By (V_2) there is $A > 0$ fulfilling $|a(t)| \leq A \ \ \forall t \in \mathbb{R}$, so

$$\int_{\mathbb{R}} a(t)|u(t)|^2 dt \leq A \int_{\mathbb{R}} |u(t)|^2 dt.$$

Therefore, by Lemma 1 and Remark 2, the embedding $X^{\alpha} \hookrightarrow L_a^2(\mathbb{R}, \mathbb{R}^n)$ is continuous.

Let $(u_k) \in X^{\alpha}$ such that $u_k \rightharpoonup 0$ in X^{α}. We are going to show that $u_k \to 0$ in $L_a^2(\mathbb{R}, \mathbb{R}^n)$. By the Banach-Steinhaus theorem we have

$$B = \sup_k \|u_k\|_{X^{\alpha}} < +\infty.$$

By (V_2), for any $\varepsilon > 0$; there is $R > 0$ such that

$$a(t) < \varepsilon, \ \forall |t| \geq R.$$

So

$$\int_{|t|>R} a(t)|u_k(t)|^2 dt \leq \frac{\varepsilon}{M} \int_{|t|>R} l(t)|u_k(t)|^2 dt \leq \frac{\varepsilon}{M} B^2. \tag{12}$$

On the other hand, by Sobolev's theorem, $\|u_k\|_{C[-R,R]} \to 0$ as $k \to \infty$, thus, there is k_0 such that

$$\int_{-R}^{R} a(t)|u_k(t)|^2 dt \leq \varepsilon, \ \ \forall \, k \geq k_0. \tag{13}$$

By (12) and (13) we get $u_k \to 0$ in $L_a^2(\mathbb{R}, \mathbb{R}^n)$ as $k \to +\infty$.

Lemma 3. *There are positive constants* c_1, c_2 *such that*

$$V(x) \geq c_1 |x|^{\mu}, \ \ |x| \geq 1 \tag{14}$$

and

$$V(x) \leq c_2 |x|^{\mu}, \ \ |x| \leq 1. \tag{15}$$

*Proof.*Let $f(\sigma)=V(\sigma x)$, then by (V_3) we have

$$\frac{d}{d\sigma}\left(\frac{f(\sigma)}{\sigma^{\mu}}\right)\geq 0. \tag{16}$$

If $|x|\leq 1$, by (16) we get

$$V(x)\leq V(\frac{x}{|x|})|x|^{\mu}. \tag{17}$$

On the other hand, if $|x|\geq 1$ by (16) we get

$$V(x)\geq |x|^{\mu}V(\frac{x}{|x|}). \tag{18}$$

Since, for all $x\in\mathbb{R}^n$ we have that $\frac{x}{|x|}\in B(0,1)$, then by the continuity of V, there are $c_1,c_2>0$ such that

$$c_1\leq V(x)\leq c_2,\ \text{ for every } x\in B(0,1).$$

Thus, the proof of the Lemma was given.

*Remark.*By using the Lemma 3, we conclude

$$V(x)=o(|x|^2) \text{ as } |x|\to 0, \tag{19}$$

Also, by (V_4), for any $x\in\mathbb{R}^n$ with $|x|\leq M_1$, there exists $d=d(M_1)>0$ fulfilling

$$|\nabla V(x)|\leq d|x|. \tag{20}$$

As in [11], we get the following result.

Lemma 4.*Suppose that (V_1) and (V_2) are satisfied. If $u_k\rightharpoonup u$ in X^{α}, then $\nabla V(u_k)\to\nabla V(u)$ in $L^2_a(\mathbb{R},\mathbb{R}^n)$.*

*Proof.*By (11) and the Banach-Steinhaus theorem, there exists $d_1>0$ such that

$$\sup_{k\in\mathbb{N}}\|u_k\|_{\infty}\leq d_1,\ \ \|u\|_{\infty}\leq d_1.$$

By (20), there exists $d_2=d_2(d_1)>0$ such that

$$|\nabla V(u_k(t))|\leq d_2|u_k(t)|,\ \ |\nabla V(u(t))|\leq d_2|u(t)|,$$

for all $k\in\mathbb{N}$ and $t\in\mathbb{R}$. So

$$|\nabla V(u_k(t))-\nabla V(u(t))|\leq d_2(|u_k(t)|+|u(t)|)\leq d_2(|u_k(t)-u(t)|+2|u(t)|).$$

Moreover, by Lemma 2, $u_k\to u$ in $L^2_a(\mathbb{R},\mathbb{R}^n)$, so up to a subsequence, it can be assumed that

$$\sum_{k=1}^{\infty}\|u_k-u\|_{2,a}<+\infty,$$

which implies $u_k(t)\to u(t)$ a.e. $t\in\mathbb{R}$ and

$$\sum_{k=1}^{\infty}|u_k(t)-u(t)|=v(t)\in L^2_a(\mathbb{R},\mathbb{R}^n).$$

Therefore

$$a(t)|\nabla V(u_k(t))-\nabla V(u(t))|^2\leq d_2^2a(t)(v(t)+2|u(t)|)^2.$$

Then, by Lebesgue's theorem we conclude.

Let $\mathfrak{B}$ be a real Banach space, $I\in C^1(\mathfrak{B},\mathbb{R})$. Thus, I denotes a continuously Fréchet-differentiable functional defined on $\mathfrak{B}$. In addition, $I\in C^1(\mathfrak{B},\mathbb{R})$ is said to fulfil the (PS) condition if any sequence $\{u_k\}_{k\in\mathbb{N}}\in\mathfrak{B}$, such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k)\to 0$ as $k\to+\infty$, admits a convergent subsequence in $\mathfrak{B}$.

Let B_r be the open ball in $\mathfrak{B}$ with the radius r and centered at 0 and ∂B_r represents its boundary. Utilising the following the Mountain Pass Theorems [19] we discuss the existence of solutions of (3).

Theorem 3. *Let $\mathfrak{B}$ denoting a real Banach space and $I \in C^1(\mathfrak{B},\mathbb{R})$ fulfilling (PS) condition. Assume that $I(0)=0$ and*

i. There are constants $\rho_1, \beta_1 > 0$ obeying $I|_{\partial B_{\rho_1}} \geq \beta_1$, and

ii. There is $e_1 \in \mathfrak{B} \setminus \overline{B_{\rho_1}}$ fulfilling $I(e_1) \leq 0$.

Then I admits a critical value $c_1 \geq \beta_1$. c_1 is given by

$$c_1 = \inf_{\gamma_1 \in \Gamma} \max_{s\in[0,1]} I(\gamma_1(s))$$

such that

$$\Gamma = \{\gamma_1 \in C([0,1],\mathfrak{B}): \ \gamma_1(0)=0, \ \gamma_1(1)=e_1\}.$$

3 Proof of theorem 1

Following the ideas of [20], consider the functional $I : X^\alpha \to \mathbb{R}$ as

$$I(u) = \int_{\mathbb{R}} \left[\frac{1}{2}|{}_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t),u(t)) - W(t,u(t))\right] dt = \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t,u(t))dt. \tag{21}$$

Here and subsequently, $W(t,u)$ denotes $a(t)V(u)$ unless otherwise specified. We consider the differentiability of the functional I, more precisely we have:

Lemma 5. *Under the conditions of Theorem 1, we conclude*

$$I'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t)) - (\nabla W(t,u(t)),v(t))]\,dt \tag{22}$$

for all $u,v \in X^\alpha$, which implies that

$$I'(u)u = \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} (\nabla W(t,u(t)),u(t))dt. \tag{23}$$

Besides, I is a continuously Fréchet differentiable functional defined on X^α, i.e., $I \in C^1(X^\alpha,\mathbb{R})$.

Proof. By (19), there is a $\delta > 0$ fulfilling

$$V(x) \leq \varepsilon |x|^2 \ \text{ for all } \ |x| \leq \delta. \tag{24}$$

Since $u \in X^\alpha$, then $u(t) \to 0$ as $|t| \to +\infty$. So, there is a constant $R_1 > 0$ obeying

$$|u(t)| \leq \delta, \ \text{ for all } \ |t| \geq R_1.$$

Hence, by (24), we conclude

$$\int_{\mathbb{R}} W(t,u(t)) \leq \int_{-R_1}^{R_1} a(t)V(u(t))dt + \varepsilon \int_{|t|\geq R_1} a(t)|u(t)|^2 dt < +\infty, \tag{25}$$

and $I : X^\alpha \to \mathbb{R}$.

Next we show that $I \in C^1(X^\alpha,\mathbb{R})$. Rewrite I as follows; $I = I_1 - I_2$, where

$$I_1 = \frac{1}{2}\int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))]dt, \ \ I_2 = \int_{\mathbb{R}} W(t,u(t))dt.$$

Then $I_1 \in C^1(X^\alpha,\mathbb{R})$ and

$$I_1'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t))]\,dt. \tag{26}$$

Now, we are going to show that

$$I_2'(u)v = \int_{\mathbb{R}} (\nabla W(t,u(t)),v(t))dt, \ \text{ for all } u,v \in X^\alpha. \tag{27}$$

In fact, for any given $u \in X^{\alpha}$, let us define $J(u) : X^{\alpha} \to \mathbb{R}$ as follows

$$J(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt.$$

By inspection we conclude that $J(u)$ is linear. By using (20), there is a constant $d_3 > 0$ obeying

$$|\nabla W(t, u(t))| \leq d_3 a(t)|u(t)|.$$

So by Hölder inequality and Lemma 2

$$|J(u)v| = \left| \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt \right| \leq d_3 \int_{\mathbb{R}} a(t)|u(t)||v(t)|dt \leq d_3 C_a^2 \|u\|_{X^{\alpha}} \|v\|_{X^{\alpha}}. \tag{28}$$

Furthermore, for u and $v \in X^{\alpha}$, by mean-value theorem, we get

$$\int_{\mathbb{R}} W(t, u(t) + v(t))dt - \int_{\mathbb{R}} W(t, u(t))dt = \int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)))dt,$$

where $h(t) \in (0, 1)$. So, by Lemma 4 and the Hölder inequality, we obtain

$$\int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)), v(t))dt - \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt = \int_{\mathbb{R}} (\nabla W(t, u(t)) + h(t)v(t) - \nabla W(t, u(t)), v(t))dt \to 0 \tag{29}$$

as $v \to 0$ in X^{α}. Therefore, by (28) and (29), the asseveration in (27) holds. It remains to prove that I_2' is continuous. Suppose that $u \to u_0$ in X^{α} and note that

$$\begin{aligned} \sup_{\|v\|_{X^{\alpha}}=1} |I_2'(u)v - I_2'(u_0)v| &= \sup_{\|v\|_{X^{\alpha}}=1} \left| \int_{\mathbb{R}} (\nabla W(t, u(t)) - \nabla W(t, u_0(t)), v(t))dt \right| \\ &\leq \sup_{\|v\|_{X^{\alpha}}=1} \|\nabla V(u(.)) - \nabla V(u_0(.))\|_{2,a} \|v\|_{2,a} \leq C_a \|\nabla V(u(.)) - \nabla V u_0(.))\|_{2,a}. \end{aligned}$$

By Lemma 4, we obtain that $I_2'(u)v - I_2'(u_0)v \to 0$ as $\|u\|_{X^{\alpha}} \to \|u_0\|_{X^{\alpha}}$ uniformly with respect to v, which implies the continuity of I_2' and $I \in C^1(X^{\alpha}, \mathbb{R})$.

Lemma 6. *Under the hypotheses of Theorem 1, I satisfies the (PS) condition.*

Proof. Suppose that $(u_k)_{k\in\mathbb{N}} \in X^{\alpha}$ be a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \leq C_1, \quad \|I'(u_k)\|_{(X^{\alpha})^*} \leq C_1, \tag{30}$$

for every $k \in \mathbb{N}$. Then by (21), (23) and (V_3), we have

$$\begin{aligned} C_1 + \|u_k\|_{X^{\alpha}} &\geq I(u_k) - \frac{1}{\mu} I'(u_k)u_k \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_k\|_{X^{\alpha}}^2 - \int_{\mathbb{R}} [W(t, u_k(t)) - \frac{1}{\mu}(\nabla W(t, u_k(t)), u_k(t))]dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_k\|_{X^{\alpha}}^2. \end{aligned} \tag{31}$$

Since $\mu > 2$, then $(u_k)_{k\in\mathbb{N}}$ is bounded in X^{α} and Lemma 2, up to a subsequence we get

$$u_k \rightharpoonup u, \quad \text{weakly in } X^{\alpha},$$
$$u_k \to u, \quad \text{strongly in } L_a^2(\mathbb{R}, \mathbb{R}^n).$$

So

$$(I'(u_k) - I'(u))(u_k - u) \to 0 \ \text{ as } \ k \to \infty.$$

Furthermore, by Lemma 4 and Hölder inequality we obtain

$$\int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t))dt \to 0 \ \text{ as } \ k \to \infty.$$

Combining these previous results with

$$(I'(u_k)-I'(u))(u_k-u) = \|u_k-u\|^2_{X^\alpha} - \int_{\mathbb{R}} [\nabla W(t,u_k)-\nabla W(t,u)](u_k-u)dt \to 0,$$

we obtain that, $\|u_k-u\|_{X^\alpha} \to 0$ as $k \to +\infty$. □
Below we present the proof of Theorem 1. The proof is divided into specific steps.
Proof of theorem 1.
Step 1. By inspection we conclude that $I(0)=0$ and $I \in C^1(X^\alpha,\mathbb{R})$ fulfills the (PS) condition by Lemma 5 and 6.
Step 2. By (19), for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|V(x)| \leq \varepsilon |u|^2, \text{ for all } |x| \leq \delta.$$

Letting $\gamma = \frac{\delta}{C_\alpha}$ and $\|u\|_\alpha = \gamma$, we have $\|u\|_\infty \leq \delta$. Hence

$$|V(u(t))| \leq \varepsilon |u(t)|^2, \text{ for all } t \in \mathbb{R}.$$

Then by Lemma 2 we conclude

$$\int_{\mathbb{R}} W(t,u(t))dt \leq \varepsilon \|u\|^2_{2,a} \leq \varepsilon C_a^2 \|u\|^2_{X^\alpha}.$$

So, by taking $\varepsilon = \frac{1}{4C_a^2}$, we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2_{X^\alpha} - \varepsilon C_a^2 \|u\|^2_{X^\alpha} \\ &= \left(\frac{1}{2} - \varepsilon C_a^2\right) \|u\|^2_\alpha \geq \frac{\delta^2}{4C^2} = \beta > 0. \end{aligned} \tag{32}$$

Then I obeys the first condition of Theorem 3.
Step 3. For any $\sigma \in \mathbb{R}$, consider the function

$$I(\sigma u) = \frac{\sigma^2}{2}\|u\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t,\sigma u(t))dt.$$

By (14), there exists $c_1 > 0$ fulfilling

$$W(t,u(t)) \geq c_1 |u(t)|^\mu \text{ for all } |u(t)| \geq 1. \tag{33}$$

Consider some $u \in X^\alpha$ obeying $\|u\|_{X^\alpha} = 1$. So, there exists an open interval $(a,b) \subset \mathbb{R}$ with $a < b$, such that $u(t) \neq 0$ for $t \in (a,b)$. Consider $\sigma > 0$ fulfilling $\sigma|u(t)| \geq 1$ for $t \in (a,b)$. As a result from(33), we conclude

$$I(\sigma u) \leq \frac{\sigma^2}{2} - c_1 \sigma^\mu \int_a^b a(t)|u(t)|^\mu dt. \tag{34}$$

Taking into account that $a(t) > 0$, $c_1 > 0$ and $\mu > 2$, (34) says that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\|\sigma u\|_{X^\alpha} > \gamma$, where γ is given in Step 2. By utilising the Theorem 3, I admits a critical value $c \geq \beta > 0$ such that

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)),$$

where

$$\Gamma = \{\gamma \in C([0,1],X^\alpha) : \ \gamma(0)=0, \ \gamma(1)=e\}.$$

Thus, there is $u \in X^\alpha$ obeying

$$I(u) = c and \ I'(u) = 0.$$

□

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