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An Uncertainty Approach for Impulsive Fractional Differential Equations: Application in Fluid Mechanics

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Abstract: In this manuscript, we will consider an uncertainty approach based on intervals for impulsive fractional differential equations (IFDEs). In this regard, we employ the Laplace transforms and the solution of IFDEs is calculated based on Riemann-Liouville differentiability. Finally, the well-known Bagley-Torvik equation (arises in fluid mechanics), which involves an additive delta function on the interval right-hand side is solved to validate the achieved results.

Keywords: Impulsive fractional differential equations, interval arithmetic, Riemann-Liouville differentiability, Bagley-Torvik equation.

1 Introduction

Fractional differentiation is an extension of integer-order differentiation. Thus, we can conclude that differential equations of fractional order (FDEs) are a general extension of differential equations of integer-order. The aim of this extension is not just for pure mathematics purposes, because the concept of FDEs has several important applications in science and technology (see [1]). For example fatigue phenomenon in a material system [2] and fractional newton mechanics [3]. For other applications see [4,5,6,7,8].

While we study many systems, we can observe that at specified points of time, the systems experience some abrupt perturbations of states. In this case, we may conclude that the perturbations act quickly. In other words, the performance of system can be considered with impulses. In this case, using impulsive fractional differential equations (IFDEs) for modeling could help find a wide range of solutions instead of local solutions (e.g. see [9] and [10]). Based on these conditions, several studies focused on IFDEs. For example, [11] addressed the stability of IFDEs with delays. The authors of [12] proved some results about the existence of solutions for an impulsive fractional integro-differential equation containing state-dependent delays. In [13] some existence and uniqueness theorems for the solutions of IFDEs were proved. For more details see [14,15,16].

In real world phenomena, uncertainty is a natural concept which arises in the system modeling. Two main popular approaches for modeling the uncertainty are fuzzy set theory as well as interval theory ([17,18,19]). Since, in practical problems, there are several sources of uncertainty, FDEs are not an exception for uncertain modeling. See, for example, [20,21,22,23,24,25,26,27,28,29] for some models and approaches which consider uncertain FDEs.

Although we see a high-growth in the papers that addressed the theory of fuzzy FDEs, the case is different for the theory of interval FDEs. Few researches addressed interval FDEs (for example see [30,31]). The theory of IFDEs is very applicable and even more effective in modeling some practical problems such as Abel equations [32].

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Considering aforementioned discussions, we are inspired to study the solution concepts of IFDEs described in [33] under interval Riemann-Liouville derivative. The uniqueness of the solution of interval FIDE is investigated under the w-monotony condition. Afterwards, a Bagley-Torvik equation which involves an additive delta function on the interval right-hand side is solved to examine the applicability of the proposed approach.

The structure of the paper is as follows: In the next section we mention some necessary preliminaries related to the interval analysis. In Section 3, a Lemma is presented to provide the unique solution of the IFDEs under interval uncertainty. Section 4 is devoted to solve the Bagley-Torvik equation using the acquired results in Section 3. Section 5 is dedicated to conclusion.

2 Preliminaries

The necessary background for this study is recalled in this section to support our theoretical results in the main body of the context. For complete details, see for example [34,35]).

In this paper, $\mathscr{I}$ denotes the family of all intervals which are subsets of real line $\mathbb{R}$ which have three properties: convexity, compactness and nonemptiness. We can define addition and multiplication in $\mathscr{I}$. In other words, for $I_1,\ I_2 \in \mathscr{I}$ which $I_1 = [\underline{I_1}, \overline{I_1}], I_2 = [\underline{I_2}, \overline{I_2}]$ with $\underline{I_1} \leq \overline{I_1}, \underline{I_2} \leq \overline{I_2}$ and for $\lambda \geq 0$ we may have:

$$I_1 + I_2 = [\underline{I_1} + \underline{I_2}, \overline{I_1} + \overline{I_2}],$$

and also for $\lambda \geq 0$ we have

$$\lambda I_1 = [\lambda \underline{I_1}, \lambda \overline{I_1}]$$

Finally, for $\mu < 0$ we have:

$$\mu I_1 = [\mu \overline{I_1}, \mu \underline{I_1}].$$

Suppose that I is an element of $\mathscr{I}$ and $\lambda_1, \lambda_2 \geq 0$ while μ_1 and μ_2 are arbitrary real numbers. Then we have:

$\lambda_1(\lambda_2 I) = (\lambda_1 \lambda_2) I$ and $(\mu_1 + \mu_2) I = \mu_1 I + \mu_2 I$.

Now, we can define the Hausdorff distance $\mathscr{D}_H$ in $\mathscr{I}$ as follows:

$$\mathscr{D}_H(I_1, I_2) = \max\{|\underline{I_1} - \underline{I_2}|, |\overline{I_1} - \overline{I_2}|\},$$

where $I_1 = [\underline{I_1}, \overline{I_1}]$ and $I_2 = [\underline{I_2}, \overline{I_2}]$. It is proved in several references (such as [34]) that the metric space $(\mathscr{I}, \mathscr{D}_H)$ has the following properties:

- Complete,
- Separable,
- Locally compact.

Suppose that $I_1, I_2, I_3, I_4 \in \mathscr{I}$ and $\lambda \in \mathbb{R}$. The Hausdorff distance $\mathscr{D}_H$ has the following properties:

$$\mathscr{D}_H(I_1 + I_2, I_3 + I_4) \leq \mathscr{D}_H(I_1, I_3) + \mathscr{D}_H(I_2, I_4),$$
$$\mathscr{D}_H(\lambda I_1, \lambda I_2) = |\lambda| \mathscr{D}_H(I_1, I_2),$$

Suppose that for $I_1, I_2 \in \mathscr{I}$, there exists an element $J \in \mathscr{I}$ such that $I_1 = I_2 + J$. In this case we call J the *Hukuhara difference* or (or for simplicity H-difference) of I_1 and I_2. We denote J by $I_1 \ominus I_2$. It must be noted that $I_1 \ominus I_2 \neq I_1 + (-1) I_2$.
Now we define the concepts length and magnitude of an element in $\mathscr{I}$. Suppose that $I = [\underline{I}, \overline{I}]$ is an element of $\mathscr{I}$. The length of I is denoted by

$L(I) := \overline{I} - \underline{I}$.
We also define the magnitude of I as $\parallel I \parallel := \mathscr{D}_H(I, \{0\}) = \max\{|\overline{I}|, |\underline{I}|\}$,
It is proved that when $L(I_1) \geq L(I_2)$ then $I_1 \ominus I_2$ exists (see [36]). If $I_1, I_2, I_3, I_4 \in \mathscr{I}$, then we have the following properties ([36]):
(i) if there exist the H-differences $I_1 \ominus I_2$ and $I_1 \ominus I_3$, then we have: $\mathscr{D}_H(I_1 \ominus I_2, I_1 \ominus I_3) = \mathscr{D}_H(I_2, I_3)$;
(ii) if there exist the H-differences $I_1 \ominus I_2$ and $I_3 \ominus I_4$, then we have: $\mathscr{D}_H(I_1 \ominus I_2,\ I_3 \ominus I_4) = \mathscr{D}_H(I_1 + I_4,\ I_2 + I_3)$;

(iii) if there exist the H-differences $I_1 \ominus I_2$ and $I_1 \ominus (I_2 + I_3)$, then there exist $(I_1 \ominus I_2) \ominus I_3$ and $(I_1 \ominus I_2) \ominus I_3 = I_1 \ominus (I_2 + I_3)$;
(iv) if there exist the H-differences $I_1 \ominus I_2$, $I_1 \ominus I_3$, $I_3 \ominus I_2$, then there exists $(I_1 \ominus I_2) \ominus (I_1 \ominus I_3)$ and we have: $(I_1 \ominus I_2) \ominus (I_1 \ominus I_3) = I_3 \ominus I_2$.

We can conclude that although H-difference is unique, it does not always exist. To overcome this shortcoming, an extension of the H-difference has been introduced by the authors of [34].

Definition 1. *The generalized H-difference (gH-difference) of two intervals $u_1, u_2 \in \mathscr{I}$ is defined as follows*

$$u_1 \ominus_g u_2 = u_3 \Leftrightarrow \begin{cases} (i) u_1 = u_2 + u_3, \\ \qquad or \\ (ii) u_2 = u_1 + (-1)u_3, \end{cases} \tag{1}$$

in which $u_3 \in \mathscr{I}$.

One may choose several possible definitions of differentiability for a function with interval values. Particularization is a key idea for fuzzy sets. For interval case, in [37] this concept is extended to introduce the generalized fuzzy differentiability.

Definition 2. *Suppose that $\mathscr{F}$ is a function from interval (a,b) to $\mathscr{I}$ and $\mathfrak{x}$ belongs to (a,b). In this situation, we will suppose that $\mathscr{F}$ is strongly gH-differentiable at the given point $\mathfrak{x}$, if there exists an element $\mathscr{F}'(\mathfrak{x}) \in \mathscr{I}$, such that $\mathscr{F}'(\omega)$ satisfies in one of the conditions (i)-(iv) which are listed below:*
(i) for all positive real numbers h which are sufficiently small, $\exists \mathscr{F}(\mathfrak{x}+h) \ominus \mathscr{F}(\mathfrak{x})$, $\exists \mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}-h)$ and

$$\begin{aligned} \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}+h) \ominus \mathscr{F}(\mathfrak{x})}{h} &= \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}-h)}{h}, \\ &= \mathscr{F}'(\mathfrak{x}) \end{aligned}$$

(ii) for each positive real numbers h which are sufficiently small, $\exists \mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}+h)$, $\exists \mathscr{F}(\mathfrak{x}-h) \ominus \mathscr{F}(\mathfrak{x})$ and

$$\begin{aligned} \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}+h)}{-h} &= \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}-h) \ominus \mathscr{F}(\mathfrak{x})}{-h}, \\ &= \mathscr{F}'(\mathfrak{x}) \end{aligned}$$

(iii) for all positive real numbers h which are sufficiently small, $\exists \mathscr{F}(\mathfrak{x}+h) \ominus \mathscr{F}(\mathfrak{x})$, $\exists \mathscr{F}(\mathfrak{x}-h) \ominus \mathscr{F}(\mathfrak{x})$ and

$$\begin{aligned} \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}+h) \ominus \mathscr{F}(\mathfrak{x})}{h} &= \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}-h) \ominus \mathscr{F}(\mathfrak{x})}{-h}, \\ &= \mathscr{F}'(\mathfrak{x}) \end{aligned}$$

(iv) for each positive real numbers h which are sufficiently small, $\exists \mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}+h)$, $\exists \mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}-h)$ and

$$\begin{aligned} \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}+h)}{-h} &= \lim_{h \searrow 0} \tfrac{\mathscr{F}(\mathfrak{x}) \ominus \mathscr{F}(\mathfrak{x}-h)}{h}, \\ &= \mathscr{F}'(\mathfrak{x}). \end{aligned}$$

In [34] the gH-differentiability was proposed using the gH-difference.

Definition 3. *Suppose that $\mathfrak{x}$ belongs to the interval (c,d) and also suppose that h be so that $\mathfrak{x}+h$ belongs to (c,d), then the gH-derivative of the function $\mathscr{F}$ from the interval (c,d) to $\mathscr{I}$ (which indicates a function with interval values) can be determined as*

$$\mathscr{F}'_{gH}(t) = \lim_{h \to 0} \frac{\mathscr{F}(\mathfrak{x}+h) \ominus_g \mathscr{F}(\mathfrak{x})}{h}. \tag{2}$$

A function $\mathscr{I}$ is said to be gH-differentiable at point $\mathfrak{x}$, if there exists $\mathscr{F}'_{gH}(\mathfrak{x}) \in \mathscr{I}$ which satisfies eq. (2). We can also suppose that $\mathscr{I}$ is [(i)-gH]-differentiable at given point $\mathfrak{x}$, if $\mathscr{F}$ satisfies in Definition (2)-(i), then we can write $\mathscr{F}'_{gH}(t) = [\mathscr{F}_1'(\mathfrak{x}), \mathscr{F}_2'(\mathfrak{x})]$, in similar way, x is [(ii)-gH]-differentiable at t, if $\mathscr{F}$ satisfies in Definition (2)-(ii), then we can write $\mathscr{F}'_{gH}(\mathfrak{x}) = [\mathscr{F}_2'(\mathfrak{x}), \mathscr{F}_1'(\mathfrak{x})]$.

Consider a function $\mathscr{F}$ from the interval $[a,b]$ to $\mathscr{I}$ (which indicates a function with interval values). We say that $\mathscr{F}$ is $\mathfrak{w}$-increasing ($\mathfrak{w}$-decreasing) on $[a,b]$ if the real function $t \to \mathfrak{w}_{\mathscr{F}}(\mathfrak{x}) := \mathfrak{w}(\mathscr{F}(t))$ is increasing (decreasing) on $[a,b]$. If $\mathscr{F}$ is $\mathfrak{w}$-increasing or $\mathfrak{w}$-decreasing on $[a,b]$, then we say that $\mathscr{F}$ is $\mathfrak{w}$-monotone on $[a,b]$ (see, [31]).

Proposition 1.*(see, [19]). Suppose that $\mathscr{F} : [c,d] \to \mathscr{I}$ is such that $\mathscr{F}(\mathfrak{x}) = [\mathscr{F}_1(\mathfrak{x}), \mathscr{F}_2(\mathfrak{x})]$, for $\mathfrak{x} \in [c,d]$. If $\mathscr{F}$ is a $\mathfrak{w}$-monotone function and gH-differentiable on $[c,d]$, then $\frac{d}{d\mathfrak{x}}\mathscr{F}^-(\mathfrak{x})$ and $\frac{d}{d\mathfrak{x}}\mathscr{F}^+(\mathfrak{x})$ exist for all $\mathfrak{x} \in [c,d]$. Moreover, we have that:*

(i) If $\mathscr{F}$ is a $\mathfrak{w}$-increasing function, then for each $\mathfrak{x}$ in interval $[c,d]$, we have $\mathscr{F}'(\mathfrak{x}) = [\frac{d}{d\mathfrak{x}}\mathscr{F}_1(\mathfrak{x}), \frac{d}{d\mathfrak{x}}\mathscr{F}_2(\mathfrak{x})]$.
(ii) If $\mathscr{F}$ is a $\mathfrak{w}$-decreasing function, then for each $\mathfrak{x}$ in interval $[c,d]$, we have $\mathscr{F}'(\mathfrak{x}) = [\frac{d}{d\mathfrak{x}}\mathscr{F}_2(\mathfrak{x}), \frac{d}{d\mathfrak{x}}\mathscr{F}_1(\mathfrak{x})]$.

3 IFDEs Under Uncertainty

In this section, we will obtain the exact solution for the following IFDE:

$$\mathfrak{D}^{\alpha} y(x) = Ay(x) + Bu(x) + \sum_{j\in J} c_j \delta(x - x_j), \tag{3}$$

where $0 < \alpha \leq 1$, $y : \mathbb{R}_+ \to \mathscr{I}^n$, $y : \mathbb{R}_+ \to \mathscr{I}^m$, A,B are constant matrices and $c_j \in \mathscr{I}^n$ are fixed interval vectors. $\mathfrak{D}$ denotes the Riemann-Liouville derivative under interval uncertainty as described in [31].
Suppose that Eq. (3) has the following initial condition:

$$\mathfrak{I}^{1-\alpha} y(x)|_{x=0^+} = y_0, \tag{4}$$

in which $\mathfrak{I}$ is the fractional integral under interval uncertainty as presented in [31].
Now, we study the uniqueness conditions of the solution of Eq. (3) under interval Riemann-Liouville differentiability.

Lemma 1.*The IFDE(3) with the initial condition (4) has a unique solution as follows:*
If $y(x)$ is w-increasing, then the unique solution of the Eqs. (3)-(4) will be as:

$$y(x) = x^{\alpha-1} E_{\alpha,\alpha}(Ax^{\alpha}) y_0 + \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}\left[A(x-t)^{\alpha}\right] Bu(t)dt +$$
$$\sum_{j\in J} \chi(x-x_j)(x-x_j)^{\alpha} E_{\alpha,\alpha}\left[A(x-x_j)^{\alpha}\right] c_j,$$

in which $E_{\alpha,\alpha}$ is the Mittag-Leffler function.

*Proof.*Applying the Laplace transforms to both sides of (3), we have:

$$s^{\alpha} Y(s) \ominus y_0 = AY(s) + BU(s) + \sum_{j\in J} e^{-x_i s} c_j,$$

or equivalently

$$\begin{cases} s^{\alpha} Y_1(s) - y_{0,1} = AY_1(s) + BU_1(s) + \sum_{j\in J} e^{-x_j s} c_{j,1}, \\ s^{\alpha} Y_2(s) - y_{0,2} = AY_2(s) + BU_2(s) + \sum_{j\in J} e^{-x_j s} c_{j,2}, \end{cases}$$

where $Y(s) = [Y_1(s), Y_2(s)]$, and all the matrices are assumed to be positive. Then,

$$\begin{cases} Y_1(s) = (s^{\alpha} I - A)^{-1} y_{0,1} + (s^{\alpha} I - A)^{-1} BU_1(s) + \sum_{j\in J} e^{-x_j s}(s^{\alpha} I - A)^{-1} c_{j,1}, \\ Y_2(s) = (s^{\alpha} I - A)^{-1} y_{0,2} + (s^{\alpha} I - A)^{-1} BU_2(s) + \sum_{j\in J} e^{-x_j s}(s^{\alpha} I - A)^{-1} c_{j,2}. \end{cases}$$

Therefore, using the fact that

$$L\{t^{\beta-1} E_{\alpha,\beta}(At^{\alpha})\} = s^{\alpha-\beta}(s^{\alpha} I - A)^{-1},$$

and the time shifting and convolution properties, we obtain the result. Indeed, the uniqueness of the solution is trivial (we conclude it using the uniqueness of the Laplace transform).

*Remark.*When $\alpha = 1$, the solution of problem (3) has the following form:

$$y(x) = e^{Ax} y_0 + \int_0^x e^{A(x-t)} Bu(t)dt + \sum_{j\in J} e^{A(x-x_j)} c_j.$$

4 Example

In this part, the Bagley-Torvik equation is solved under interval uncertainty based on the interval Riemann-Liouville derivative.

Example 1. Now, consider the Bagley-Torvik equation which involves additive delta function under interval uncertainty as:

$$ay''(x) + b\mathfrak{D}^{\frac{3}{2}}y(x) + cy(x) = u(x) + d\delta(x - x_1). \tag{5}$$

Using the following notations

$$\begin{aligned}
\mathscr{V}_1(x) &= y(x),\\
\mathscr{V}_2(x) &= \mathfrak{D}^{\frac{1}{2}}y(x),\\
\mathscr{V}_3(x) &= y'(x),\\
\mathscr{V}_4(x) &= \mathfrak{D}^{\frac{3}{2}}y(x).
\end{aligned}$$

Then, Eq. (5) reduces to the following uncertain system:

$$\mathfrak{D}^{\frac{1}{2}}\mathscr{V} = A\mathscr{V} + BU + C\delta(x - x1),$$

where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{c}{a} & 0 & 0 & -\frac{b}{a} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{a} \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{d}{a} \end{bmatrix}$ and $\mathscr{V} = \begin{bmatrix} \mathscr{V}_1 \\ \mathscr{V}_2 \\ \mathscr{V}_3 \\ \mathscr{V}_4 \end{bmatrix}$ under the initial conditions $y_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Using Lemma (1), the solution is achieved as:

$$\mathscr{V}(x) = \int_0^x \frac{1}{\sqrt{x-t}} E_{\frac{1}{2},\frac{1}{2}}(A\sqrt{x-t})Bu(t)dt + \frac{\chi(x-x_1)}{\sqrt{x-x_1}} E_{\frac{1}{2},\frac{1}{2}}(A\sqrt{x-t})C$$

5 Conclusion

In this paper we developed some conditions for the uniqueness of the IFDEs which was illustrated in [33] under interval uncertainty based on the Riemann-Liouville differentiability. The results were tested by solving the interval Bagley-Torvik equation involving additive delta function to depict the correctitude of the approach.

Conflict of Interest

The authors declare that they have no conflict of interest.

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