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RBF-PS Scheme for the Numerical Solution of the Complex Modified Korteweg-de Vries equation

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Abstract: RBF-Pseudospectral is used for the numerical solution of complex modified Kortewege-de Vries (CmKdV) equation. The numerical scheme is fast and accurate. There is no linearization of the nonlinear terms. The scheme is tested for single solitary wave, two and three solitary waves interaction. The results of the numerical scheme are compared with other meshless method of lines and the earlier work.

Keywords: Meshless Method, CmKdV Equation, RBF-PS Scheme

1 Introduction

During the last two decades the meshless methods have been developed and effectively applied to solve many engineering and science problems [1-4]. There is a class of meshless methods that focus on the use of radial basis functions [5], such as radial basis function collocation method (RBFCM) [6-10]. The radial basis functions (RBFs) have been under intensive research in multivariate data and Kansa used them for scattered data approximation in [7] and pioneered the solution of PDEs [8], that is why the method is some time called the Kansa's method. The key point of the Kansa's method for solving the PDEs is the approximation of the fields on the boundary and in the domain by a set of global approximation functions. The convergence theory of Kansa's approach was provided by Schaback [11]. The main advantage of using the RBFCM for solution of PDEs is its simplicity, applicability to various PDEs, and effective-ness in dealing with high dimensional problems and complicated domains. The main disadvantage of RBFCM represents the related full matrices that are very sensitive to the choice of the free parameter in RBFs and difficult to solve for problems with a large number of unknowns. This is because the use of the radial basis function interpolation increases the condition numbers of the related matrices with increasing number of nodes. This is especially true for a bad choice of data centers and when infinitely smooth basic functions such as multiquadrics are used with extreme values of their associated shape parameter. There are several methods to circumvent this issue such as domain decomposition [12, 13] the greedy algorithm [14, 15], etc. One of the possibilities for mitigating computational cost for large-scale problems is to employ the domain decomposition by Mai-Duy and Tran-Cong [16], multi-grid approach and compactly supported RBFs by Chen et al. [5] in 2002.

G. Fasshauer [17] connected the radial basis functions collocation method to the pseudo-spectral method, known as RBF-PS method. G. Fasshauer used RBF-PS method for solving Allen-Cahn equation, 2D Helmholtz equation and 2D Laplace equation with piecewise boundary conditions [2]. Ferreira et al. [18, 19] used RBF-PS method for solving beams, plates and shells problems. Roque et al. [20,21] applied RBF-PS method for composite and sandwich plates problems.

In the present work we extended the approach of G. Fasshauer [17] and developed a kernel based meshless scheme for the complex modified Korteweg-de Vries (CmKdV). Given a set of centers $\{x_1, x_2, ..., x_N\} \subseteq \Omega$. The RBF approximation of a function $u(x,t)$ takes the form

$$u(x,t) = \sum_{j=1}^{N} \lambda_j(t) K(x, x_j), \; x \in \Omega, \qquad (1)$$

where the radial kernels $K(x, x_j)$ are defined by $K(x,x_j) : \varphi(\|x - x_j\|), 1 \le j \le N$, and $r = \|x - x_j\|$

denote the Euclidean distance between two points x and x_j and $\varphi(r)$ is a function defined for $r \geq 0$. Interpolation of the function $u:\Omega \to R$ on the set of points $\{x_1, x_2, ..., x_N\}$ is done by solving the system of equations

$$u(x_i,t) = \sum_{j=1}^{N} \lambda_j(t) K(x_i, x_j), 1 \leq i \leq N, \quad (2)$$

In matrix form, we have

$$u = A\lambda, \quad (3)$$

where the entries of the matrix A are $K(x_i, x_j), 1 \leq i, j \leq N,$ and vector expansion coefficients is $\{\lambda_1, \lambda_2, ..., \lambda_N\}^T$.

Using equation (1) the derivatives u_x may be obtained by differentiating the kernel functions and then evaluating at each point x_i, we have in matrix-vector notation

$$u_x = A_x \lambda, \quad (4)$$

where the entries of the matrix A_x

$$\frac{d}{dx} K(x, x_j)_{x=x_i}, 1 \leq i, j \leq N.$$

The differentiation matrix can be obtained by solving equations (5)-(6) for the value of λ. Thus, we have

$$u_x = A_x A^{-1} u = D_x u, \quad (5)$$

where $D_x = A_x A^{-1}$ is the differentiation matrix. It should be noted that the differentiation matrix depends on the invertibility of the matrix A. It is well known that the matrix A is always invertible for distinct set of collocation points. In a similar way, we can write

$$u_{xx} = A_{xx} A^{-1} u = D_{xx} u, \quad (6)$$

Where $D_{xx} = A_{xx} A^{-1}$, and the entries of matrix A_{xx} are $\frac{d^2}{dx^2} K(x, x_j)_{x=x_i}, 1 \leq i, j \leq N.$

Similarly we can compute differentiation matrices of higher order.

2 RBF-PS scheme for CmKdV equation

The numerical solution of nonlinear wave equations has been the subject of many studies in recent years. There is little numerical analysis literature for non integrable wave equations. The complex modified Korteweg-de Vries (CmKdV) equation is known as non integrable equation. Only a few analytical solutions corresponding to some special cases of CmKdV equation are available. The complex modified Korteweg-de Vries (CmKdV) equation given by

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} + \alpha \frac{\partial}{\partial x}(|w|^2 w) = 0, -\infty < x < \infty, t > 0, \quad (7)$$

where w is complex valued function of the spatial coordinate x and time t, α is a real parameter. Equation (1) has an exact solitary wave solution [22]

$$w(x,t) = \sqrt{\frac{2C}{\alpha}}\, \text{sech}[\sqrt{C}(x - x_0 - Ct)] \exp(i\theta_0) \quad (8)$$

This solitary wave is centered at x_0 and moving to the right with velocity C. CmKdV equation is a model for the propagation of transverse waves in a molecular chain model [23]. The CmKdV equation (7) has been solved numerically by Taha [24], M. S. Ismail [22, 26], G. M. Muslu [23], M. Uddin et al. [10] and analytically by Wazwaz [26]. The CmKdV equation is transformed into nonlinear coupled equations by decomposing w into its real and imaginary parts i.e.

$$w(x,t) = u(x,t) + iv(x,t), i^2 = -1, \quad (9)$$

where $u(x,t)$ and $v(x,t)$ are real functions. As a result we obtained the following coupled pair of equations

$$u_t + u_{xxx} + \alpha[(3u^2 + v^2)u_x + 2uvv_x] = 0, \quad (10)$$

$$v_t + v_{xxx} + \alpha[(3v^2 + u^2)v_x + 2uvu_x] = 0, \quad (11)$$

In vector form the system of equations (10)-(11) may be written as

$$w_t + w_{xxx} + G(u,v) w_x = 0, \quad (12)$$

where $w = [u, v]^t$,

$$G = \alpha \begin{bmatrix} 3u^2 + v^2 & 2uv \\ 2uv & u^2 + 3v^2 \end{bmatrix},$$

Using the above differentiation matrices, the kernel-based meshless schemes corresponding to equations (10)-(11) are given as

$$\frac{du}{dt}=-D_{xxx}u-\alpha[(3u^2+v^2)*D_xu+2uv*D_xv] \quad (13)$$

$$\frac{dv}{dt}=-D_{xxx}v-\alpha[(3v^2+u^2)*D_xv+2uv*D_xu] \quad (14)$$

$$\frac{dw}{dt}=-L_{xxx}w-G(u,v)*L_xw, \quad (15)$$

Where L_x and L_{xxx} are the corresponding differentiation matrices and G and w are defined in (12). In more compact form we have

$$\frac{dw}{dt}=Dw, \quad (16)$$

where $D=-L_{xxx}-G(u,v)*L_x$.

The scheme (16) is the ODE system generated by the meshless method of lines. For time integration we can use any ODE solver. In our computations we used Runge-Kutta method of order four.

In the present scheme the differentiation matrices D_x, and D_{xxx} are computed only once outside the time-stepping procedure. Inside the time-stepping we require only matrix-vector multiplications. So this approach is much faster than the approach used in [27-29, 10], where the interpolation coefficients are computed at each time-step. We can choose any differential kernel function which is decaying towards infinity. To keep the matrices A, D_x and D_{xxx} sparse, we can use compactly supported radial kernel functions. A radial kernel is given as

$$K(x,x_j):\varphi(\|x-x_j\|),1\le j\le N.$$

The are a variety of radial kernel functions in the literature. In our computation we used the multiquadric $\varphi(r)=\sqrt{r^2+c^2}$ a globally supported radial kernel and the Wendland's

$$\varphi_{3,2}(r)=(1-cr)_+^6(35(cr)^2+18cr+3)$$

a compactly supported radial kernel. As usual these kernel functions contain a shape parameter c. An algorithm for finding optimal values of c is proposed by G. Fasshauer [2] for such type of meshless method of lines.

3 Stability of the scheme

In the present technique the time-depend PDE is transformed into a system of ODEs in time. The method of lines refers to the idea of solving this coupled system of ODEs by a finite difference formula in t (e.g Runge-Kutta). The numerical stability of the method of lines is investigated by the Rule of Thumb, which is given by

The method of lines is stable if the eigenvalues of the (linearized) spatial discretization operator, scaled by δt, *lie in the stability region of the time-discretization operator* [30]. The stability region is the subset of complex plane consisting of those eigenvalues for which the technique produces bounded solution.

4 Numerical results

In this section the meshless method of lines is applied for the numerical solution of CmKdV equation. The accuracy of the method is tested in terms of the error norms and the three invariants given by

$$I_1=\int_{-\infty}^{\infty}w\,dx,$$

$$I_2=\int_{-\infty}^{\infty}|w|^2dx,$$

$$I_3=\int_{-\infty}^{\infty}\left(\frac{\alpha}{2}|w|^4-|w_x|^2\right)dx, \quad (17)$$

Single soliton: We consider equation (7) with the initial condition

$$w(x,0)=\sqrt{\frac{2C}{\alpha}}\operatorname{sech}[\sqrt{C}(x-x_0)]\exp(i\theta_0)$$

The problem is solved in spatial domain $-20\le x\le 40$, and time domain $[0,20]$. The two radial kernels the Hardy's MQ and the Wendland's $\varphi_{3,2}$ are used in this computations, other radial kernels produced the same results. For time integration RK4 method is used. The parameters $\alpha=2, C=1, \delta t=0.0001, N=600, x_0=0, (MQ)_c=0.5$, and $(\varphi_{3,2})_c=0.22$ are used in this computations.

1- Corresponding to $\theta_0=\pi/4$, the results are given in Table 1. In comparison the present scheme produced better results than the other numerical schemes [22, 25]. The motion of solitary wave is shown in Figure 1.

2- When $\theta_0 = 0$, then we are left only with the real part u . The results are listed in Table 2. In comparison the results of the present scheme are better than those given in [22,10].

MQ	t	L_∞	L_2	RMS	I_1	I_2	I_3
	5	1.067×10^{-5}	5.378×10^{-5}	2.196×10^{-6}	3.13642	1.99667	0.66556
	10	1.063×10^{-5}	5.890×10^{-5}	2.404×10^{-6}	3.13643	1.99667	0.66556
	15	1.534×10^{-5}	7.314×10^{-5}	2.980×10^{-6}	3.13642	1.99667	0.66556
	20	2.412×10^{-5}	9.065×10^{-5}	3.701×10^{-6}	3.13645	1.99667	0.66556
$\varphi_{3,2}$	5	1.988×10^{-5}	9.385×10^{-5}	3.831×10^{-6}	3.13637	1.99667	0.66556
	10	3.676×10^{-5}	1.833×10^{-4}	7.482×10^{-6}	3.13637	1.99667	0.66556
	15	5.505×10^{-5}	2.751×10^{-4}	1.123×10^{-6}	3.13637	1.99667	0.66556
	20	7.242×10^{-5}	3.661×10^{-4}	1.495×10^{-6}	3.13637	1.99667	0.66556
[22]	20	1.490×10^{-4}	-	-	3.14160	1.99999	0.66976
[25]	20	3.080×10^{-4}	-	-	-	1.99999	-

Table 1: Single soliton: $\alpha = 2, C = 1, \theta_0 = \pi/4, (MQ)_c = 0.5, (\varphi_{3,2})_c = 0.2, \delta t = 0.0001, N = 600$. corresponding to (1).

MQ	t	L_∞	L_2	RMS	I_1	I_2	I_3
	5	1.067×10^{-5}	5.378×10^{-5}	2.196×10^{-6}	3.13642	1.99667	0.66556
	10	1.063×10^{-5}	5.890×10^{-5}	2.404×10^{-6}	3.13643	1.99667	0.66555
	15	1.534×10^{-5}	7.314×10^{-5}	2.986×10^{-6}	3.13642	1.99667	0.66556
	20	2.412×10^{-5}	9.065×10^{-5}	3.701×10^{-6}	3.13645	1.99667	0.66556
$\varphi_{3,2}$	5	1.988×10^{-5}	9.385×10^{-5}	3.831×10^{-6}	3.13637	1.99667	0.66556
	10	3.676×10^{-5}	1.833×10^{-4}	7.482×10^{-6}	3.13637	1.99667	0.66556
	15	5.505×10^{-5}	2.751×10^{-4}	1.123×10^{-5}	3.13637	1.99667	0.66556
	20	7.242×10^{-5}	3.661×10^{-4}	1.495×10^{-5}	3.13637	1.99667	0.66556
[22]	20	2.180×10^{-4}	-	-	3.14159	1.99999	0.66976
[10]	20	2.731×10^{-4}	1.572×10^{-5}	-	3.14159	1.99999	0.66666

Table 2: Single soliton: $\alpha = 2, C = 1, \theta_0 = \pi/4, (MQ)_c = 0.5, (\varphi_{3,2})_c = 0.2, \delta t = 0.0001, N = 600$. corresponding to (1).

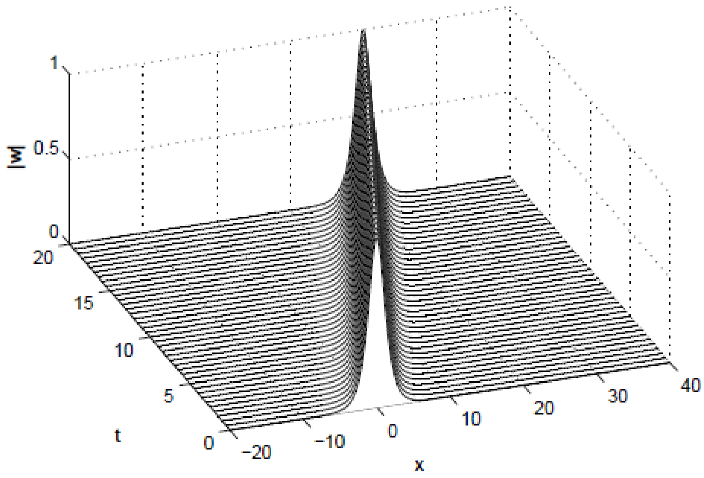

Figure 1: Interaction of two solitary waves: $\alpha = 2$, $C_1 = 2$, $C_2 = 0.5$, $\theta_1 = 0$, $\theta_2 = \pi/2$, $\delta t = 0.0001$, $N = 500$.

Two soliton interaction: Now we consider interaction of two solitary waves, with the initial condition of the form

$$w(x,0) = \sum_{j=1}^{2} \sqrt{\frac{2C_j}{\alpha}} \operatorname{sec}h[\sqrt{C_j}(x - x_j)]\exp(i\theta_j) \tag{19}$$

This problem is solved in $0 \le x \le 100$, over time $[0,30]$. The Wendland's $\varphi_{3,2}$ compactly supported kernels function is used in this computations. For time integration RK4 method is used. The parameters $\alpha = 2$, $C_1 = 2$, $C_2 = 0.5$, $\delta t = 0.0001$, $N = 500$, $x_1 = 25$, $x_2 = 50$, and $(\varphi_{3,2})_c = 0.22$ are used in this computations.

1- We examined the interaction of orthogonally polarized waves when $\theta_1 = 0$, $\theta_2 = \pi/2$. The results are fully agreed with [25] and are shown in Figure 2.
2- Next we studied the interaction of two y-polarized solitary waves when $\theta_1 = 0$, and $\theta_2 = 0$. Here again the results are shown in Figure 3, and are agreed with [25].
3- Finally we consider the interaction of two solitary waves when $\theta_1 = \pi/4$, $\theta_2 = \pi/4$. These results are displayed in Figure 4, and is agreed with [25].

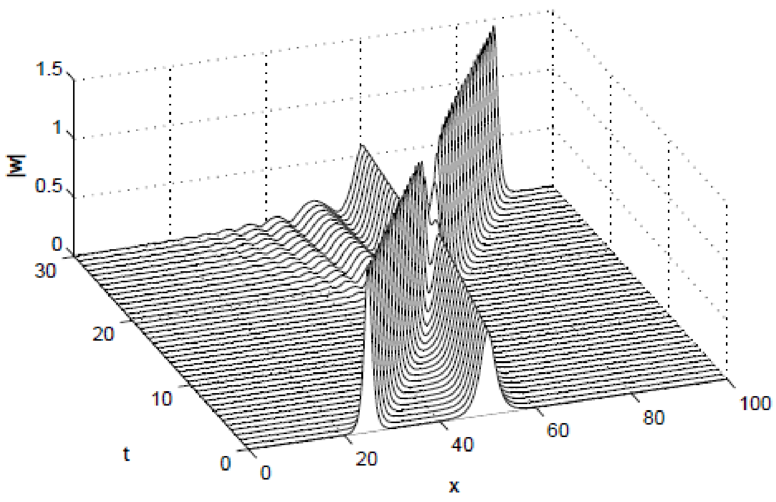

Figure 2: Interaction of two solitary waves:
$\alpha = 2,\ C_1 = 2,\ C_2 = 0.5,\ \theta_1 = 0,\ \theta_2 = \pi/2,\ (\varphi_{3,2})_c = 0.22,\ N = 500,\ \delta t = 0.0001,$

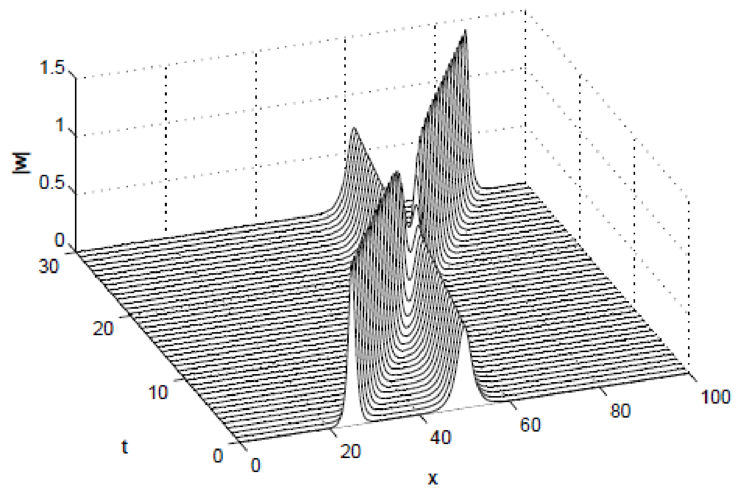

Figure 3. Interaction of two solitary waves:
$\alpha = 2,\ C_1 = 2,\ C_2 = 0.5,\ \theta_1 = 0,\ \theta_2 = 0,\ (\varphi_{3,2})_c = 0.22,\ N = 500.$

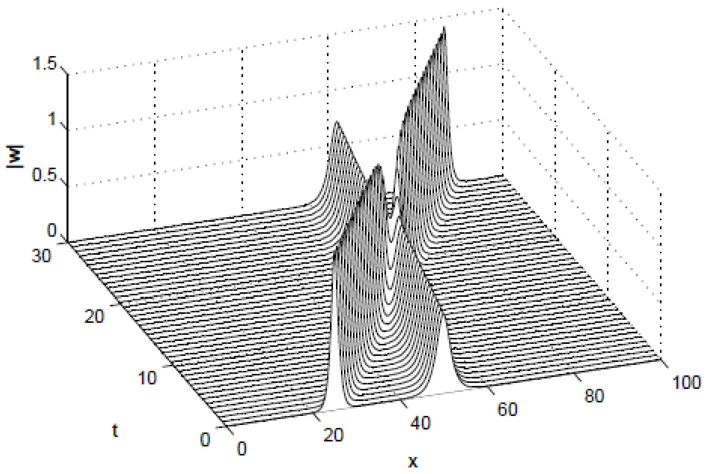

Figure 4. Interaction of two solitary waves:
$\alpha = 2,\ C_1 = 2,\ C_2 = 0.5,\ \theta_1 = \pi/4,\ \theta_2 = \pi/4,\ (\varphi_{3,2})_c = 0.22,\ N = 500.$

Three soliton interaction: Finally we consider the interaction of three solitary waves, with the initial condition

$$w(x,0) = \sum_{j=1}^{3} \sqrt{\frac{2C_j}{\alpha}}\, \mathrm{sech}[\sqrt{C_j}(x - x_j)] \exp(i\theta_j) \tag{20}$$

We solved the problem in spatial domain $0 \le x \le 100,$ over time $[0,30].$ The Wendland's $\varphi_{3,2}$ compactly supported kernels function is used. RK4 scheme is used for time integration. The parameters
$\alpha = 2,\ C_1 = 1,\ C_2 = 0.5,\ C_3 = 0.3,\ \delta t = 0.0001,\ N = 500,\ x_1 = 10,\ x_2 = 30,$
$x_3 = 50,\ (\varphi_{3,2})_c = 0.22,\ \theta_i = \pi/4, i = 1,2,3.$
are used in this computations. The results are shown in Figure 5, and the scheme successfully resolved the motion and interaction of the three solitary waves.

5 Concluding remarks

In this paper, RBF-PS scheme is used for the approximate solution of complex modified Korteweg-de Vries equation. The present scheme is much faster and accurate than the other meshless method of lines [10]. The present scheme performed well in terms of accuracy and robustness. To reduce computation time further we
used Wendland's compactly supported function. The technique used in this paper provides an efficient alternative for the solution of such type of nonlinear partial differential equations.

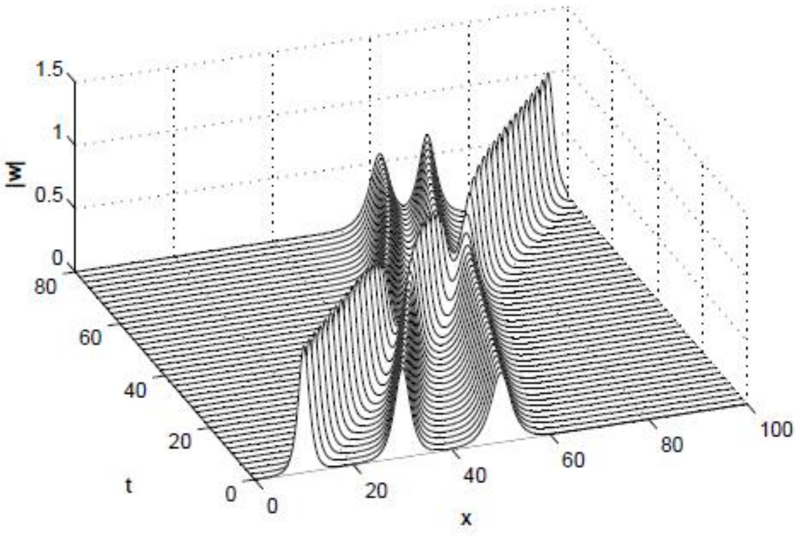

Figure 5. Interaction of three solitary waves:
$\alpha = 2,\ C_1 = 2,\ C_2 = 0.5,\ C_3 = 0.3,\ \theta_1 = \pi/4,\ \theta_2 = \pi/4,\ \theta_3 = \pi/4,\ (\varphi_{3,2})_c = 0.22,\ N = 500.$

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