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An Extension of Pareto Distribution

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Abstract: In this paper, a new generalized version of the Pareto distribution, which is called cubic transmuted Pareto distribution (CTPD), is introduced and studied. Most of the mathematical and statistical properties are studied and the model parameters are estimated by the maximum likelihood method. Finally, an application of CTPD to some real data sets and compare it with some distributions based on Pareto distribution is illustrated.
Keywords: Pareto distribution; Cubic transmutation; Maximum likelihood estimation; Order statistic; Hazard function.

1 Introduction

The Pareto distribution is named after an Italian-born Swiss professor of economics, Vilfredo Pareto (1848-1923). It is used in modelling the distribution of incomes and other financial variables, and in the description of social and other phenomena. More recently, attempts have been made to explain many empirical phenomena using the Pareto distribution or some closely related form. For more detail see [5]. Shaw and Buckley [9] proposed the transmutation maps to solve financial mathematics problems. Merovci and Puka [6] studied the transmuted Pareto distribution using the quadratic rank transmutation map studied by Shaw and Buckley [9]. Salma [8] developed L-Quadratic (LQ) distribution by generalizing U-Quadratic distribution using the quadratic rank transmutation map. Granzottoa *et al*. [4] developed the cubic ranking transmutation map, or the transmuted distributions of order 2. This new parametric family offers tractable distributions and is able to fit complex data sets such as ones with bimodal distribution or bimodal hazard rates. They focused on the cubic ranking on Weibull and log-logistic distributions. Al-Kadim and Mohammed [1] proposed cubic transmuted Weibull distribution (CTWD), and discussed some of its statistical properties.
In this article we use cubic ranking transmutation map suggested by Al- Kadim [2] to propose a new model which generalizes the Pareto model. This new version of the Pareto distribution called cubic transmuted Pareto distribution (CTPD). Most of the mathematical and statistical properties are studied and the model parameters are estimated by the maximum likelihood method. Moreover, an application to some real data sets is illustrated.

1.1 Pareto Distribution

Let X be a random variable with the Pareto distribution. The probability density function (pdf) and the cumulative distribution function (cdf) are defined, respectively, as

$$\mathrm{g}(x) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}} \tag{1}$$

and

$$\mathrm{G}(x) = 1 - \left(\frac{x_0}{x}\right)^\alpha, \quad \alpha > 0, \qquad x \geq x_0 \tag{2}$$

where x_0 is the (necessarily positive) minimum possible value of X, and α is a shape parameter.

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1.2 Cubic Ranking Transmutation Map

A random variable X is said to have a cubic transmuted distribution Al- Kadim [2], if its cdf is given by

$$F(x) = (1+\lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)\,, \qquad |\lambda| \le 1 \tag{3}$$

and the pdf is given by

$$f(x) = (1+\lambda)\mathrm{g}(x) - 4\lambda \mathrm{g}(x)\mathrm{G}(x) + 3\lambda G^2(x)\mathrm{g}(x) \tag{4}$$

where $\mathrm{G}(x)$ is the cdf of the base distribution. Observe that at $\lambda = 0$, $F(x) = \mathrm{G}(x)$.

The rest of this paper is organized as follows. The new proposed distribution, the CTPD and its mathematical and statistical properties are presented in Section 2. An application of the CTPD to two real datasets for the purpose of illustration is conducted in Section 3. Finally, in Section 4, we present some concluding remarks.

2 Cubic Transmuted Pareto Distribution

In this Section, the new proposed distribution, CTPD and its mathematical and statistical properties are demonstrated.

Theorem 2.1. Let X be a random variable with the cubic transmuted Pareto distribution. The cdf and pdf are defined, respectively, as

$$F(x) = 1 - \left(\frac{x_0}{x}\right)^{\alpha}\left(1 - \lambda\left(\frac{x_0}{x}\right)^{\alpha} + \lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right), \qquad \alpha > 0, \qquad x \ge x_0, \ \ |\lambda| \le 1 \tag{5}$$

and

$$f(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right) \tag{6}$$

Proof. Part 1.

Consider the cdf of cubic transmuted distribution of Eq. (3) namely

$$\begin{aligned} F(x) &= (1+\lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x) \\ &= G(x)[(1+\lambda) - 2\lambda G(x) + \lambda G^2(x)] \end{aligned} \tag{7}$$

Substitute $G(x)$ of Eq. (2) into Eq. (7), we get

$$\begin{aligned} F(x) &= \left(1 - \left(\frac{x_0}{x}\right)^{\alpha}\right)\left[(1+\lambda) - 2\lambda\left(1 - \left(\frac{x_0}{x}\right)^{\alpha}\right) + \lambda\left(1 - \left(\frac{x_0}{x}\right)^{\alpha}\right)^2\right] \\ &= \left(1 - \left(\frac{x_0}{x}\right)^{\alpha}\right)\left[(1+\lambda) - 2\lambda\left(1 - \left(\frac{x_0}{x}\right)^{\alpha}\right) + \lambda\left(1 + \left(\frac{x_0}{x}\right)^{2\alpha} - 2\left(\frac{x_0}{x}\right)^{\alpha}\right)\right] \\ &= \left(1 - \left(\frac{x_0}{x}\right)^{\alpha}\right)\left[1 + \lambda - 2\lambda + 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + \lambda + \lambda\left(\frac{x_0}{x}\right)^{2\alpha} - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha}\right] \\ &= \left(1 - \left(\frac{x_0}{x}\right)^{\alpha}\right)\left(1 + \lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right) \\ &= 1 - \left(\frac{x_0}{x}\right)^{\alpha} + \lambda\left(\frac{x_0}{x}\right)^{2\alpha} - \lambda\left(\frac{x_0}{x}\right)^{3\alpha} \\ &= 1 - \left(\frac{x_0}{x}\right)^{\alpha}\left(1 - \lambda\left(\frac{x_0}{x}\right)^{\alpha} + \lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right) \end{aligned}$$

Hence

$$F(x) = 1 - \left(\frac{x_0}{x}\right)^{\alpha}\left(1 - \lambda\left(\frac{x_0}{x}\right)^{\alpha} + \lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)$$

and this complete the proof.

Proof. Part 2.

To prove part 2 we use the following

$$f(x) = \frac{d\,F(x)}{dx}$$

Recall Eq. (5) namely

$$\begin{aligned} F(x) &= 1 - \left(\frac{x_0}{x}\right)^{\alpha}\left(1 - \lambda\left(\frac{x_0}{x}\right)^{\alpha} + \lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right) \\ &= 1 - \left(\frac{x_0}{x}\right)^{\alpha} + \lambda\left(\frac{x_0}{x}\right)^{2\alpha} - \lambda\left(\frac{x_0}{x}\right)^{3\alpha} \end{aligned}$$

Directly

$$\frac{d\,F(x)}{dx} = \alpha x_0^{\alpha} x^{-\alpha-1} - 2\lambda\alpha x_0^{2\alpha} x^{-2\alpha-1} + 3\lambda\alpha x_0^{3\alpha} x^{-3\alpha-1}$$

$$= \alpha x_0^{\alpha} x^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)$$

Hence

$$f(x) = \alpha x_0^{\alpha} x^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)$$

and this completes the proof.

The shape of pdf with selected parameter values are shown in Figure 1 and Figure 2.

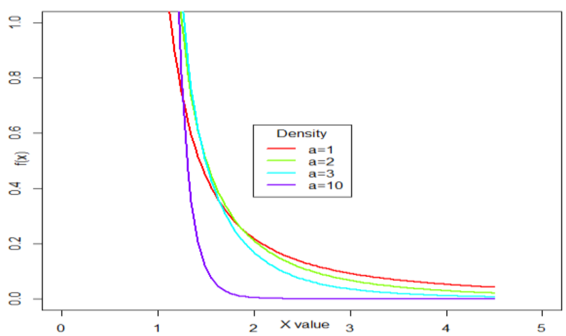

Fig.1. The pdf of the CTPD at $\alpha = 1,2,3,100$, $\lambda = 1$

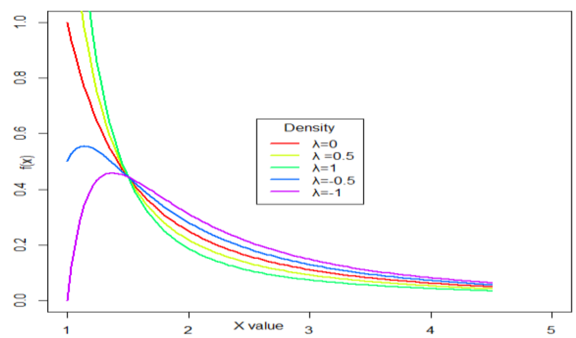

Fig. 2. The pdf of CTPD at $\alpha = 1$, λ=0, 0.5, 1, -0.5, -1.

Remark 1. To prove that $f(x)$ in Eq. (6) is pdf we need to prove $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $f(x) \geq 0$.

2.1 Limits of the Probability Density and Distribution Functions

Proposition 2.1. The limit of the density function $f(x)$ for CTPD as $x \to x_0$ is $\frac{\alpha(1+\lambda)}{x_0}$ and the limit as $x \to \infty$ is 0.

Proof.

$$\lim_{x\to x_0} f(x;\alpha,x_0,\lambda) = \lim_{x\to x_0}\left[\alpha x_0^{\alpha} x^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]$$

$$= \alpha x_0^{\alpha} x_0^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{x_0}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x_0}\right)^{2\alpha}\right)$$

$$= \frac{\alpha}{x_0}\,(1 - 2\lambda + 3\lambda)$$

$$= \frac{\alpha(1+\lambda)}{x_0}$$

Since $\alpha > 0, x_0 > 0, |\lambda| < 1$ then $\frac{\alpha(1+\lambda)}{x_0} > 0$.

and

$$\lim_{x\to\infty} f(x;\alpha,x_0,\lambda) = \lim_{x\to\infty}\left[\alpha x_0^{\alpha} x^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]$$

$$= \alpha x_0^{\alpha}\infty^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{\infty}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{\infty}\right)^{2\alpha}\right)$$

$$= 0$$

(1) Proof of $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{x_0}^{\infty} f(x;\alpha,x_0,\lambda)\,dx = \int_{x_0}^{\infty}\left[\alpha x_0^{\alpha}x^{-\alpha-1}\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]dx$$
$$= \int_{x_0}^{\infty}[(\alpha x_0^{\alpha}x^{-\alpha-1} - 2\alpha\lambda x_0^{2\alpha}x^{-2\alpha-1} + 3\alpha\lambda x_0^{3\alpha}x^{-3\alpha-1})]\,dx$$
$$= \alpha\left[x_0^{\alpha}\left(\frac{x^{-\alpha}}{-\alpha}\right)\right]_{x_0}^{\infty} - 2\lambda\alpha x_0^{2\alpha}\left[\left(\frac{x^{-2\alpha}}{-2\alpha}\right)\right]_{x_0}^{\infty} + 3\lambda\alpha x_0^{3\alpha}\left[\left(\frac{x^{-3\alpha}}{-3\alpha}\right)\right]_{x_0}^{\infty}$$
$$= [-x_0^{\alpha}(x^{-\alpha})]_{x_0}^{\infty} + \lambda x_0^{2\alpha}[(x^{-2\alpha})]_{x_0}^{\infty} - \lambda x_0^{3\alpha}[(x^{-3\alpha})]_{x_0}^{\infty}$$
$$= [-x_0^{\alpha}(0-x_0^{-\alpha})] + \lambda x_0^{2\alpha}[(0-x_0^{-2\alpha})] - \lambda x_0^{3\alpha}[0-x_0^{-3\alpha}]$$
$$= 1+\lambda-\lambda = 1$$

(2) Proof of $\mathbf{f(x) \geq 0}$

$$f(x) = \alpha x_0^{\alpha}x^{-\alpha-1}\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)$$
$$= \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}\left(1-\frac{2\lambda x_0^{\alpha}}{x^{\alpha}}+\frac{3\lambda x_0^{2\alpha}}{x^{2\alpha}}\right)$$

now $f(x_0) = \frac{\alpha(1+\lambda)}{x_0} \geq 0$; $\alpha > 0, x_0 > 0, |\lambda| \leq 1$ and $f(\infty) = 0$, implies that $f(x) \geq 0$.

From 1 and 2 we conclude that $f(x)$ is pdf.

Proposition 2.2 .The limit of the distribution function $F(x)$ for CTPD as $x \to x_0$ is 0 and the limit as $x \to \infty$ is 1.

Proof.

$$\lim_{x\to x_0} F(x;\alpha,x_0,\lambda) = \lim_{x\to x_0}\left[1-\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}-\lambda\left(\frac{x_0}{x}\right)^{3\alpha}\right]$$
$$= 1-\left(\frac{x_0}{x_0}\right)^{\alpha}+\lambda\left(\frac{x_0}{x_0}\right)^{2\alpha}-\lambda\left(\frac{x_0}{x_0}\right)^{3\alpha}$$
$$= (1-1+\lambda-\lambda)$$
$$= 0$$

and

$$\lim_{x\to\infty} F(x;\alpha,x_0,\lambda) = \lim_{x\to\infty}\left[1-\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}-\lambda\left(\frac{x_0}{x}\right)^{3\alpha}\right]$$
$$= 1-\left(\frac{x_0}{\infty}\right)^{\alpha}+\lambda\left(\frac{x_0}{\infty}\right)^{2\alpha}-\lambda\left(\frac{x_0}{\infty}\right)^{3\alpha} = 1$$

Hence $F(x)$ is a cumulative distribution function.

Reliability Analysis

2.1.1 Reliability Function

Proposition 2.3. The reliability function of CTPD is given as

$$R(x) = \left(\frac{x_0}{x}\right)^{\alpha}\left(1-\lambda\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right) \tag{8}$$

Proof.

The reliability function for a given distribution is

$$R(x) = 1 - F(x)$$

Substitute the cdf of CTPD of Eq. (5) in Eq. (8) we get

$$R(x) = 1-\left[1-\left(\frac{x_0}{x}\right)^{\alpha}\left(1-\lambda\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]$$
$$= \left(\frac{x_0}{x}\right)^{\alpha}\left(1-\lambda\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)$$

2.1.2 Hazard Function

Proposition 2.4. The Hazard function of CTPD is given as

$$h(x)=\frac{\alpha}{x}\left\{\frac{1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}}{1-\lambda\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}}\right\} \tag{9}$$

Proof. The proof is simple.
Hint. Use the formula

$$h(x)=\frac{f(x)}{1-F(x)}$$

2.1.3 Reverse Hazard Function

Proposition 2.5. The Reverse Hazard function of CTPD is given as

$$r_h(x)=\alpha x_0^{\alpha}x^{-\alpha-1}\left\{\frac{\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)}{1-\left(\frac{x_0}{x}\right)^{\alpha}\left(1-\lambda\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)}\right\} \tag{10}$$

Proof.
The Reverse Hazard function for a given distribution is

$$r_h(x)=\frac{f(x)}{F(x)}$$

Substitute the $f(x)$ and $F(x)$ of CTPD of Eq.(6) and Eq.(5) respectively in the above equation to directly

$$r_h(x)=\alpha x_0^{\alpha}x^{-\alpha-1}\left\{\frac{\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)}{1-\left(\frac{x_0}{x}\right)^{\alpha}\left(1-\lambda\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)}\right\}$$

2.1.4 Odds Function

Proposition 2.6. The Odds function of CTPD is given as

$$O(x)=\left[\left\{1-\left(\frac{x_0}{x}\right)^{\alpha}\left(1-\lambda\left(\frac{x_0}{x}\right)^{\alpha}+\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right\}^{-1}-1\right]^{-1} \tag{11}$$

Proof. The proof is simple.
Hint. Use the formula

$$O(x)=\frac{F(x)}{1-F(x)}$$

2.2 The Moments

Proposition 3.1. If X has the CTPD with $|\lambda|\le 1$ then the r^{th} moment of X about the origin is

$$E(X^r)=\alpha x_0^r\left[\frac{6\alpha^2-r\alpha(5+\lambda)+r^2(1+\lambda)}{(\alpha-r)(2\alpha-r)(3\alpha-r)}\right] \tag{12}$$

Proof.
We know that

$$E(X^r)=\int_{-\infty}^{\infty}x^r f(x)\,dx$$

Substitute $f(x)$ in above equation by its value in Eq. (6) to get

$$E(X^r) = \int_{x_0}^{\infty}\left[x^r \alpha x_0^{\alpha} x^{-\alpha-1}\,\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]dx$$
$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty}\left[x^{r-\alpha-1}\,\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]dx$$
$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty}(x^{r-\alpha-1}-2\lambda x_0^{\alpha}x^{r-2\alpha-1}+3\lambda x_0^{2\alpha}x^{r-3\alpha-1})\,dx$$
$$= \alpha x_0^{\alpha}\left[\int_{x_0}^{\infty}x^{r-\alpha-1}\,dx-2\lambda x_0^{\alpha}\int_{x_0}^{\infty}x^{r-2\alpha-1}\,dx+3\lambda x_0^{2\alpha}\int_{x_0}^{\infty}x^{r-3\alpha-1}\,dx\right]$$
$$= \alpha x_0^{\alpha}\left[\left[\frac{x^{r-\alpha}}{r-\alpha}\right]_{x_0}^{\infty}-2\lambda x_0^{\alpha}\left[\frac{x^{r-2\alpha}}{r-2\alpha}\right]_{x_0}^{\infty}+3\lambda x_0^{2\alpha}\left[\frac{x^{r-3\alpha}}{r-3\alpha}\right]_{x_0}^{\infty}\right]$$
$$= \frac{\alpha x_0^{\alpha}}{r-\alpha}\,[x^{r-\alpha}]_{x_0}^{\infty}-\frac{2\lambda\alpha x_0^{2\alpha}}{r-2\alpha}\,[x^{r-2\alpha}]_{x_0}^{\infty}+\frac{3\lambda\alpha x_0^{3\alpha}}{r-3\alpha}\,[x^{r-3\alpha}]_{x_0}^{\infty}$$
$$= -\frac{\alpha x_0^{r}}{r-\alpha}+\frac{2\lambda\alpha x_0^{r}}{r-2\alpha}-\frac{3\lambda\alpha x_0^{r}}{r-3\alpha}$$
$$= \alpha x_0^{r}\left[-\frac{1}{r-\alpha}+\frac{2\lambda}{r-2\alpha}-\frac{3\lambda}{r-3\alpha}\right]\;;\qquad r<\alpha$$

by performing some algebra calculation we get

$$E(X^r) = \alpha x_0^{r}\left[\frac{6\alpha^2-r\alpha(5+\lambda)+r^2(1+\lambda)}{(\alpha-r)(2\alpha-r)(3\alpha-r)}\right]$$

when $r=1$ we get the mean

$$\mu_X = E(X) = \alpha x_0\left[\frac{6\alpha^2-\alpha(5+\lambda)+(1+\lambda)}{(\alpha-1)(2\alpha-1)(3\alpha-1)}\right]$$

when $r=2$ we get the $E(X^2)$

$$E(X^2) = 2\alpha x_0^{2}\left[\frac{3\alpha^2-\alpha(5+\lambda)+2(1+\lambda)}{(\alpha-2)(2\alpha-2)(3\alpha-2)}\right]$$

and $Var(X)=E(X^2)-[E(X)]^2$ is

$$Var(X) = 2\alpha x_0^{2}\left[\frac{3\alpha^2-\alpha(5+\lambda)+2(1+\lambda)}{(\alpha-2)(2\alpha-2)(3\alpha-2)}\right]-\left\{\alpha x_0\left[\frac{6\alpha^2-\alpha(5+\lambda)+(1+\lambda)}{(\alpha-1)(2\alpha-1)(3\alpha-1)}\right]\right\}^2$$

2.3 The Moment Generating Function

Proposition 4.1. If X is a random variable has the CTPD with $|\lambda|\le 1$ then the moment generating function of X is

$$M_X(t) = \alpha x_0^{\alpha}\sum_{m=0}^{\infty}\frac{t^m}{m!\,(\alpha-m)}x_0^{m-\alpha}-2\lambda\alpha x_0^{2\alpha}\sum_{m=0}^{\infty}\frac{t^m}{m!\,(2\alpha-m)}x_0^{m-2\alpha}$$
$$+3\lambda\alpha x_0^{3\alpha}\sum_{m=0}^{\infty}\frac{t^m}{m!\,(3\alpha-m)}x_0^{m-3\alpha}\;,\qquad m<\alpha \tag{13}$$

Proof.
We know that

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty}e^{tx}f(x)\,dx$$

Substitute $f(x)$ in above equation by its value in Eq. (6) to get

$$M_X(t) = \int_{x_0}^{\infty}e^{tx}\left[\alpha x_0^{\alpha}x^{-\alpha-1}\,\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]dx$$
$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty}e^{tx}\,(x^{-\alpha-1}-2\lambda x_0^{\alpha}x^{-2\alpha-1}+3\lambda x_0^{2\alpha}x^{-3\alpha-1})dx$$

$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty} e^{tx}x^{-\alpha-1}dx - 2\lambda\alpha x_0^{2\alpha}\int_{x_0}^{\infty} e^{tx}x^{-2\alpha-1}dx + 3\lambda\alpha x_0^{3\alpha}\int_{x_0}^{\infty} e^{tx}x^{-3\alpha-1}dx$$

Let $I_1 = \int_{x_0}^{\infty} e^{tx}\,x^{-\alpha-1}dx$, $I_2 = \int_{x_0}^{\infty} e^{tx}\,x^{-2\alpha-1}dx$ and , $I_3 = \int_{x_0}^{\infty} e^{tx}\,x^{-3\alpha-1}dx$ then

$$M_X(t) = \alpha x_0^{\alpha} I_1 - 2\lambda\alpha x_0^{2\alpha} I_2 + 3\lambda\alpha x_0^{3\alpha} I_3$$

$$I_1 = \int_{x_0}^{\infty} e^{tx}\,x^{-\alpha-1}dx$$

$$= \int_{x_0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(tx)^m}{m!}\right)x^{-\alpha-1}dx$$

$$= \sum_{m=0}^{\infty}\frac{t^m}{m!}\int_{x_0}^{\infty} x^{m-\alpha-1}dx$$

$$= \sum_{m=0}^{\infty}\frac{t^m}{m!\,(\alpha-m)}\,x_0^{m-\alpha}$$

Following the same way we get

$$I_2 = \sum_{m=0}^{\infty}\frac{t^m}{m!\,(2\alpha-m)}\,x_0^{m-2\alpha} \quad \text{and} \quad I_3 = \sum_{m=0}^{\infty}\frac{t^m}{m!\,(3\alpha-m)}\,x_0^{m-3\alpha}$$

therefore,

$$M_X(t) = \alpha x_0^{\alpha}\sum_{m=0}^{\infty}\frac{t^m}{m!\,(\alpha-m)}x_0^{m-\alpha} - 2\lambda\alpha x_0^{2\alpha}\sum_{m=0}^{\infty}\frac{t^m}{m!\,(2\alpha-m)}x_0^{m-2\alpha} + 3\lambda\alpha x_0^{3\alpha}\sum_{m=0}^{\infty}\frac{t^m}{m!\,(3\alpha-m)}x_0^{m-3\alpha}$$

2.4 Median and Mode

2.4.1 Median

Proposition 5.1. The median of CTPD is calculated using the following formula

$$\lambda m^{-3\alpha}x_0^{3\alpha} - \lambda m^{-2\alpha}x_0^{2\alpha} + x_0^{\alpha}m^{-\alpha} - \tfrac{1}{2} = 0 \qquad (14)$$

Proof.

The following formula used to get median of a given distribution

$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$$

Where m is median.

Substitute the pdf of CTPD of Eq. (6) in the above equation we get

$$\int_{x_0}^{m}\left[\alpha x_0^{\alpha}x^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]dx = \frac{1}{2}$$

$$\alpha x_0^{\alpha}\int_{x_0}^{m}\left(x^{-\alpha-1} - 2\lambda x_0^{\alpha}x^{-2\alpha-1} + 3\lambda x_0^{2\alpha}x^{-3\alpha-1}\right)dx = \frac{1}{2}$$

$$\alpha x_0^{\alpha}\int_{x_0}^{m} x^{-\alpha-1}dx - 2\lambda\alpha x_0^{2\alpha}\int_{x_0}^{m} x^{-2\alpha-1}dx + 3\lambda\alpha x_0^{3\alpha}\int_{x_0}^{m} x^{-3\alpha-1}dx = \frac{1}{2}$$

$$\alpha x_0^{\alpha}\left[\frac{x^{-\alpha}}{-\alpha}\right]_{x_0}^{m} - 2\lambda\alpha x_0^{2\alpha}\left[\frac{x^{-2\alpha}}{-2\alpha}\right]_{x_0}^{m} + 3\lambda\alpha x_0^{3\alpha}\left[\frac{x^{-3\alpha}}{-3\alpha}\right]_{x_0}^{m} = \frac{1}{2}$$

$$-x_0^{\alpha}\left[x^{-\alpha}\right]_{x_0}^{m} + \lambda x_0^{2\alpha}\left[x^{-2\alpha}\right]_{x_0}^{m} - \lambda x_0^{3\alpha}\left[x^{-3\alpha}\right]_{x_0}^{m} = \tfrac{1}{2}$$

$$\lambda m^{-3\alpha}x_0^{3\alpha} - \lambda m^{-2\alpha}x_0^{2\alpha} + x_0^{\alpha}m^{-\alpha} - \tfrac{1}{2} = 0$$

Another formula can be obtained after performing some algebra calculation that is

$$m^{3\alpha} - 2m^{2\alpha}x_0^{\alpha} + 2\lambda m^{\alpha}x_0^{2\alpha} - 2\lambda x_0^{3\alpha} = 0 \qquad (15)$$

2.4.2 Mode

Proposition 5.2. The Mode of CTPD is given as

$$x=\left(\frac{-4\lambda^2(2\alpha+1)^2x_0^{2\alpha}\pm x_0^{\alpha}\sqrt{\left(4\lambda(4\lambda-9)\alpha^2+16\lambda(\lambda-3)\alpha+4\lambda(\lambda-3)\right)}}{-2(\alpha+1)}\right)^{1/\alpha} \quad (16)$$

Proof.
The mode or modal value of a continuous random variable X with a probability density function $f(x)$ is the value of x for which $f(x)$ takes a maximum value that is,

$$f'(x)=0$$

Recall Eq. (6) namely

$$f(x)=\alpha x_0^{\alpha}x^{-\alpha-1}\left(1-2\lambda\left(\frac{x_0}{x}\right)^{\alpha}+3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)$$
$$=\alpha x_0^{\alpha}x^{-\alpha-1}-2\lambda\alpha x_0^{2\alpha}x^{-2\alpha-1}+3\lambda x_0^{3\alpha}\alpha x^{-3\alpha-1}$$

Differentiate $f(x)$ to get

$$f'(x)=-\alpha(\alpha+1)x_0^{\alpha}x^{-\alpha-2}+2\lambda\alpha(2\alpha+1)x_0^{2\alpha}x^{-2\alpha-2}-3\lambda\alpha(3\alpha+1)x_0^{3\alpha}x^{-3\alpha-2}$$
$$=\alpha x_0^{\alpha}x^{-\alpha-2}[-(\alpha+1)+2\lambda(2\alpha+1)x_0^{\alpha}x^{-\alpha}-3\lambda(3\alpha+1)x_0^{2\alpha}x^{-2\alpha}]$$

Equate

$$\alpha x_0^{\alpha}x^{-\alpha-2}[-(\alpha+1)+2\lambda(2\alpha+1)x_0^{\alpha}x^{-\alpha}-3\lambda(3\alpha+1)x_0^{2\alpha}x^{-2\alpha}]=0$$
$$-(\alpha+1)+\frac{2\lambda(2\alpha+1)x_0^{\alpha}}{x^{\alpha}}-\frac{3\lambda(3\alpha+1)x_0^{2\alpha}}{x^{2\alpha}}=0$$
$$-(\alpha+1)x^{2\alpha}+2\lambda(2\alpha+1)x_0^{\alpha}x^{\alpha}-3\lambda(3\alpha+1)x_0^{2\alpha}=0$$

we know that

$$Ax^2+Bx+C=0$$
$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2a}$$

Let $A=-(\alpha+1)$ $\quad B=2\lambda(2\alpha+1)x_0^{\alpha}$ $\quad C=-3\lambda(3\alpha+1)x_0^{2\alpha}$

$$B^2-4AC=[2\lambda(2\alpha+1)x_0^{\alpha}]^2-4(\alpha+1)(3\lambda(3\alpha+1)x_0^{2\alpha})$$
$$=4\lambda^2(2\alpha+1)^2x_0^{2\alpha}-12\lambda(\alpha+1)\left((3\alpha+1)x_0^{2\alpha}\right)$$
$$=x_0^{2\alpha}\left(4\lambda^2(2\alpha+1)^2-12\lambda(\alpha+1)(3\alpha+1)\right)$$
$$=x_0^{2\alpha}\left(4\lambda^2(4\alpha^2+4\alpha+1)-12\lambda(3\alpha^2+4\alpha+1)\right)$$
$$=x_0^{2\alpha}(16\lambda^2\alpha^2+16\lambda^2\alpha+4\lambda^2-36\lambda\alpha^2-48\lambda\alpha-12\lambda)$$
$$=x_0^{2\alpha}(16\lambda^2\alpha^2-36\lambda\alpha^2+16\lambda^2\alpha-48\lambda\alpha+4\lambda^2-12\lambda)$$
$$=x_0^{2\alpha}\left(4\lambda(4\lambda-9)\alpha^2+16\lambda(\lambda-3)\alpha+4\lambda(\lambda-3)\right)$$

$$\frac{-B\pm\sqrt{B^2-4AC}}{2a}=\frac{-4\lambda^2(2\alpha+1)^2x_0^{2\alpha}\pm x_0^{\alpha}\sqrt{\left(4\lambda(4\lambda-9)\alpha^2+16\lambda(\lambda-3)\alpha+4\lambda(\lambda-3)\right)}}{-2(\alpha+1)}$$

$$x^{\alpha}=\frac{-4\lambda^2(2\alpha+1)^2x_0^{2\alpha}\pm x_0^{\alpha}\sqrt{\left(4\lambda(4\lambda-9)\alpha^2+16\lambda(\lambda-3)\alpha+4\lambda(\lambda-3)\right)}}{-2(\alpha+1)}$$

$$x=\left(\frac{-4\lambda^2(2\alpha+1)^2x_0^{2\alpha}\pm x_0^{\alpha}\sqrt{\left(4\lambda(4\lambda-9)\alpha^2+16\lambda(\lambda-3)\alpha+4\lambda(\lambda-3)\right)}}{-2(\alpha+1)}\right)^{1/\alpha}$$

2.5 Geometric and Harmonic Mean

Proposition 6.1. The Geometric Mean of CTPD is given as

$$G = \text{Antilog}\left(\log x_0 + \left(\frac{6-\lambda}{6\alpha}\right)\right) \tag{17}$$

Proof.

The logarithm of Geometric mean of a continuous random variable X with a probability density function $f(x)$ is,

$$\log G = \int_{-\infty}^{\infty} \log x \, f(x)\, dx$$

Substitute the pdf of CTPD of Eq. (6) in the above equation to get

$$\log G = \int_{x_0}^{\infty} \log x \left(\alpha x_0^{\alpha} x^{-\alpha-1}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right) dx$$

$$= \alpha x_0^{\alpha}\int_{x_0}^{\infty} \log x\,(x^{-\alpha-1} - 2\lambda x_0^{\alpha} x^{-2\alpha-1} + 3\lambda x_0^{2\alpha} x^{-3\alpha-1})\ dx$$

$$= \alpha x_0^{\alpha}\left[\int_{x_0}^{\infty}(\log x)x^{-\alpha-1}dx - 2\lambda x_0^{\alpha}\int_{x_0}^{\infty}(\log x)x^{-2\alpha-1}dx + 3\lambda x_0^{2\alpha}\int_{x_0}^{\infty}(\log x)x^{-3\alpha-1}dx\right]$$

Let $\log G = \alpha x_0^{\alpha}(I_1 - 2\lambda x_0^{\alpha} I_2 + 3\lambda x_0^{2\alpha} I_3)$

where $I_1 = \int_{x_0}^{\infty}(\log x)x^{-\alpha-1}dx$, $I_2 = \int_{x_0}^{\infty}(\log x)x^{-2\alpha-1}dx$ and $I_3 = \int_{x_0}^{\infty}(\log x)x^{-3\alpha-1}dx$

now

$$I_1 = \int_{x_0}^{\infty}(\log x)x^{-\alpha-1}dx$$

$$= \left[(\log x)\frac{x^{-\alpha}}{-\alpha}\right]_{x_0}^{\infty} - \int_{x_0}^{\infty}\frac{1}{x}\frac{x^{-\alpha}}{-\alpha}dx$$

$$= -\frac{1}{\alpha}[(\log x)\, x^{-\alpha}]_{x_0}^{\infty} + \frac{1}{\alpha}\int_{x_0}^{\infty} x^{-\alpha-1}\,dx$$

$$= -\frac{1}{\alpha}[(\log x)\, x^{-\alpha}]_{x_0}^{\infty} - \frac{1}{\alpha^2}[\, x^{-\alpha}]_{x_0}^{\infty}$$

$$= \frac{1}{\alpha x_0^{\alpha}}\left(\log x_0 + \frac{1}{\alpha}\right)$$

Following the same way we get

$I_2 = \frac{1}{2\alpha x_0^{2\alpha}}\left(\log x_0 + \frac{1}{2\alpha}\right)$ and $I_3 = \frac{1}{3\alpha x_0^{3\alpha}}\left(\log x_0 + \frac{1}{3\alpha}\right)$

Therefore

$$\log G = \alpha x_0^{\alpha}\left[\left(\frac{1}{\alpha x_0^{\alpha}}\left(\log x_0 + \frac{1}{\alpha}\right)\right) - 2\lambda x_0^{\alpha}\left(\frac{1}{2\alpha x_0^{2\alpha}}\left(\log x_0 + \frac{1}{2\alpha}\right)\right) + 3\lambda x_0^{2\alpha}\left(\frac{1}{3\alpha x_0^{3\alpha}}\left(\log x_0 + \frac{1}{3\alpha}\right)\right)\right]$$

$$= \left(\log x_0 + \frac{1}{\alpha}\right) - \lambda\left(\log x_0 + \frac{1}{2\alpha}\right) + \lambda\left(\log x_0 + \frac{1}{3\alpha}\right)$$

$$= \log x_0 + \frac{6-\lambda}{6\alpha}$$

and

$$G = \text{Antilog}\left(\log x_0 + \left(\frac{6-\lambda}{6\alpha}\right)\right)$$

Proposition 6.2. The Harmonic Mean of CTPD is given as

$$H = \frac{x_0}{\alpha}\left(\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{6\alpha^2 + (5+\lambda)\alpha + (1+\lambda)}\right) \tag{18}$$

Proof.

The inverse of Harmonic mean of a continuous random variable X with a probability density function $f(x)$ is,

$$H^{-1} = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\, dx$$

Substitute the pdf of CTPD of Eq. (6) in the above equation to get

$$H^{-1} = \int_{x_0}^{\infty} \frac{1}{x}\left(\alpha x_0^{\alpha} x^{-\alpha-1} \left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right) dx$$

$$= \alpha x_0^{\alpha} \int_{x_0}^{\infty} \left(x^{-\alpha-2} \left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right) dx$$

$$= \alpha x_0^{\alpha} \int_{x_0}^{\infty} x^{-\alpha-2} dx - 2\lambda\alpha x_0^{2\alpha} \int_{x_0}^{\infty} x^{-2\alpha-2} dx + 3\lambda\alpha x_0^{3\alpha} \int_{x_0}^{\infty} x^{-3\alpha-2} dx$$

$$= \alpha x_0^{\alpha} \left[\frac{x^{-\alpha-1}}{-\alpha-1}\right]_{x_0}^{\infty} - 2\lambda\alpha x_0^{2\alpha} \left[\frac{x^{-2\alpha-1}}{-2\alpha-1}\right]_{x_0}^{\infty} + 3\lambda\alpha x_0^{3\alpha} \left[\frac{x^{-3\alpha-1}}{-3\alpha-1}\right]_{x_0}^{\infty}$$

$$= \alpha x_0^{\alpha} \left(\frac{x_0^{-\alpha-1}}{\alpha+1}\right) - 2\lambda\alpha x_0^{2\alpha} \left(\frac{x_0^{-2\alpha-1}}{2\alpha+1}\right) + 3\lambda\alpha x_0^{3\alpha} \left(\frac{x_0^{-3\alpha-1}}{3\alpha+1}\right)$$

$$= \frac{\alpha}{x_0}\left[\left(\frac{1}{\alpha+1}\right) - 2\lambda\left(\frac{1}{2\alpha+1}\right) + 3\lambda\left(\frac{1}{3\alpha+1}\right)\right]$$

After performing some algebra calculation we get

$$H^{-1} = \frac{\alpha}{x_0}\left(\frac{6\alpha^2 + (5+\lambda)\alpha + (1+\lambda)}{(\alpha+1)(2\alpha+1)(3\alpha+1)}\right)$$

Therefore

$$H = \frac{x_0}{\alpha}\left(\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{6\alpha^2 + (5+\lambda)\alpha + (1+\lambda)}\right)$$

2.6 Parameters Estimation

Maximum likelihood approach can be used for the estimation of model parameters. The maximum likelihood estimates MLE of the parameters that are inherent within the CTPD is given by the following.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a cubic transmuted Pareto distribution. Then the likelihood function is given by

$$L = \prod_{i=1}^{n} f(x_i; \alpha, x_0, \lambda) = \alpha^n x_0^{n\alpha}\left[\prod_{i=1}^{n} x_i^{-(\alpha+1)}\right]\left[\prod_{i=1}^{n}\left(1 - 2\lambda\left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\right]$$

so, the log likelihood function is

$$\ln L = n\ln\alpha + n\alpha\ln x_0 - (\alpha+1)\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\ln(1 - 2\lambda x_0^{\alpha} x_i^{-\alpha} + 3\lambda x_0^{2\alpha} x_i^{-2\alpha}) \quad (19)$$

Therefore, the maximum likelihood estimates of α and λ which maximize Eq. (19), must satisfy the following normal equations,

$$\frac{\partial \ln L}{\partial \alpha} = 2\lambda \sum_{i=1}^{n} \frac{\left(\frac{x_0}{x_i}\right)^{\alpha}\left(1 - 3\left(\frac{x_0}{x_i}\right)^{\alpha}\right)\ln\left(\frac{x_0}{x_i}\right)}{1 - 2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x_i}\right)^{2\alpha}} - \sum_{i=1}^{n}\ln x_i + \frac{n}{\alpha} + n\ln x_0 = 0 \quad (20)$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\left(\frac{x_0}{x_i}\right)^{\alpha}\left(2 - 3\left(\frac{x_0}{x_i}\right)^{\alpha}\right)}{1 - 2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x_i}\right)^{2\alpha}} = 0 \quad (21)$$

The maximum likelihood estimates $\hat{\theta} = (\hat{\alpha}, \hat{\lambda})$ of $\theta = (\alpha, \lambda)$ is obtained by solving the above nonlinear system of Eq. (20) and Eq. (21). It is more convenient to use nonlinear optimization algorithms such as the quazi-Newton or Newton-Raphson to numerically maximize the log-likelihood function in Eq. (19).

2.7 Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a CTPD. Then the pdf of the j^{th} order statistic is given by

$$f_{(j)}(x) = \frac{n!}{(j-1)!\,(n-j)!} f(x)\big(F(x)\big)^{j-1}\big(1-F(x)\big)^{n-j}$$

The pdf's of smallest and largest order statistic respectively are

$$f_{(1)}(x) = nf(x)\big(1-F(x)\big)^{n-1}$$

$$f_{(n)}(x) = nf(x)\big(F(x)\big)^{n-1}$$

Using pdf and cdf of CTPD we get

$$f_{(1)}(x) = \frac{n\alpha x_0^\alpha}{x^{\alpha+1}}\left(1-2\lambda\left(\frac{x_0}{x}\right)^\alpha + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\left(\left(\frac{x_0}{x}\right)^\alpha - \lambda\left(\frac{x_0}{x}\right)^{2\alpha} + \lambda\left(\frac{x_0}{x}\right)^{3\alpha}\right)^{n-1} \quad (22)$$

$$f_{(n)}(x) = \frac{n\alpha x_0^\alpha}{x^{\alpha+1}}\left(1-2\lambda\left(\frac{x_0}{x}\right)^\alpha + 3\lambda\left(\frac{x_0}{x}\right)^{2\alpha}\right)\left(1-\left(\frac{x_0}{x}\right)^\alpha + \lambda\left(\frac{x_0}{x}\right)^{2\alpha} - \lambda\left(\frac{x_0}{x}\right)^{3\alpha}\right)^{n-1} \quad (23)$$

3 Application of Transmuted Pareto Distribution

In this section, we provide an application of CTPD . Moreover, compare CTPD with the Pareto distribution and Transmuted Pareto distribution. For this purposes we consider two real data sets.

3.1 Wheaton River Flood Peaks Data Set (WRFP)

The data are the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place. This data were analyzed by Choulakian and Stephens [3] and Merovci and Puka [6]. For purpose of the analysis we set $x_0 = 0.1$. Table 1, shows values of the parameter estimates, standard errors (SE), - log likelihood (-LL), and 95% confidence intervals (CI) for Pareto, TP and CTP distributions. In order to compare the CTPD with Pareto and TPD we use some different comparison measures includes -2×log-likelihood (-2log(L)), Akaike's information criterion (AIC), Corrected Akaike's information criterion (AICC) and Schwarz's Bayesian information criterion (BIC). Table 2 represents -2log(L), AIC, AICC and BIC for Pareto, TPD and CTPD.

Table 1: Parameter estimates, SE, -LL and CI for Pareto, TPD and CTPD for WRFP data set

	Parameter estimates	**SE**	**-LL**	**95% CI**
Pareto	$\hat{\alpha} = 0.243863$	0.028751	303.0642	(0.1875 , 0.3002)
Transmuted Pareto	$\hat{\alpha} = 0.349941$	0.031068	286.2009	(0.289 , 0.443)
	$\hat{\lambda} = -0.952417$	0.047477		(-1.0133 , -0.8594)
Cubic Transmuted Pareto	$\hat{\alpha} = 0.256395$	0.02426	289.8275	(0.2088 , 0.3851)
	$\hat{\lambda} = -0.933943$	0.065645		(-0.9815 , -0.80153)

Table 2: 2log (L), AIC, AICC and BIC for Pareto, TPD and CTPD for WRFP data set

Distribution	**-2log(L)**	**AIC**	**AICC**	**BIC**
Pareto	606.1283	610.1283	610.3022	614.6816
Transmuted Pareto	572.4018	578.4018	578.7547	585.2318
Cubic Transmuted Pareto	579.655	585.655	586.008	592.485

From Table 2, concerning the WRFP data set, we observe that the calculated values of the four comparison criteria (the smaller the better) all reveal that the cubic transmuted Pareto distribution is the most appropriate model than Pareto distribution while transmuted Pareto is the best one.

3.2 Floyd River Flood Data Set (FRF)

The second data set is for the Floyd River located in James, Iowa, USA. The Floyd River flood data set consist of 39 annual flood discharge rate for the years 1935–1973. This data were analyzed by Mudholkar and Hutson [7] and Merovci & Puka [6]. For purpose of the analysis we set $x_0 = 318$. Table 3, explains values of the parameter estimates, standard errors (SE), - log likelihood (-LL), and 95% confidence intervals (CI) for Pareto, TPD and CTPD. Table 4 represents -2log(L), AIC, AICC and BIC for Pareto, TPD and CTPD.

Table 3: Parameter estimates, SE, -LL and CI for Pareto, TPD and CTPD for FRF data set.

Distribution	Parameter estimates	SE	-LL	95% CI
Pareto	$\hat{\alpha} = 0.412471$	0.066064	392.8099	(0.283 , 0.542)
Transmuted Pareto	$\hat{\alpha} = 0.585919$	0.072021	385.3491	(0.4448 , 0.761)
	$\hat{\lambda} = -0.91024$	0.089307		(-1.0514 , -0.7352)
Cubic Transmuted Pareto	$\hat{\alpha} = 0.435662$	0.056971	387.0518	(0.324 , 0.6757)
	$\hat{\lambda} = -0.875966$	0.122455		(-0.9876 , -0.636)

Table 4: 2log (L), AIC, AICC and BIC for Pareto, TPD and CTPD for FRF data set.

Distribution	-2log(L)	AIC	AICC	BIC
Pareto	785.6199	789.6199	789.9532	792.947
Transmuted Pareto	770.6983	776.6983	777.384	781.689
Cubic Transmuted Pareto	774.1036	780.1036	780.7893	785.0943

Regarding the FRF data set, from Table 4, we observe that the calculated values of the four comparison criteria (the smaller the better) all reveal that the cubic transmuted Pareto distribution is the more appropriate model than Pareto distribution while transmuted Pareto is the best one.

4 Conclusions

In this paper, a new model called the Cubic Transmuted Pareto distribution (CTPD) which based on the Pareto distribution is proposed. Some mathematical and statistical properties for CTPD is obtained such: reliability function, hazard function, inverse hazard function, odd function, moments, expectation, variance, mode, median, geometric and harmonic mean, moment generating function, MLE of the unknown parameters. An application of the cubic transmuted Pareto distribution to real data shows that the new distribution can be used quite effectively to provide better fits than the Pareto distribution.

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