Journal of Ecology of Health & Environment
An International Journal

http://dx.doi.org/10.18576/jehe/090101

The Effective of Wind Speed and Eddy Diffusivities Variation on Time-Dependent Advection-Diffusion Equation

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Received: 23 Feb. 2020, Revised: 22 Oct. 2020, Accepted: 27 Nov. 2020.
Published online: 1 Jan. 2021.

Abstract: In this paper, Advection - diffusion equation (ADE) has been solved in two dimensions using the separation technique to obtain the crosswind integrated ground concentration considering vertical eddy diffusivity and mean wind speed depend on power law and time. We compared between estimated concentrations and observed which conducted in the Northern part of Copenhagen, Denmark of sulfur hexafluoride (SF_6) which released from a tower at a height of 115m without buoyancy. Comparison between estimated and observed crosswind integrated of pollutant concentration per emission rate, there are some present data which are agreement with observed data (one to one), others lie inside a factor of two and four.

Keywords: Separation of Variables Technique[1], Atmosphere Diffusion Experiments[2], Advection-Diffusion Equation[3].

1 Introduction

Different shapes of the atmospheric diffusion equation has been solved depend on Gaussian and non-Gaussian solutions. Study of mathematical solutions with wind speed for power law and the realistic assumption for eddy diffusivity by [1]. The solutions have been implemented in the KAPPA-G model [2]. We has been extended the solutions under boundary conditions assumption for dry deposition at the ground. The modeling of atmospheric dispersion has been solved by [3,4]. Solved the Advection–diffusion equation with variable coefficients for three dispersion problems in One-dimensional: (i) the dispersion solute along steady flow through an inhomogeneous medium, (ii) the dispersion of temporally dependent solute along uniform flow through the homogeneous medium. (iii) The dispersion along temporally dependent flow through an inhomogeneous medium has been soluted. Using Laplace transformation technique to obtain the Analytical solutions for Continuous point sources of uniform and increasing nature was considered in an initially solute-free semi-infinite medium. Inhomogeneity of the medium is expressed by the spatially dependent flow. The dispersion was considered proportional to the square of the spatially dependent velocity. Atmospheric dispersion modeling refers to the mathematical description of contaminant transport in the atmosphere. The term dispersion in this context used to describe the combination of diffusion (due to turbulent eddy motion) and advection (due to the wind). Mathematical and approximate solutions for the atmospheric dispersion problem have been derived under a wide range of simplifying assumptions, as well as various boundary conditions and parameter dependencies. Study Mathematical Solution is especially useful to engineers and environmental scientists who study pollutant transport, since they allow parameter sensitivity and source estimation by [5]. Both our scientific understanding and technical developments have greatly increased by using of empirical, Mathematical and numerical models to present the air pollution concentration in the atmosphere. For this purpose, the advection – diffusion equation Wide spread in operational atmospheric dispersion models, in principal; from this equation it is possible to obtain the dispersion from a source given appropriate boundary and initial conditions plus knowledge of the mean wind velocity and concentration turbulent fluxes [6].

In this paper, we solved advection-diffusion equation (ADE) in two dimensions using separation

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technique to obtain the crosswind integrated ground concentration considering vertical eddy diffusivity and mean wind speed depend on power law and time.

2 Mathematical Solutions

The basic gradient transport model can be written [7]:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right) \quad (1)$$

Where,

C is the average concentration of diffusing for continuous point source (x, y, and z) (kg/m3).

U is mean wind velocity along the x-axis (m/s).

K_x, k_y and k_z are the eddy diffusivities coefficients along x, y and z-axes respectively (m /s).

x wind coordinate measured in wind direction from the source (m).

y is crosswind coordinate direction (m).

z is vertical coordinate measured from the ground (m).

Equation (1) is impossible to solve Mathematical for complete general functional forms for the diffusivity K and wind speeds u, v and w. Integrating equation (1) respect to y from -∞ to ∞ to obtain the crosswind integrated of pollutant concentration as follows and the diffusion in direction x is neglected, then:

$$\frac{\partial \bar{C}_y}{\partial t}+u\frac{\partial \bar{C}_y}{\partial x}=\frac{\partial}{\partial z}\left(K_z\frac{\partial \bar{C}_y}{\partial z}\right)=K_z\frac{\partial^2 \bar{C}_y}{\partial z^2}+\frac{\partial K_z}{\partial z}\frac{\partial \bar{C}_y}{\partial z} \quad (2)$$

Where, K_z and u are function of z in power law [7]:

$$K_z=K_1\left(\frac{z}{z_1}\right)^n,\ \frac{\partial K_z}{\partial z}=\frac{nK_1}{z_1}\left(\frac{z}{z_1}\right)^{n-1}$$
$$u(z)=u_1\left(\frac{z}{z_1}\right)^m,\ \frac{u(z)}{K_z}=\frac{u_1}{k_1}\left(\frac{z}{z_1}\right)^{m-n} \quad (3)$$

Dividing Eq. (2) on K_z, one gets that:

$$\frac{\partial \bar{C}_y}{\partial t}+u(z)\frac{\partial \bar{C}_y}{\partial x}=K_z\frac{\partial^2 \bar{C}_y}{\partial z^2}+\frac{\partial K_z}{\partial z}\frac{\partial \bar{C}_y}{\partial z} \quad (4)$$

Substituting from Eq. (3) in Eq. (4), gives us:

$$\frac{\partial \bar{C}_y}{\partial t}+u_1\left(\frac{z}{z_1}\right)^m\frac{\partial \bar{C}_y}{\partial x}=K_1\left(\frac{z}{z_1}\right)^n\frac{\partial^2 \bar{C}_y}{\partial z^2}+\frac{nK_1}{z_1}\left(\frac{z}{z_1}\right)^{n-1}\frac{\partial \bar{C}_y}{\partial z} \quad (5)$$

Eq. (5) is subjected to the following boundary conditions

1-The flux at the ground and the top of the boundary layer is equal zero given by: -

$$k_z\frac{\partial \bar{C}_y}{\partial z}=0 \quad \text{at} \quad z=0 \quad (i)$$

2-The mass continuity is written in the form:-

$$u(z)\ \overline{C}_y\ (x,z,t)=Q\,\delta(t)\,\delta(z-h) \quad \text{at } x=0,\ t=0 \quad (ii)$$

δ () is Dirac delta function

3-The concentration of the pollutant tends to zero at large distance of the source, i.e.

$$\overline{C}_y\ (x,z,t)=0 \quad \text{at } x,z\to\infty,\ t>0 \quad (iii)$$

Using the method of separating the variables, and thus suppose the experimental solution to Eq. (2) with the following formula

$$\bar{\bar{C}}_y=(x,z,t)=T(t)\,G(x,z) \quad (6)$$

Where, G(x, z) =X(x) Z (z)

Substituting from Eq. (6) in Eq. (5) and divided on T (t) G(x, z), one gets that: -

$$\frac{\partial T(t)}{T(t)\,\partial t}+u_1\left(\frac{z}{z_1}\right)^m\frac{\partial G(x,z)}{G(x,z)\,\partial x}=K_1\left(\frac{z}{z_1}\right)^n\frac{\partial^2 G(x,z)}{G(x,z)\,\partial z^2}+\frac{nK_1}{z_1}\left(\frac{z}{z_1}\right)^{n-1}\frac{\partial G(x,z)}{G(x,z)\,\partial z}=-\mu^2 \quad (7)$$

Where, μ is a constant of separation variable.

Eq. (7) are divided into three Eqns. as follows

$$\frac{\partial T(t)}{\partial t}=-\mu^2\,T(t)\Rightarrow T(t)=c_1e^{-\mu^2 t} \quad (8)$$

Substituting from Eq. (iii) in Eq. (8), one gets that

$$\mu^2=\frac{1}{t} \quad (8.1)$$

Eq. (8) becomes:

$$T(t)=c_1e^{-1} \quad (8.2)$$

$$\frac{\partial X(x)}{\partial x}=-\mu^2\,X(x)\Rightarrow X(x)=c_2e^{-\mu^2 x} \quad (9)$$

Substituting from Eq. (ii) in Eq. (9), one gets that

$$c_2=\frac{Q\delta(t)\delta(z-h)}{u(z)} \quad (9.1)$$

Eq. (9.1) becomes:

$$X(x)=\frac{Q\delta(t)\delta(z-h)}{u(z)}e^{-\mu^2 x} \quad (9.1)$$

$$\frac{\partial^2 Z(z)}{\partial z^2}+\frac{nu_1\left(\frac{z}{z_1}\right)^{m-1}}{z_1}\frac{\partial Z(z)}{\partial z}+\frac{\mu^2 u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}Z(z)=0 \quad (10)$$

Then Eq. (10) is Bessel Equation. The general solution is

$$Z(z)=A\,J_0\left(\frac{u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu\right)+BY_0\left(\frac{u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu\right) \quad (10.1)$$

Where, J_0 and Y_0 are Bessel function of order zero, A and B are constants.

Since, C(x, z, t) is a finite at z=0 then B =0

The Eq. (10.1) becomes:

$$Z(z)=A\,J_0\left(\frac{u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu\right) \quad (10.2)$$

Where, J_0 is taken from [8] as follows:

$$J_{n=0}\left(\frac{u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu\right)=\sqrt{\frac{2u_1}{\pi K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu}\ \cos\left(\frac{u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu-\frac{\pi}{4}-\frac{n\pi}{2}\right)$$

The general solution of Eq. (6) as follows:

$$\frac{\overline{C}_y\,(x,z,t)}{Q} = c_1 A \frac{\delta(t)\delta(z-h)}{u(z)}\left(\mathrm{Exp} - \mu^2(t + (11)x)\right)\sqrt{\frac{2u_1}{\pi K_1}\left(\frac{z}{z_1}\right)^{m-n}}\;\mu\,\cos\left(\frac{u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu - \frac{\pi}{4}\right)$$

Where,

$c_3 = A\, c_1$

Substituting from Eq. (ii) in Eq. (11), one gets that

$$C_3 = \frac{1}{\sqrt{\frac{2u_1}{\pi K_1}\left(\frac{z}{z_1}\right)^{m-n}}\;\mu\,\cos\left(\frac{u_1}{K_1}\left(\frac{z}{z_1}\right)^{m-n}\mu - \frac{\pi}{4}\right)} \qquad (11.1)$$

Eq. (11) becomes:

$$\frac{\overline{C}_y\,(x,z,t)}{Q} = \frac{\delta(t)\delta(z-h)}{u_1\left(\frac{z}{z_1}\right)^{m}}\left(\mathrm{Exp} - \mu^2(t+x)\right) \qquad (12)$$

Where, $\delta(z-h)$ at m=1 gives [8]:

$$\delta(z-h) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty}(\cos(z-h))$$

3 Results and Discussion

The used data set was taken from the atmospheric diffusion experiments conducted in the Northern part of Copenhagen, Denmark, under unstable conditions [9, 10]. The tracer sulfur hexafluoride (SF_6) was eased from a tower at a height of 115m without buoyancy. The values of different parameters such as stability, wind speed, and downwind distance during the experiment are represented in (Table 1). Comparison between the predicted and observed crosswind normalized integrated concentration at a different downwind distance, wind speed and distance for the different runs and "m", "n" depend on stability classes. The two figures shows that the Comparison between present and observed crosswind normalized integrated of pollutant concentration, one finds that some present data which are agreement with observed crosswind normalized integrated concentration (one to one) and others lie inside factor of two and four. This present crosswind normalized integrated concentration is agreement with previous work.

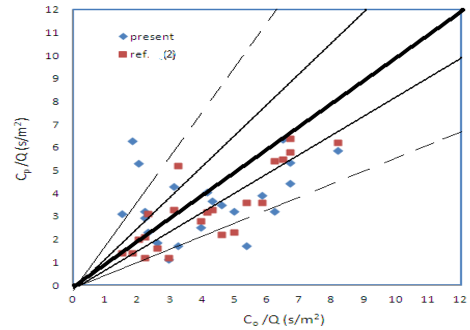

Fig.1: Comparison between predicted and observed concentrations per emission rate.

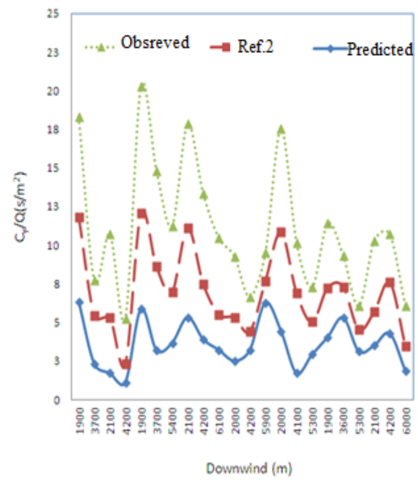

Fig. 2: Comparison between concentrations per emission rate and downwind distance.

Table 1: Comparison between the predicted and observed Crosswind- normalized integrated concentration at a different downwind distance, wind speed.

Run.	m	n	Speed (m/s)	Effective Height (m)	Distance (m)	$\frac{\overline{C}_y}{Q}$ (10^{-4}s/m^2)	
						Observe	Present	Ref.[2]
1	0.90	0.10	2.1	1980	1900	6.48	6.34	4.13
1	0.80	0.20	4.9	1980	3700	2.31	2.33	2.00
2	0.70	0.30	4.9	1920	2100	5.38	1.73	2.28
2	0.60	0.40	2.4	1920	4200	2.95	1.13	5.69
3	0.50	0.50	2.4	1120	1900	8.2	5.87	6.83
3	0.40	0.60	2.4	1120	3700	6.22	3.21	4.54
3	0.30	0.70	2.5	1120	5400	4.3	3.67	5.71
5	0.20	0.98	3.1	820	2100	6.72	5.33	6.90

5	0.10	0.90	3.1	820	4200	5.84	3.91	3.80
5	0.10	0.10	3.1	820	6100	4.97	3.23	4.02
6	0.90	0.10	7.2	1300	2000	3.96	2.52	3.62
6	0.80	0.20	7.2	1300	4200	2.22	3.22	2.70
6	0.70	0.30	7.2	1300	5900	1.83	6.27	6.78
7	0.60	0.40	4.1	1850	2000	6.7	4.45	1.59
7	0.50	0.50	4.1	1850	4100	3.25	1.73	2.91
7	0.40	0.60	4.1	1850	5300	2.23	2.93	2.61
8	0.30	0.70	4.2	810	1900	4.16	4.04	3.40
8	0.20	0.98	4.2	810	3600	2.02	5.31	5.10
8	0.20	0.98	4.2	810	5300	1.52	3.13	4.63
9	0.10	0.90	5.1	2090	2100	4.58	3.51	2.63
9	0.10	0.90	5.1	2090	4200	3.11	4.29	4.63
9	0.10	0.90	5.1	2090	6000	2.59	3.98	1.63

4 Conclusions

Using separation of variables technique is estimated advection-diffusion equation (ADE) in two dimensions to obtain the crosswind integrated ground level concentration by taking vertical eddy diffusivity and mean wind velocity depend on power law dependent on time. The used dataset was observed from the atmospheric diffusion experiments conducted in the Northern part of Copenhagen, Denmark. The tracer sulfur hexafluoride (SF_6) was released from a tower at a height of 115m without buoyancy. Comparison between present and observed crosswind integrated of pollutant concentration per emission rate, there are some present data which are agreement with observed data (one to one), others lie inside a factor of two also factor of four.

Acknowledgement

All thanks and return to God Almighty who helped me accomplish this work. I would like to Thank Dr. Khaled Sadek Mohamed Essa for his help.

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