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# A New Integrator of Runge-Kutta Type for Directly Solving General Third-order ODEs with Application to Thin Film Flow Problem

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**Abstract:** In this study, a five-stage fourth-order Runge-Kutta type method for directly solving general third-order ordinary differential equations (ODEs) of the form $y''' = f(x,y,y',y'')$ which is denoted as RKTGG method is constructed. The order conditions of RKTGG method up to order four are derived. Based on the order conditions developed, five-stage fourth-order explicit Runge-Kutta type method is constructed. Zero-stability of the current method is shown. The various type of general third-order ODEs has been solved using new method and numerical comparisons are made when the same problem is reduced to the first-order system of equations which are solved using existing Runge-Kutta methods. The numerical study of a third-order ODE arising in thin film flow of viscous fluid in physics is also discussed. Numerical results show that the new method is more efficient in terms of accuracy and number of function evaluations.

**Keywords:** Runge-Kutta type methods, General third-order ordinary differential equations, Order conditions, Thin film flow

## 1 Introduction

In this paper, we are considering to the general third-order ordinary differential equations (ODEs) of the form

$$\begin{aligned} y'''(x) &= f\big(x,y(x),y'(x),y''(x)\big), \\ y(x_0) &= y_0, y'(x_0) = y_0', y''(x_0) = y_0'' \end{aligned} \tag{1}$$

where $y,y',y'' \in \mathbb{R}^d, f:\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector-valued function. Specifically, third order differential equations emerge in numerous physical problems, for example, thin film flow, gravity-driven flows and electromagnetic wave. The general solution of (1) is done by reducing it to an equivalent first-order system which is three times and can be using standard Runge-Kutta method or multi-step method. A lot of researches have being solved problem (1) by converting the problem to a system of first-order equations. Furthermore, there are several authors who study on numerical methods which solve problem (1) directly, for instance Jator [1], Awoyemi and Idowu [2], proposed a class of hybrid collocation methods for the direct solution of higher-order ODEs. You and Chen [3], constructed direct integrations of RK type for special third-order ODEs. Waeleh et al. [4], proposed a new algorithm for solving higher-order IVPs of ordinary differential equations. Jator [5], constructed hybrid multi-step method for solving second-order IVPs without predictors. Samat and Ismail [6], developed a block multi-step method which can directly solve general third-order equations. Furthermore, Ibrahim et al. [7], found a way using multi-step method that can directly solve stiff third-order differential equations. Mechee et al. [8], constructed a three-stage fifth-order RK method for directly solving special third-order ODEs. Kasim et al. [9], proposed integration for special third-order ODEs using improved Runge-Kutta direct method. Mechee et al. [10], suggested a new four-stage sixth order Runge-Kutta method for direct integration of special third-order (ODEs). Subsequently, Senu et al. [11] constructed embedded explicit Runge-Kutta methods for directly solving special third-order differential equations. In this paper, the main

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aim is to construct a one-step method of order four to solve third-order ODEs directly. The derivation of order conditions are given in Section 2. In Section 3, the zero-stability of the new method is given. Five-stage fourth-order are constructed in Section 4. The efficiency of the new method, when compared with existing method is given in Section 5. The thin film flow problem is discussed in Section 6.

## 2 Derivation of the New Method

The general form of RKTGG method with $m$-stage for solving the IVPs (1) can be expressed as follows:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + h^3\sum_{i=1}^{m} b_i k_i, \tag{2}$$

$$y'_{n+1} = y'_n + hy''_n + h^2\sum_{i=1}^{m} b'_i k_i, \tag{3}$$

$$y''_{n+1} = y''_n + h\sum_{i=1}^{m} b''_i k_i, \tag{4}$$

where

$$\begin{aligned} k_1 &= f\left(x_n, y_n, y'_n, y''_n\right),\\ k_i &= f\Bigg(x_n + c_i h,\, y_n + c_i hy'_n + \frac{h^2}{2}c_i^2 y''_n + h^3\sum_{j=1}^{i-1} a_{ij}k_j,\\ &\qquad y'_n + c_i hy''_n + h^2\sum_{j=1}^{i-1}\bar{a}_{ij}k_j,\, y''_n + h\sum_{j=1}^{i-1}\bar{\bar{a}}_{ij}k_j\Bigg) \end{aligned} \tag{5}$$

for $i = 2,3,\dots,m$. The parameters $b_i, b'_i, b''_i, a_{ij}, \bar{a}_{ij}, \bar{\bar{a}}_{ij}$ and $c_i$ of the RKTGG method assumed to be real and used for $i, j = 1,2,\dots,m$. This method is an explicit form if $a_{ij} = \bar{a}_{ij} = \bar{\bar{a}}_{ij} = 0$ for $i \le j$ and it is an implicit one if $a_{ij} \neq 0$, $\bar{a}_{ij} \neq 0$ and $\bar{\bar{a}}_{ij} \neq 0$ for $i \le j$. The new method can be represented by Butcher tableau as follows:

$$\begin{array}{c|c|c|c} c & A & \bar{A} & \bar{\bar{A}} \\ \hline & b^T & b'^T & b''^T \end{array} \tag{6}$$

To determine the parameters of the new method given by (2)-(5), the RKTGG method expression is expanded using Taylor's series expansion. After performing a few algebraic manipulations, this expansion is equated to the true solution that is given by Taylor's series expansion. The direct expansion of the local truncation error utilized to derive the general order conditions for the new method. This idea based on the derivation of order conditions for the RK method suggested by Dormand [12]. The new method RKTGG can be expressed as follows:

$$\begin{aligned} y_{n+1} &= y_n + h\Phi(x_n, y_n, y'_n, y''_n),\\ y'_{n+1} &= y'_n + h\Phi'(x_n, y_n, y'_n, y''_n),\\ y''_{n+1} &= y''_n + h\Phi''(x_n, y_n, y'_n, y''_n). \end{aligned} \tag{7}$$

where the increment functions are

$$\begin{aligned} \Phi(x_n, y_n, y'_n, y''_n) &= y'_n + \frac{h}{2}y''_n + h^2\sum_{i=1}^{m} b_i k_i,\\ \Phi'(x_n, y_n, y'_n, y''_n) &= y''_n + h\sum_{i=1}^{m} b'_i k_i,\\ \Phi''(x_n, y_n, y'_n, y''_n) &= \sum_{i=1}^{m} b''_i k_i. \end{aligned} \tag{8}$$

where $k_i$ is given in (5). If we assume that $\Delta$, $\Delta'$ and $\Delta''$ are the Taylor series increment function. Thus, the local truncation errors of $y(x)$, $y'(x)$ and $y''(x)$ can be obtained by substituting the accurate solution of (1) into (8) as follows:

$$\begin{aligned} \tau_{n+1} &= h[\Phi - \Delta],\\ \tau'_{n+1} &= h[\Phi' - \Delta'],\\ \tau''_{n+1} &= h[\Phi'' - \Delta'']. \end{aligned} \tag{9}$$

**Definition 1.***A RKTGG method* (2) - (4) *has order $p$ if for sufficiently smooth problems* (1)

$y(x_n + h) - y_{n+1} = \mathbf{O}(h^{p+1}), y'(x_n + h) - y'_{n+1} = \mathbf{O}(h^{p+1}), y''(x_n + h) - y''_{n+1} = \mathbf{O}(h^{p+1})$.

In the terms of elementary differentials, the expressions (9) are best given and the Taylor series can be expressed as follows:

$$\begin{aligned} \Delta &= y' + \frac{1}{2}hy'' + \frac{1}{6}h^2F_1^{(3)} + \frac{1}{24}h^3F_1^{(4)} + O(h^4),\\ \Delta' &= y'' + \frac{1}{2}hF_1^{(3)} + \frac{1}{6}h^2F_1^{(4)} + \frac{1}{24}h^3F_1^{(5)} + O(h^4),\\ \Delta'' &= F_1^{(3)} + \frac{1}{2}hF_1^{(4)} + \frac{1}{6}h^2F_1^{(5)} + O(h^3). \end{aligned} \tag{10}$$

The first few elementary differentials for the scalar case are

$$\begin{aligned} F_1^{(3)} &= f,\\ F_1^{(4)} &= f_x + f_y y_x + f_{y'} y_{xx} + f_{y''} f,\\ F_1^{(5)} &= f_{xx} + y_x f_{xy} + f_{xy'} y_{xx} + f_{xy''} f + y_{xx}^2 f_{yy} + y_x f_{y'y''} f\\ &\quad + f_{y'y'} y_{xx}^2 + f_{y'y''} f + f_{y'} f + f_{y''y''} f^2 + f_{y''} f_x\\ &\quad + f_{y''} f_y y_x + f_{y''} f_{y'} y_{xx} + f_{y''y''} f \end{aligned} \tag{11}$$

Substituting (11) into (8), the increment functions $\Phi, \Phi'$ and $\Phi''$ for new method becomes

$$
\begin{aligned}
\sum_{i=1}^{m} b_i k_i = {} & \sum_{i=1}^{m} b_i f + \sum_{i=1}^{m} b_i c_i \big( f_x + f_y y_x + f_{y'} y_{xx} + f_{y''} f \big) h \\
& + \frac{1}{2} \sum_{i=1}^{m} b_i c_i^2 \big( f_{xx} + y_x f_{xy} + f_{xy'} y_{xx} + f_{xy''} f \\
& + y_{xx}^2 f_{yy} + y_x f_{y' y''} f + f_y y_{xx} + f_{y' y'} y_{xx}^2 + f_{y' y''} f \\
& + f_{y'} f + f_{y'' y''} f^2 + f_{y''} f_x + f_{y''} f_y y_x \\
& + f_{y''} f_{y'} y_{xx} + f_{y'' y''} f \big) h^2 + O(h^3), \\
\sum_{i=1}^{m} b'_i k_i = {} & \sum_{i=1}^{m} b'_i f + \sum_{i=1}^{m} b'_i c_i \big( f_x + f_y y_x + f_{y'} y_{xx} + f_{y''} f \big) h \\
& + \frac{1}{2} \sum_{i=1}^{m} b_i c_i^2 \big( f_{xx} + y_x f_{xy} + f_{xy'} y_{xx} + f_{xy''} f \\
& + y_{xx}^2 f_{yy} + y_x f_{y' y''} f + f_y y_{xx} + f_{y' y'} y_{xx}^2 + f_{y' y''} f \\
& + f_{y'} f + f_{y'' y''} f^2 + f_{y''} f_x + f_{y''} f_y y_x \\
& + f_{y''} f_{y'} y_{xx} + f_{y'' y''} f \big) h^2 + O(h^3), \\
\sum_{i=1}^{m} b''_i k_i = {} & \sum_{i=1}^{m} b''_i f + \sum_{i=1}^{m} b''_i c_i \big( f_x + f_y y_x + f_{y'} y_{xx} + f_{y''} f \big) h \\
& + \frac{1}{2} \sum_{i=1}^{m} b_i c_i^2 \big( f_{xx} + y_x f_{xy} + f_{xy'} y_{xx} + f_{xy''} f \\
& + y_{xx}^2 f_{yy} + y_x f_{y' y''} f + f_y y_{xx} + f_{y' y'} y_{xx}^2 + f_{y' y''} f \\
& + f_{y'} f + f_{y'' y''} f^2 + f_{y''} f_x + f_{y''} f_y y_x \\
& + f_{y''} f_{y'} y_{xx} + f_{y'' y''} f \big) h^2 + O(h^3).
\end{aligned} \tag{12}
$$

From (10) and (12), the local truncation error (9) can be expressed as follows:

$$
\begin{aligned}
\tau_{n+1} &= h^3 \left[ \sum_{i=1}^{m} b_i k_i - \left( \frac{1}{6} F_1^{(3)} + \frac{1}{24} h F_1^{(4)} + \ldots \right) \right], \\
\tau'_{n+1} &= h^2 \left[ \sum_{i=1}^{m} b'_i k_i - \left( \frac{1}{2} F_1^{(3)} + \frac{1}{6} h F_1^{(4)} + \ldots \right) \right], \\
\tau''_{n+1} &= h \left[ \sum_{i=1}^{m} b''_i k_i - \left( F_1^{(3)} + \frac{1}{2} h F_1^{(4)} + \frac{1}{6} h^2 F_1^{(5)} + \ldots \right) \right].
\end{aligned} \tag{13}
$$

Substituting (12) into (13) and expanding as a Taylor expansion using Maple package (see [13]), the local truncation errors or the order conditions for $m$-stage up to order four for new method can be expressed as follows:

**Order conditions for y:**

Order 3

$$\sum b_i = \frac{1}{6} \tag{14}$$

Order 4

$$\sum b_i c_i = \frac{1}{24}, \; \sum b_i \bar{\bar{a}}_{ij} = \frac{1}{24} \tag{15}$$

**Order conditions for y′:**

Order 2

$$\sum b'_i = \frac{1}{2} \tag{16}$$

Order 3

$$\sum b'_i c_i = \frac{1}{6}, \; \sum b'_i \bar{\bar{a}}_{ij} = \frac{1}{6} \tag{17}$$

Order 4

$$\sum b'_i c_i^2 = \frac{1}{12}, \; \sum b'_i c_i \bar{\bar{a}}_{ij} = \frac{1}{12} \tag{18}$$

$$\sum b'_i \bar{a}_{ij} = \frac{1}{24}, \; \frac{1}{2} \sum b'_i \bar{\bar{a}}_{ij}^2 + \sum b'_i \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} = \frac{1}{24} \tag{19}$$

$$\sum b'_i \bar{\bar{a}}_{ij} c_j = \frac{1}{24}, \sum b'_i \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} = \frac{1}{24} \tag{20}$$

**Order conditions for y″:**

Order 1

$$\sum b''_i = 1 \tag{21}$$

Order 2

$$\sum b''_i c_i = \frac{1}{2}, \; \sum b''_i \bar{\bar{a}}_{ij} = \frac{1}{2} \tag{22}$$

Order 3

$$\sum b''_i c_i^2 = \frac{1}{3}, \; \sum b''_i c_i \bar{\bar{a}}_{ij} = \frac{1}{3} \tag{23}$$

$$\sum b''_i \bar{\bar{a}}_{ij} c_j = \frac{1}{6}, \; \sum b''_i \bar{a}_{ij} = \frac{1}{6} \tag{24}$$

$$\frac{1}{2} \sum b''_i \bar{\bar{a}}_{ij}^2 + \sum b''_i \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} = \frac{1}{6}, \; \sum b''_i \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} = \frac{1}{6}. \tag{25}$$

Order 4

$$\sum b''_i \bar{a}_{ij} c_j = \frac{1}{24}, \; \sum b''_i \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} c_k = \frac{1}{24}, \tag{26}$$

$$\sum b''_i c_i \bar{\bar{a}}_{ij} c_j = \frac{1}{8}, \; \sum b''_i \bar{\bar{a}}_{ij} c_j^2 = \frac{1}{12} \tag{27}$$

$$\sum b''_i c_i^3 = \frac{1}{4}, \; \sum b''_i c_i^2 \bar{\bar{a}}_{ij} = \frac{1}{4}, \tag{28}$$

$$\sum b''_i \bar{a}_{ij} c_i = \frac{1}{8}, \sum b''_i \bar{a}_{ij} \bar{\bar{a}}_{jk} + \sum b''_i \bar{\bar{a}}_{ij} \bar{a}_{jk} = \frac{1}{12} \tag{29}$$

$$
\begin{aligned}
& \frac{1}{2} \sum b''_i c_i \bar{\bar{a}}_{ij}^2 + \sum b''_i c_i \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} = \frac{1}{8}, \; \sum b''_i \bar{\bar{a}}_{ij}^2 c_j \\
& + \sum b''_i \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} c_k = \frac{1}{8}
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
& \sum b''_i \bar{\bar{a}}_{ij} c_j \bar{\bar{a}}_{jk} + \sum b''_i c_i \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} = \frac{5}{24}, \; \frac{1}{2} \sum b''_i \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jL} \bar{\bar{a}}_{jk} \\
& + \sum b''_i \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jL} = \frac{1}{6}
\end{aligned} \tag{31}
$$

$$\sum b_i'' a_{ij} = \frac{1}{24}, \sum b_i'' \bar{\bar{a}}_{ij} \bar{a}_{ij} = \frac{1}{8} \tag{32}$$

$$\frac{1}{6} \sum b_i'' \bar{\bar{a}}_{iL} \bar{\bar{a}}_{ik} \bar{\bar{a}}_{ij} = \frac{1}{24}, \sum b_i'' \bar{\bar{a}}_{ij} \bar{\bar{a}}_{jk} \bar{\bar{a}}_{kL} = \frac{1}{24} \tag{33}$$

All indexes are run from one to $m$. To obtain the higher-order RKTGG method, the following simplifying assumption is used in order to reduce the number of equations to be solved:

$$\begin{aligned} \sum \bar{\bar{a}}_{ij} &= c_i, \\ b_i' &= b_i''(1 - c_i), \\ b_i &= b_i'' \frac{(1 - c_i)^2}{2}. \quad i = 1, \ldots, m. \end{aligned} \tag{34}$$

## 3 Zero-Stability of the New Method

In this section, we discuss the concept of zero-stability of new method to be convergent. Zero-stability is one of significant tool to prove the convergence of multi-step methods and stability (see [14,15]). Hairer et al. [16], discussed zero-stability to determine an upper bound on the order of convergence of linear multi-steps methods. Now, the first characteristic polynomial for the RKTGG method (2)-(5) is based on the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2 y''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \end{bmatrix},$$

where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix coefficient of $y_{n+1}, hy'_{n+1}$ and $h^2 y''_{n+1}$

and $A = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is matrix coefficient of $y_n, hy'_n$ and $h^2 y''_n$, respectively.

Then, the first characteristic polynomial of new method is

$$\rho(\zeta) = det[I\zeta - A] = \begin{vmatrix} \zeta - 1 & -1 & -\frac{1}{2} \\ 0 & \zeta - 1 & -1 \\ 0 & 0 & \zeta - 1 \end{vmatrix}.$$

thus,

$$\rho(\zeta) = (\zeta - 1)^3.$$

Hence, the method is zero stable since the roots, $\zeta_{1,2,3}$ are equal to one.

## 4 Construction of the RKTGG Methods

In this section, based on the order conditions which have been derived in Section 2, we proceed to construct explicit RKTGG methods. The global local truncation error for the $p$ order RKTGG method is defined as follows:

$$\| \tau_g^{(p+1)} \|_2 = \left( \sum_{i=1}^{n_p+1} \left(\tau_i^{(p+1)}\right)^2 + \sum_{i=1}^{n'_p+1} \left(\tau_i'^{(p+1)}\right)^2 + \sum_{i=1}^{n''_p+1} \left(\tau_i''^{(p+1)}\right)^2 \right)^{\frac{1}{2}} \tag{35}$$

where $\tau^{(p+1)}, \tau'^{(p+1)}$ and $\tau''^{(p+1)}$ are the local truncation error terms for $y, y'$ and $y''$ respectively, $\tau_g{}^{(p+1)}$ is the global local truncation error.

We then focus on the derivation of five-stage RKTGG method of order four and the algebraic conditions ((14)–(15), (16)–(20), (21)–(33)) are used because of the high number of the resulting system of equations which consists of 37 nonlinear equations. Therefore, we use the simplifying assumption (34) reducing the system of equations to 25 equations with 34 unknowns and left with 9 degree of freedom. Solving the system simultaneously and the family of solution in term of $a_{21}, a_{31}, a_{42}, a_{43}, a_{54}, \bar{a}_{42}, \bar{a}_{54}, \bar{\bar{a}}_{32}$ and $c_2$ are given as follows:

$$a_{32} = -\frac{(-1 + c_2)}{12\,(6c_2^2 + 1 - 4c_2)}, a_{41} = 0, a_{51} = 0, a_{52} = 0,$$

$$\begin{aligned} a_{53} = -&\frac{1}{18\,(6c_2^2 + 1 + 4c_2)\,(-1 + c_2)\,(-1 + 4c_2)} (2 + \\ &15\,a_{31} - 12\,a_{42} - 12\,a_{43} - 324\,a_{42}\,c_2^4 + 162\,a_{54}\,c_2 \\ &- 540\,a_{54}\,c_2^2 - 432\,a_{54}\,c_2^4 + 828\,a_{54}\,c_2^3 - 39\,c_2^3 + 33\,c_2^2 \\ &- 18\,a_{54} + 18\,c_2^4 + 648\,a_{43}\,c_2^3 - 3\,a_{21} - 14\,c_2 - 648\,a_{31} \\ &c_2^3 - 648\,a_{31}\,c_2^3 + 432\,a_{31}\,c_2^2 - 132\,a_{31}\,c_2 + 324\,a_{31}\,c_2^4 \\ &+ 120\,a_{43}\,c_2 - 432\,a_{43}\,c_2^2 - 324\,a_{43}\,c_2^4 + 12\,a_{21}\,c_2 \\ &+ 120\,a_{42}\,c_2 - 432\,a_{42}\,c_2^2 + 648\,c_2^3), \end{aligned}$$

$$\bar{a}_{31} = 0, \bar{a}_{32} = \frac{1}{6\left(6\,c_2^2 + 1 - 4\,c_2\right)}, \bar{a}_{41} = 0,$$

$$\begin{aligned} \bar{a}_{42} = -&\frac{1}{48\left(-1 + c_2\right)^2 \left(3c_2 - 1\right)^4} (648\,c_2^5 \\ &- 3456\,\bar{a}_{54}\,c_2^5 - 1566\,c_2^4 + 7488\,c_2^4 + 7488\,\bar{a}_{54}\,c_2^4 \\ &+ 1512\,c_2^3 - 5976\,\bar{a}_{54}\,c_2^3 + 2376\,\bar{a}_{54}\,c_2^2 - 771\,c_2^2 \\ &+ 144\,c_2^2\,\bar{a}_{21} - 84\,c_2\,\bar{a}_{21} + 196\,c_2 - 468\,\bar{a}_{54}\,c_2 \\ &- 19 + 12\,\bar{a}_{21} + 36\,\bar{a}_{54}), \end{aligned}$$

$$\bar{a}_{43}=-\frac{1}{48\left(9c_2^2-6c_2+1\right)\left(-1+c_2\right)^2\left(3c_2-1\right)^2}\Big(-324\,c_2^6+432\,c_2^5+3456\,\bar{a}_{54}c_2^5+144\,c_2^4+216\,c_2^4\bar{a}_{21}-7488\,\bar{a}_{54}c_2^4-576\,c_2^3\bar{a}_{21}+5976\,\bar{a}_{54}c_2^3-552\,c_2^3+384\,c_2^2\bar{a}_{21}-411\,c_2^2-2376\,\bar{a}_{54}c_2^2-108\,\bar{a}_{21}c_2-124\,c_2+468\,\bar{a}_{54}c_2-36\,\bar{a}_{54}+13+12\,\bar{a}_{21}\Big),$$

$$\bar{a}_{51}=0,\ \bar{a}_{52}=0,$$

$$\bar{a}_{53}=\frac{1}{36\left(-1+4c_2\right)\left(-1+c_2\right)\left(6c_2^2+1-4c_2\right)}\Big(-864\,\bar{a}_{54}c_2^4+90\,c_2^4+1656\,\bar{a}_{54}c_2^3-168\,c_2^3-1080\,\bar{a}_{54}c_2^2+123\,c_2^2+36\,\bar{a}_{21}c_2^2+324\,\bar{a}_{54}c_2-54\,c_2-24\,\bar{a}_{21}-36\,\bar{a}_{54}+8+6\,\bar{a}_{21}\Big),\ \bar{\bar{a}}_{21}=c_2,\ \bar{\bar{a}}_{31}=1-\bar{\bar{a}}_{32},\ \bar{\bar{a}}_{41}=0,$$

$$\bar{\bar{a}}_{42}=-\frac{18c_2^2-16c_2+3}{\left(-1+c_2\right)\left(9c_2^2-6c_2+1\right)},$$

$$\bar{\bar{a}}_{43}=\frac{1}{8\left(-1+c_2\right)\left(9c_2^2-6c_2+1\right)},$$

$$\bar{\bar{a}}_{51}=-\frac{-108c_2^4+108\bar{\bar{a}}_{32}c_2^4+216\,c_2^3-216\,\bar{\bar{a}}_{32}c_2^2}{6\left(6c_2^2+1-4\right)\left(-1+c_2\right)\left(-1+4c_2\right)}+\frac{144\,\bar{\bar{a}}_{32}c_2^2-44\,\bar{\bar{a}}_{32}c_2+45c_2+5\bar{\bar{a}}_{32}}{6\left(6c_2^2+1-4\right)\left(-1+c_2\right)\left(-1+4c_2\right)},$$

$$\bar{\bar{a}}_{52}=\frac{216\,\bar{\bar{a}}_{32}c_2^4+54\,c_2^3-432\,\bar{\bar{a}}_{32}c_2^3-58\,c_2^2+288\,\bar{\bar{a}}_{32}c_2^2}{12\left(-1+4c_2\right)\left(6c_2^2+1-4c_2\right)\left(-1+c_2\right)}+\frac{+15\,c_2-88\,\bar{\bar{a}}_{32}c_2+10\,\bar{\bar{a}}_{32}-1}{12\left(-1+4c_2\right)\left(6c_2^2+1-4c_2\right)\left(-1+c_2\right)},$$

$$\bar{\bar{a}}_{53}=-\frac{72\,c_2^4-102\,c_2^3+50\,c_2^2-11\,c_2+1}{12\left(6c_2^2+1-4c_2\right)\left(-1+c_2\right)\left(-1+4c_2\right)},$$

$$\bar{\bar{a}}_{54}=\frac{9c_2^2-6c_2+1}{3\left(6c_2^2+1-4c_2\right)},\ b'_1=0,\ b'_3=0,\ b'_5=0,$$

$$b'_2=\frac{1}{6\left(6c_2^2+1-4c_2\right)},\ b'_4=\frac{9c_2^2-6c_2+1}{3\left(6c_2^2+1-4c_2\right)},$$

$$c_1=0,\ c_3=1,\ c_4=\frac{-1+2c_2}{2\left(3c_2-1\right)},\ c_5=1,$$

$$b_1=0,\ b_2=-\frac{-1+c_2}{12\left(6c_2^2+1-4c_2\right)},\ b_3=0,$$

$$b_4=\frac{12c_2^2-7c_2+1}{12(6c_2^2+1-4c_2)},\ b_5=0,\ b''_1=0,$$

$$b''_2=-\frac{1}{6\left(-1+c_2\right)\left(6c_2^2+1-4c_2\right)},$$

$$b''_3=-\frac{18\,c_2^2-24\,c_2+5}{6\left(-1+c_2\right)\left(-1+4c_2\right)},$$

$$b''_4=\frac{2\left(3c_2-1\right)^3}{3\left(-1+4c_2\right)\left(6c_2^2+1-4c_2\right)},\ b''_5=1.$$

By letting $a_{21}=\frac{1}{2}, a_{31}=\frac{1}{4}, a_{42}=\frac{1}{6}, a_{43}=\frac{1}{8}, a_{54}=\frac{1}{10}, \bar{a}_{42}=\frac{1}{3}, \bar{a}_{54}=\frac{1}{5}, \bar{\bar{a}}_{32}=\frac{3}{4}, c_2=\frac{1}{6}$. Then, the coefficients of five-stage fourth-order RKTGG method denoted by RKTGG4 can be represented as follows (see Table 1):

**Table 1: The RKTGG4 Method**

$$\begin{array}{c|ccccc|ccccc|ccccc}
0 & 0 & & & & & 0 & & & & & 0 & & & & \\
\frac{1}{6} & \frac{1}{2} & 0 & & & & \frac{61}{144} & 0 & & & & \frac{1}{6} & 0 & & & \\
1 & \frac{1}{4} & \frac{5}{36} & 0 & & & 0 & \frac{1}{3} & 0 & & & \frac{1}{4} & \frac{3}{4} & 0 & & \\
\frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{8} & 0 & & 0 & \frac{1}{3} & -\frac{67}{72} & 0 & & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \\
1 & 0 & 0 & -\frac{13}{240} & \frac{1}{10} & 0 & 0 & 0 & \frac{19}{48} & \frac{1}{5} & 0 & \frac{19}{120} & \frac{91}{120} & -\frac{1}{12} & \frac{1}{6} & 0 \\
\hline
 & 0 & \frac{5}{36} & 0 & \frac{1}{36} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & \frac{2}{5} & -\frac{9}{10} & \frac{1}{2} & 1
\end{array}$$

## 5 Numerical Experiments

In this segment, all the problems including $y'''=f(x,y,y',y'')$ are tested upon. The numerical outcomes are compared with the results obtained when the same set of problems is reduced to a system of first-order equations and is solved utilizing the existing Runge-Kutta of the same order.

(i) **RKTGG4:** The new fourth-order five-stage RKTGG method derived in this paper.

(ii) **RKS4:** The four-stage fourth-order RK method as in Butcher [14].

(iii) **RKZ4:** The five-stage fourth-order RK method given in Hairer et al. [16].

(iv) **RKE4:** The six-stage fourth-order RK method given in Lambert [15].

**Problem 1** (Homogeneous Linear Problem)

$$y'''(x)=-6y''(x),$$
$$y(0)=1, y'(0)=-1, y''(0)=2,$$

Theoretical solution :

$$y(x)=\frac{17}{18}-\frac{2}{3}x+\frac{1}{18}e^{-6x}.$$

**Problem 2** (Inhomogeneous Linear Problem)

$$y'''(x)=y''(x)+\cos^2(x)+\sin(x)+1,$$
$$y(0)=0, y'(0)=1, y''(0)=0,$$

Theoretical solution :

$$y(x) = \frac{1}{2}\cos(2x) - \frac{1}{20}\sin(2x) + \frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) + \frac{21}{10}e^x - \frac{3}{4}x^2 - \frac{3}{2}x - \frac{21}{8}.$$

**Problem 3** (Homogeneous Linear Problem)

$$y'''(x) = -y(x) + 2y''(x),$$
$$y(0) = 0, y'(0) = 1, y''(0) = 0,$$

Theoretical solution :

$$y(x) = \frac{1}{5}\sqrt{5}e^{\frac{1}{2}(\sqrt{5}+1)x} - \frac{1}{5}\sqrt{5}e^{-\frac{1}{2}(\sqrt{5}-1)x}.$$

**Problem 4** (Homogeneous Nonlinear Problem)

$$y'''(x) = -\frac{3y''(x)}{2(y(x))^2},$$
$$y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = -\frac{1}{4},$$

Theoretical solution :

$$y(x) = \sqrt{x+1}.$$

**Problem 5** (Nonlinear System)

$$y_1'''(x) = -\frac{1}{4}e^{4x}y_3(x)y_2''(x),$$
$$y_1(0) = 1, y_1'(0) = -1, y_1''(0) = 1,$$
$$y_2'''(x) = -\frac{8}{9}e^{2x}y_1(x)y_3''(x),$$
$$y_2(0) = 1, y_2'(0) = -2, y_2''(0) = 4,$$
$$y_3'''(x) = -27y_2(x)y_1''(x),$$
$$y_3(0) = 1, y_3'(0) = -3, y_3''(0) = 9,$$

Theoretical solution :

$$y_1(x) = e^{-x},$$
$$y_2(x) = e^{-2x},$$
$$y_3(x) = e^{-3x}.$$

## 6 An Application to a Problem in Thin Film Flow

In this part, we use the suggested method to a well-known problem in physics regarding the thin film flow of a liquid. Several researchers have discussed this problem. Momoniat and Mahomed [17], constructed symmetry reduction and numerical solution of a third-order ODE from thin film flow. Tuck and Schwartz [18], discussed the flow of a thin film of viscous fluid over a solid surface. Tension and gravity, as well as viscosity, are taken into account. The problem was formulated using

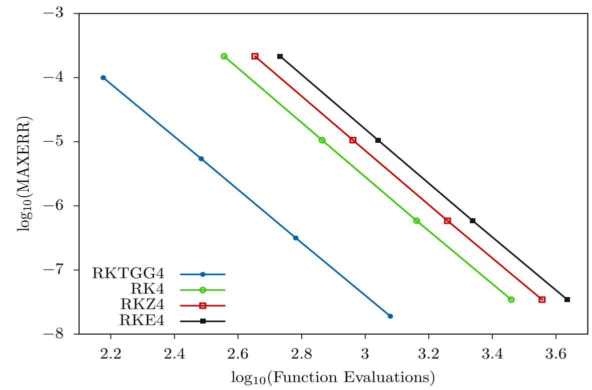


**Fig. 1: Comparison for RKTGG4, RK4, RKZ4 and RKE4 Problem 1 with $X_{end}$=3**

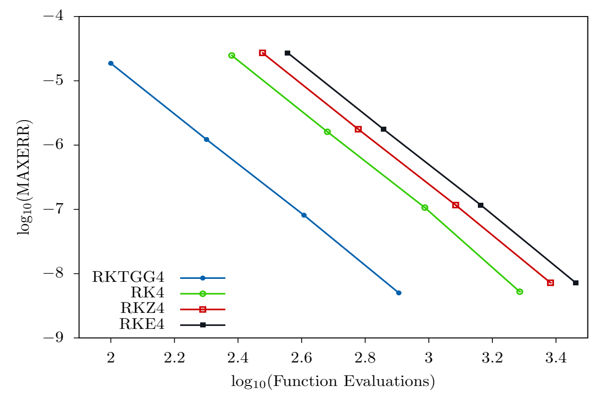


**Fig. 2: Comparison for RKTGG4, RK4, RKZ4 and RKE4 Problem 2 with $X_{end}$=2**

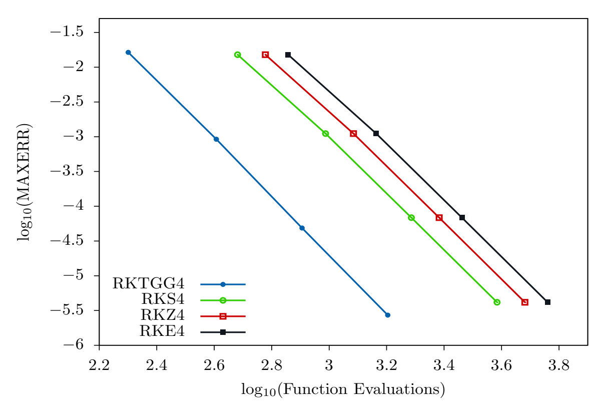


**Fig. 3: Comparison for RKTGG4, RK4, RKZ4 and RKE4 Problem 3 with $X_{end}$=4**

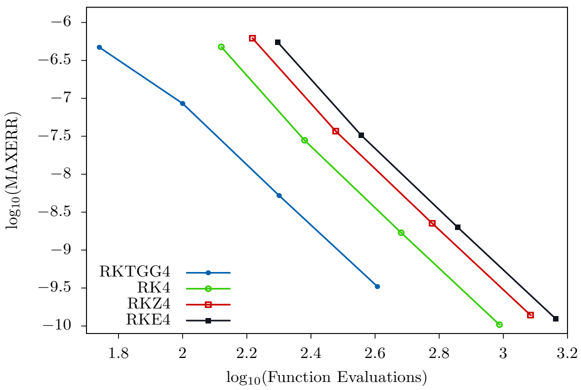


**Fig. 4: Comparison for RKTGG4, RK4, RKZ4 and RKE4 Problem 4 with $X_{end}$=1**

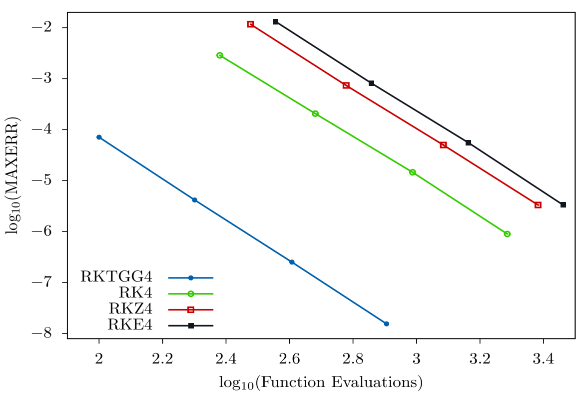


**Fig. 5: Comparison for RKTGG4, RK4, RKZ4 and RKE4 Problem 5 with $X_{end}$=2**

third-order ODE as follows:

$$\frac{d^3y}{dx^3} = f(y) \tag{36}$$

for the film profile $y(x)$ in a coordinate frame moving with the fluid. The form of $f(y)$ varies according to the physical context. Different forms of the function $f$ are studied in [18]. For drainage down a dry surface, the form of $f(y)$ is given as:

$$\frac{d^3y}{dx^3} = -1 + \frac{1}{y^2}. \tag{37}$$

When the surface is pre-wetted by a thin film with thickness $\xi > 0$ (where $\xi > 0$ is very small), the function $f$ is given by

$$f(y) = -1 + \frac{1+\xi+\xi^2}{y^2} - \frac{\xi+\xi^2}{y^3}. \tag{38}$$

Problems concerning the flow of thin films of viscous fluid with a free surface in which surface tension effects play a role typically lead to third-order ordinary differential equations governing the shape of the free surface of the fluid, $y = y(x)$. According to [18], one such equation is

$$y''' = y^{-k}, x \geq x_0 \tag{39}$$

**Table 2:** Table comparing values of the numerical solution, a fourth-order Runge-Kutta method (RK4, RKE4), and our new method (RKTGG4) method at $x \in [0, 0.2, 0.4, 0.6, 0.8, 1.0]$ taking $h = 0.1$ and $k = 2$ with the initial conditions $y(0) = y'(0) = y''(0) = 1$.

| $x$ | Exact solution | RK4 | RKE4 | RKTGG4 |
|---|---|---|---|---|
| 0.0 | 1.000000000 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 0.2 | 1.221211030 | 1.2212105060 | 1.2212107764 | 1.2212093404 |
| 0.4 | 1.488834893 | 1.4888356990 | 1.4888512316 | 1.4888322182 |
| 0.6 | 1.807361404 | 1.8073626884 | 1.8074900091 | 1.8073559531 |
| 0.8 | 2.179819234 | 2.1798208831 | 2.1803395852 | 2.1798100221 |
| 1.0 | 2.608275822 | 2.6082768844 | 2.6097383193 | 2.6082610510 |

**Table 3:** Table comparing values of the numerical solution, a fourth-order Runge-Kutta method (RK4, RKE4), and our new method (RKTGG4) method at $x \in [0, 0.2, 0.4, 0.6, 0.8, 1.0]$ taking $h = 0.01$ and $k = 2$ with the initial conditions $y(0) = y'(0) = y''(0) = 1$.

| $x$ | Exact solution | RK4 | RKE4 | RKTGG4 |
|---|---|---|---|---|
| 0.0 | 1.000000000 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 0.2 | 1.221211030 | 1.2212100046 | 1.2212103652 | 1.2212100045 |
| 0.4 | 1.488834893 | 1.4888347800 | 1.4888507105 | 1.4888347796 |
| 0.6 | 1.807361404 | 1.8073613978 | 1.8074895517 | 1.8073613971 |
| 0.8 | 2.179819234 | 2.1798192341 | 2.1803393119 | 2.1798192330 |
| 1.0 | 2.608275822 | 2.6082748678 | 2.6097383271 | 2.6082748662 |

**Table 4:** Table comparing values of the numerical solution, a fifth-order Runge-Kutta method (RK4, RKE4), and our new method (RKTGG4) method at $x \in [0, 0.2, 0.4, 0.6, 0.8, 1.0]$ taking $h = 0.1$ and $k = 3$ with the initial conditions $y(0) = y'(0) = y''(0) = 1$.

| $x$ | RK4 | RKE4 | RKTGG4 |
|---|---|---|---|
| 0.0 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 0.2 | 1.2211559590 | 1.2211564251 | 1.2211541652 |
| 0.4 | 1.4881067401 | 1.4881307936 | 1.4881016329 |
| 0.6 | 1.8042645823 | 1.8044430234 | 1.8042548878 |
| 0.8 | 2.1715254210 | 2.1721919823 | 2.1715098965 |
| 1.0 | 2.5909615178 | 2.5927033256 | 2.5909389202 |

**Table 5:** Table comparing values of the numerical solution, a fifth-order Runge-Kutta method (RK4, RKE4), and our new method (RKTGG4) method at $x \in [0, 0.2, 0.4, 0.6, 0.8, 1.0]$ taking $h = 0.01$ and $k = 3$ with the initial conditions $y(0) = y'(0) = y''(0) = 1$.

| $x$ | RK4 | RKE4 | RKTGG4 |
|---|---|---|---|
| 0.0 | 1.0000000000 | 1.0000000000 | 1.0000000000 |
| 0.2 | 1.2211551425 | 1.2211557726 | 1.2211551423 |
| 0.4 | 1.4881052844 | 1.4881300313 | 1.4881052839 |
| 0.6 | 1.8042625484 | 1.8044424292 | 1.8042625474 |
| 0.8 | 2.1715227984 | 2.1721917529 | 2.1715227969 |
| 1.0 | 2.59095825948 | 2.5927036287 | 2.5909582573 |

with initial conditions

$$y(x_0) = \delta, y'(x_0) = \zeta, y''(x_0) = \lambda, \tag{40}$$

where $\delta$, $\zeta$, and $\lambda$ are constants, is of particular importance because it describes the dynamic balance between surface and viscous forces in a thin fluid layer in the neglect of gravity. For comparison purposes, we use Runge-Kutta methods which are fourth-order (RK4 and RKE4) methods, respectively. To use Runge-Kutta methods we write (1) as a system of three first-order equations. Following [19], we can write (39) as the following system:

$$\frac{dy_1}{dx} = y_2(x), \frac{dy_2}{dx} = y_3(x), \frac{dy_3}{dx} = y_1^{-k}(x), \tag{41}$$

where

$$y_1(0) = 1, y_2(0) = 1, y_3(x) = 1, \tag{42}$$

we have taken $x_0 = 0$ and $\delta = \zeta = \lambda$=1. Unfortunately, for general $k$, (39) cannot be solved analytically. However, we can use these reductions to determine an the efficient way to solve (1) numerically.

We focus on the cases $k$ = 2 and $k$= 3. The results are displayed in Tables 2 and 3 for the case $k$= 2 and Tables 4 and 5 for the case $k$= 3.

## 7 Discussion and Conclusion

In this study, a five-stage fourth-order explicit RKTGG method denoted RKTGG4 for directly solving general third-order differential equations of the form $y''' = f(x, y, y', y'')$ has been derived and the comparison are made with existing RK methods that have the same algebraic order which are found in [14,15] and [16]. Furthermore, numerical comparison is based on the computation of the maximum global error of the solution $(max(\mid y(x_n) - y_n \mid))$ which is equal to the maximum absolute errors of the actual and computed solutions. In general, the numerical results show graphically as displayed in Figures 1–5 show that the global error of the

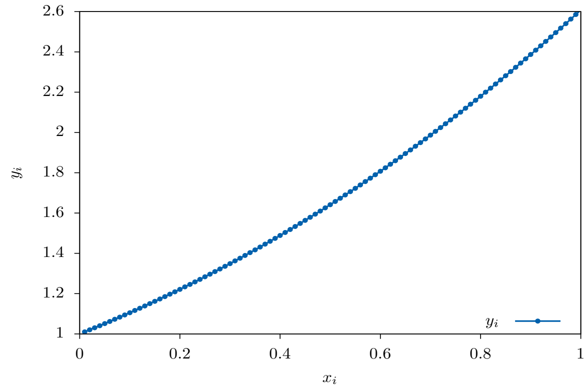


**Fig. 6: Plot of the solution $y_i$ for problem (39) for $k = 2, h = 0.01$**

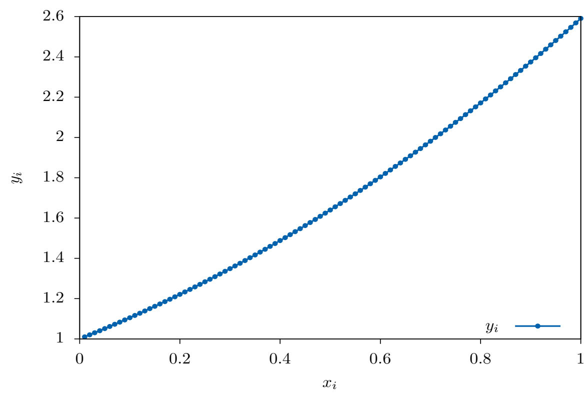


**Fig. 7: Plot of the solution $y_i$ for problem (39) for $k = 3, h = 0.01$**

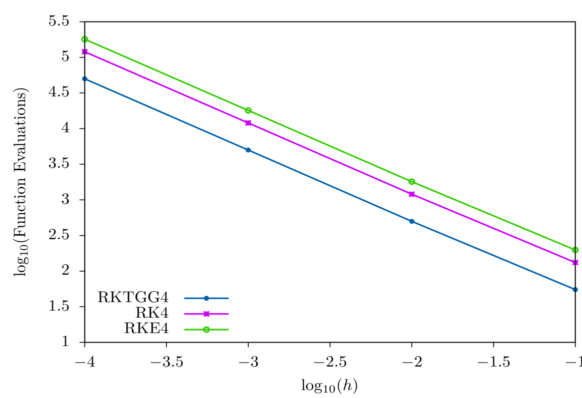


**Fig. 8: Plot of graph for function evaluations against step size, $h$ for $k = 3, h = 1/10^i, i = 1...4$.**

new method. In Figures 6 and 7, we plot the numerical solution, $y_i$ for $k = 2$ and $k = 3$, respectively, with $h = 0.01$ . Figure 8 shows that the new RKTGG4 method requires less function evaluations than the RK4 and RKE4 methods. This is because when problem (39) is solved using RK4 and RKE4 method, it needs to be reduced to a system of first-order equations which is three times the dimension. From numerical results, we notice that the new RKTGG4 method is more efficient compared with existing RK methods and it has been shown that the new method is more accurate and competent when solving general third-order ODEs.

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