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Continuous Family of Solutions for Fractional Integro-Differential Inclusions of Caputo-Katugampola Type

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Abstract: Investigating the properties of the new fractional operators is an important issue within the fractional calculus. In this manuscript a continuous family of solutions for a fractional integro-differential inclusion involving Caputo-Katugampola fractional derivative is obtained.

Keywords: Fractional derivative, differential inclusion, initial value problem.

1 Introduction

A strong development of the theory of differential equations and inclusions of fractional order can be seen during the last years [1,2,3,4,5] . We recall that fractional differential equations can model better many physical phenomena.

Recently, a generalized Caputo-Katugampola fractional derivative was suggested in [6] by Katugampola and afterwards he provided the existence of solutions for fractional differential equations defined by this derivative. This Caputo-Katugampola fractional derivative extends the well known Caputo and Caputo-Hadamard fractional derivatives. Recently, several qualitative properties of solutions of fractional differential equations defined by Caputo-Katugampola derivative were obtained [7,8].

In the present paper we study the following Cauchy problem

$$D_c^{\alpha,\rho}x(t)\in F(t,x(t),W(x)(t)) \quad a.e.\ ([0,T]), \quad x(0)=x_0, \tag{1.1}$$

where $\alpha\in(0,1]$, $\rho>0$, $J=[0,T]$, $D_c^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative, $FJ\times\mathbf{R}\times\mathbf{R}\to\mathscr{P}(\mathbf{R})$ is a set-valued map, $W:C(J,\mathbf{R})\to C(J,\mathbf{R})$ is the nonlinear $W(x)(t)=\int_0^t v(t,s,x(s))ds$, $v(.,.,.):J\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ and $x_0\in\mathbf{R}$.

The goal of this paper is to prove the existence of solutions continuously depending on a parameter for problem (1.1). Our main theorem is, at the same time, a continuous version of Filippov's theorem [9] for problem (1.1). On the other hand, as a consequence of this result we obtain a continuous selection of the solution set of problem (1.1). The proof is essentially based on the Bressan-and Colombo selection theorem [10].

We note that similar results for other classes of fractional differential inclusions defined by Riemann-Liouville, Caputo or Hadamard fractional derivatives exists in the literature [11,12,13] . The present paper extends and unifies all these results in the case of the more general problem (1.1).

The manuscript is organized as follows: in Section 2 we present some preliminary results and Section 3 is devoted to our main results. The conclusions are presented in Section 4.

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2 Preliminaries

Let $T > 0$, $J := [0,T]$. In what follows $\mathscr{L}(J)$ is the σ-algebra of all Lebesgue measurable subsets of J, X is a real separable Banach space. As usual, $\mathscr{P}(X)$ is the set of all nonempty subsets of X and $\mathscr{B}(X)$ is the set of all Borel subsets of X. If $C \subset J$ then $\chi_C(.) : J \to \{0,1\}$ is the characteristic function of C. If $C \subset X$, its closure is denoted by cl(C).

The Hausdorff distance between the closed sets $C,D \subset X$ is $d_H(C,D) = \max\{d^*(C,D), d^*(D,C)\}$, where $d^*(C,D) = \sup\{d(c,D);\, c \in C\}$ and $d(y,C) = \inf\{|y-c|;\, c \in C\}$.

By $C(J,X)$ we understand the Banach space of all continuous functions $y(.) : J \to X$. Its norm is $|y(.)|_C = \sup_{t\in J}|x(t)|$. $L^1(J,X)$ is the Banach space of all (Bochner) integrable functions $y(.) : J \to X$ endowed with the norm $|y(.)|_1 = \int_0^T |y(t)|dt$.

Some preliminary results that needed the sequel are presented. The following lemma is proved in [14].

Lemma 2.1. *Consider $x : J \to X$ a measurable function and consider $H : J \to \mathscr{P}(X)$ set-valued which has closed values and is measurable.*

Then, if $\varepsilon : J \to (0,\infty)$ is measurable, there exists a measurable selection $h : J \to X$ of $H(\cdot)$ which satisfies

$$|x(t) - h(t)| < d(x(t), H(t)) + \varepsilon(t) \quad a.e.\ (J).$$

Definition 2.2. The set $A \subset L^1(J,X)$ is called *decomposable* if for any $b(\cdot), c(\cdot) \in A$ and any subset $D \in \mathscr{L}(J)$ one has $a\chi + b\chi_{J\setminus A} \in A$.

$\mathscr{D}(J,X)$ denotes the set of all decomposable closed subsets of $L^1(J,X)$.

In what follows (S,d) is a separable metric space. The next two lemmas are proved in [10].

Lemma 2.3. *Consider $H(.,.) : J \times S \to \mathscr{P}(X)$, $\mathscr{L}(J) \otimes \mathscr{B}(S)$-measurable set-valued map with closed values such that $H(t,.)$ is lower semicontinuous for all $t \in J$.*

Then the set-valued map $H^*(.) : S \to \mathscr{D}(J,X)$

$$H^*(s) = \{f \in L^1(J,X); \quad f(t) \in H(t,s) \quad a.e.\ (J)\}$$

has nonempty closed values and is lower semicontinuous iff there exists $q(.) : S \to L^1(J,X)$ continuous that verifies

$$d(0, H(t,s)) \le q(s)(t) \quad a.e.\ (J),\ \forall s \in S.$$

Lemma 2.4. *Consider $H(.) : S \to \mathscr{D}(J,X)$ a set-valued map with closed decomposable values that is lower semicontinuous, consider $a(.) : S \to L^1(J,X)$, $b(.) : S \to L^1(J,\mathbf{R})$ continuous functions such that the values of the set-valued map $F(.) : S \to \mathscr{D}(J,X)$ defined by*

$$F(s) = cl\{f \in H(s); \quad |f(t) - a(s)(t)| < b(s)(t) \quad a.e.\ (J)\}$$

are nonempty.

Then $F(.)$ has a continuous selection.

The following notions were introduced [6]. Let $\rho > 0$.

Definition 2.5. a) *The generalized left-sided fractional integral of order* $\alpha > 0$ of a Lebesgue integrable function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$I^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds, \tag{2.1}$$

providing the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is Gamma function.

b) *The generalized fractional derivative*, corresponding to the generalized left-sided fractional integral in (2.1) of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$D^{\alpha,\rho} f(t) = (t^{1-\rho}\frac{d}{dt})^n (I^{n-\alpha,\rho})(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} (t^{1-\rho}\frac{d}{dt})^n \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds$$

if the integral exists and $n = [\alpha]$.

c) *The Caputo-Katugampola* generalized fractional derivative is defined by

$$D_c^{\alpha,\rho} f(t) = (D^{\alpha,\rho}[f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} s^k])(t)$$

We note that if $\rho = 1$, the Caputo-Katugampola fractional derivative becomes the well- known Caputo fractional derivative. On the other hand, passing to the limit with $\rho \to 0+$, the above definition yields the Hadamard fractional derivative.

In what follows $\rho > 0$ and $\alpha \in [0,1]$.

Lemma 2.6. *For a given integrable function $h(.) : [0,T] \to \mathbf{R}$, the unique solution of the initial value problem*

$$D_c^{\alpha,\rho} x(t) = h(t) \quad a.e. \ ([0,T]), \quad x(0) = x_0,$$

is given by

$$x(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} h(s) ds$$

For the proof of Lemma 2.6, see [6]; namely, Lemma 4.2.

By a solution of the problem (1.1) we mean a function $x \in C(J,\mathbf{R})$ for which there exists a function $h \in L^1(J,\mathbf{R})$ satisfying $h(t) \in F(t,x(t),W(x)(t))$ $a.e.$ (J), $D_c^{\alpha,\rho} x(t) = h(t)$ $a.e.$ (J) and $x(0) = x_0$.
The solution set of (1.1) is then denoted with $\mathscr{S}(x_0)$.

3 Main results

Below we assume the following hypotheses.

Hypothesis 3.1. i) $F(.,.) : J \times \mathbf{R} \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is $\mathscr{L}(J) \otimes \mathscr{B}(\mathbf{R} \times \mathbf{R})$ measurable with nonempty closed values.
ii) There exists $l(.) \in L^1(J,(0,\infty))$ in such a way that, for almost all $t \in J$

$$d_H(F(t,u_1,v_1),F(t,u_2,v_2)) \le l(t)(|u_1 - u_2| + |v_1 - v_2|) \quad \forall\, u_1,u_2,v_1,v_2 \in \mathbf{R}.$$

iii) The mapping $v(.,.,.) : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ verifies: $\forall y \in \mathbf{R}$, $(s,t) \to v(s,t,y)$ is measurable.
iv) $|v(s,t,y) - v(s,t,x)| \le l(t)|y - x| \quad a.e. \ (s,t) \in J \times J, \quad \forall y,x \in \mathbf{R}$.

Hypothesis 3.2. (i) S is a separable metric space, the mappings $a(.) : S \to \mathbf{R}$ and $\varepsilon(.) : S \to (0,\infty)$ are continuous.
(ii) There exists $g(.),q(.) : S \to L^1(J,\mathbf{R})$, $y(.) : S \to C(J,\mathbf{R})$ continuous that satisfy

$$(Dy(s))_c^{\alpha,\rho}(t) = g(s)(t) \quad a.e. \ t \in J, \quad \forall s \in S,$$

$$d(g(s)(t),F(t,y(s)(t),W(y(s)(.))(t)) \le q(s)(t) \quad a.e. \ t \in J, \ \forall\, s \in S.$$

We utilize below the following notation

$$k(t) := l(t)(1 + \int_0^t l(u)du), \quad t \in J,$$

$$\xi(s) = \frac{1}{1 - I^{\alpha,\rho}k}(|a(s) - y(s)(0)| + \varepsilon(s) + I^{\alpha,\rho}q(s)), \ s \in S,$$

where $I^{\alpha,\rho}k := \sup_{t \in J} |I^{\alpha,\rho}k(t)|$ and $I^{\alpha,\rho}q(s) := \sup_{t \in J} |I^{\alpha,\rho}q(s)(t)|$.

Theorem 3.3. *Hypotheses 3.1 and 3.2 are verified.*
If $I^{\alpha,\rho}k < 1$, then there exists $x(.) : S \to C(J,\mathbf{R})$ continuous, $x(s)(.)$ denotes a solution of

$$D_c^{\alpha,\rho} z(t) \in F(t,z(t),W(z)(t)), \quad z(0) = a(s)$$

such that, $\forall (t,s) \in J \times S$,

$$|y(s)(t) - x(s)(t)| \le \xi(s).$$

Proof. In what follows we consider the notations $b(s) = |a(s) - y(s)(0)| + \varepsilon(s)$, $q_n(s) := (I^{\alpha,\rho}k)^{n-1}(b(s) + I^{\alpha,\rho}q(s))$, $n \ge 1$, $x_0(s)(t) = y(s)(t)$, $\forall s \in S$. Define the set-valued maps

$$A_0(s) = \{ f \in L^1(J,\mathbf{R}); \quad f(t) \in F(t,y(s)(t),W(y(s)(.))(t)) \quad a.e. \ (J) \},$$

$$B_0(s) = \text{cl}\{f \in A_0(s); \quad |f(t) - g(s)(t)| < q(s) + \frac{\rho^\alpha \Gamma(\alpha+1)}{T^{\rho\alpha}}\varepsilon(s)\}.$$

By our assumptions, $d(g(s)(t), F(t, y(s)(t), W(y(s)(.))(t)) \leq q(s)(t) < q(s)(t) + \frac{\rho^\alpha \Gamma(\alpha+1)}{T^{\rho\alpha}}\varepsilon(s)$, so according with Lemma 2.1, $B_0(s)$ is not empty.

Put $G_0(t,s) = F(t, y(s)(t), V(y(s)(.))(t))$ and one has

$$d(0, G_0(t,s)) \leq |g(s)(t)| + q(s)(t) = q^*(s)(t)$$

with $q^*(.) : S \to L^1(J, \mathbf{R})$ being continuous.

Taking into account Lemmas 2.3 and 2.4 we deduce that there exists h_0 a selection of B_0 that is continuous, i.e.

$$h_0(s)(t) \in F(t, y(s)(t), W(y(s)(.))(t)) \quad a.e.\,(J),\ \forall s \in S,$$

$$|h_0(s)(t) - g(s)(t)| \leq q(s)(t) + \frac{\rho^\alpha \Gamma(\alpha+1)}{T^{\rho\alpha}}\varepsilon(s) \quad \forall t \in J,\ s \in S.$$

Set $x_1(s)(t) = a(s) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} h_0(s)(u)du$ and we have

$$|x_1(s)(t) - x_0(s)(t)| \leq |a(s) - y(s)(0)| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} |h_0(s)(v) - g(s)(v)| dv \leq |a(s) - y(s)(0)| +$$

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} (q(s)(u) + \frac{\rho^\alpha \Gamma(\alpha+1)}{T^{\rho\alpha}}\varepsilon(s))du \leq |a(s) - y(s)(0)| + I^{\alpha,\rho} q(s) + \frac{\rho^\alpha \Gamma(\alpha+1)}{T^{\rho\alpha}\Gamma(\alpha)}\varepsilon(s)\cdot$$

$$\int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} du \leq b(s) + I^{\alpha,\rho} q(s) = q_1(s).$$

Following an idea in [7], we define the sequences $h_n(.) : S \to L^1(J, \mathbf{R})$, $x_n(.) : S \to C(J, \mathbf{R})$ such that

a) $x_n(.) : S \to C(J, \mathbf{R})$, $h_n(.) : S \to L^1(J, \mathbf{R})$ are continuous.

b) $h_n(s)(t) \in F(t, x_n(s)(t), W(x_n(s)(.))(t))$, $s \in S$, a.e. (J).

c) $|h_n(s)(t) - h_{n-1}(s)(t)| \leq k(t) q_n(s)$, $s \in S$, a.e. (J).

d) $x_{n+1}(s)(t) = a(s) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} h_n(s)(u)du.$

If we assume that $h_i(.), x_i(.)$ are already constructed with a)-c) and define $x_{n+1}(.)$ as in d). It follows from c) and d) that

$$\begin{aligned} &|x_{n+1}(s)(t) - x_n(s)(t)| \leq \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} |h_n(s)(u) - h_{n-1}(s)(u)| du \\ &\leq \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} k(u) q_n(s) du \leq I^{\alpha,\rho} k \cdot q_n(s) = q_{n+1}(s). \end{aligned} \tag{3.1}$$

Also we have

$$\begin{aligned} &d(h_n(s)(t), F(t, x_{n+1}(s)(t), W(x_{n+1}(s)(.))(t)) \leq l(t)(|x_{n+1}(s)(t) - x_n(s)(t)| + \textstyle\int_0^t l(u)|x_{n+1}(s)(v) - x_n(s)(v)|dv) \leq \\ &l(t)(1 + \textstyle\int_0^t l(u)du) q_{n+1}(s) = k(t) q_{n+1}(s). \end{aligned}$$

For $s \in S$ we define

$$A_{n+1}(s) = \{f \in L^1(J, \mathbf{R}); f(t) \in F(t, x_{n+1}(s)(t), W(x_{n+1}(s)(.))(t))\ a.e.\,(J)\},$$

$$B_{n+1}(s) = \text{cl}\{f \in A_{n+1}(s); \quad |f(t) - h_n(s)(t)| < k(t) q_{n+1}(s)\ a.e.\,(J)\}.$$

In order to prove that $B_{n+1}(s)$ is nonempty we point out that function $t \to p_n(s)(t) = ((I^{\alpha,\rho}k)^{n-1} - (I^{\alpha,\rho}k)^n)(b(s) + I^{\alpha,\rho} q(s)) l(t)$ is strictly positive and measurable for any s. We have

$$d(h_n(s)(t), F(t, x_{n+1}(s)(t), W(x_{n+1}(s)(.))(t)) \leq k(t)|x_{n+1}(s)(t) - x_n(s)(t)| - p_n(s)(t) \leq k(t) q_{n+1}(s)$$

With Lemma 2.1 we find $w(.) \in L^1(J, \mathbf{R})$ such that $w(t) \in F(t, x_n(s)(t), W(x_{n+1}(s)(.))(t))$ a.e. (J) and

$$|w(t) - h_n(s)(t)| < d(h_n(s)(t), F(t, x_n(s)(t), W(x_{n+1}(s)(.))(t)) + p_n(s)(t)$$

i.e., $B_{n+1}(s)$ is nonempty.

Put $G_{n+1}(t,s) = F(t, x_{n+1}(s)(t), W(x_{n+1}(s)(.))(t))$. One may estimate

$$d(0, G_{n+1}(t,s)) \leq |h_n(s)(t)| + k(t)|x_{n+1}(s)(t) - x_n(s)(t)| \leq |h_n(s)(t)| + k(t)q_{n+1}(s) = q^*_{n+1}(s)(t) \quad a.e. (I)$$

with $q^*_{n+1}(.) : S \to L^1(J, \mathbf{R})$ being continuous.

As above we find $h_{n+1}(.) : S \to L^1(I, \mathbf{R})$ being continuous such that

$$h_{n+1}(s)(t) \in F(t, x_{n+1}(s)(t), W(x_{n+1}(s)(.))(t)) \quad \forall s \in S, \ a.e. (J),$$

$$|h_{n+1}(s)(t) - h_n(s)(t)| \leq k(t)q_{n+1}(s) \quad \forall s \in S, \ a.e. (J).$$

Taking into account conditions c), d) and (3.1) one has

$$|x_{n+1}(s)(.) - x_n(s)(.)|_C \leq I^{\alpha,\rho} k q_n(s) = q_{n+1}(s) = (I^{\alpha,\rho}k)^n (b(s) + I^{\alpha,\rho} q(s)) \tag{3.2}$$

$$|h_{n+1}(s)(.) - h_n(s)(.)|_1 \leq |k(.)|_1 q_n(s) = |k(.)|_1 (I^{\alpha,\rho}k)^n (b(s) + I^{\alpha,\rho} q(s)). \tag{3.3}$$

Therefore sequences $h_n(s)(.)$, $x_n(s)(.)$ are Cauchy in spaces $L^1(J, \mathbf{R})$ and $C(J, \mathbf{R})$, respectively. Denote by $h(.) : S \to L^1(J, \mathbf{R})$, $x(.) : S \to C(J, \mathbf{R})$ with their limits. The mapping $s \to b(s) + |I^{\alpha,\rho} q(s)|$ is continuous, therefore locally is bounded. Thus from (3.3) we deduce the continuity of $s \to h(s)(.)$ from S into $L^1(J, \mathbf{R})$.

As above, from (3.2), we obtain that the Cauchy condition is satisfied for the sequence $x_n(s)(.)$ locally uniformly with respect to s. Thus, the mapping $s \to x(s)(.)$ is continuous. At the same time, since the convergence of $x_n(s)(.)$ to $x(s)(.)$ is uniform and

$$d(h_n(s)(t), F(t, x(s)(t), W(x(s)(.))(t)) \leq M(t)|x_n(s)(t) - x(s)(t)| \quad a.e. \ (J),$$

$\forall s \in S$ we may pass to the limit and deduce that

$$h(s)(t) \in F(t, x(s)(t), W(x(s)(.))(t)) \quad \forall s \in S, \ a.e. \ (J).$$

We have

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} h_n(s)(u) du - \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} h(s)(u) du \right| \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} |h_n(s)(u) -$$

$$h(s)(u)| du \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} k(u). |x_{n+1}(s)(.) - x_n(s)(.)|_C du \leq I^{\alpha,\rho} k. |x_{n+1}(s)(.) - x_n(s)(.)|_C.$$

Passing to the limit in d) we find

$$x(s)(t) = a(s) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - u^\rho)^{\alpha-1} u^{\rho-1} h(s)(u) du.$$

We add for all $n \geq 1$ inequalities (3.1) and we get

$$|x_{n+1}(s)(t) - y(s)(t)| \leq \sum_{l=1}^{n} q_l(s) \leq \xi(s).$$

Finally, passing to the limit in the last inequality we end the proof of the theorem.

From Theorem 3.3 we may find a selection of the solution set of problem (1.1) that is continuous.

Hypothesis 3.4. Hypothesis 3.1 is fulfilled, $I^{\alpha,\rho}k < 1$, $q_0(.) \in L^1(J, \mathbf{R}_+)$ exists and $\mathrm{d}(0, F(t, 0, W(0)(t)) \leq q_0(t)$ a.e. (J).

Corollary 3.5. *Hypothesis 3.4 is verified.*

Then there exists a function $s(.,.) : J \times \mathbf{R} \to \mathbf{R}$ *such that*

a) $s(., x) \in \mathscr{S}(x)$, $\forall x \in \mathbf{R}$.

b) $x \to s(., x)$ *from* $\mathbf{R}$ *into* $C(J, \mathbf{R})$ *is continuous.*

Proof. It is enough to put in Theorem 3.3 $S = \mathbf{R}$, $a(x) = x$, $\forall x \in \mathbf{R}$, $\varepsilon(.) : \mathbf{R} \to (0, \infty)$ a given continuous mapping, $g(.) = 0$, $y(.) = 0$, $q(x)(t) = q_0(t)$ $\forall x \in \mathbf{R}$, $t \in J$.

4 Conclusion

We discussed the existence of solutions continuously depending on a parameter corresponding to the problem which can be seen in (1.1). The theorem 3.3 contains the main results of this manuscript. Besides, we obtain a continuous selection of the solution set of problem (1.1).

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