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Reformulation Complex Scalar Field Interacting With the Electromagnetic Lagrangian Density by Riemann- Liouville Factional Derivative

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Abstract: We recast complex scalar fields as interacting field using fractional derivatives. to be more developed By applying the Hamiltonian formulation using fractional derivatives to the complex scalar fields, we applied the Hamiltonian formulation using fractional derivatives to the complex scalar fields. In addition, we observed that the Euler-Lagrange equation and the Hamiltonian equation yield the same result. Finally, we studied an example to elucidate the results
Keywords: Fractional Derivatives; Hamiltonian Formulation; Euler Lagrange Equations ; complex scalar field interacting with the electromagnetic.

1 Introduction

Fractional calculus is an extension of classical calculus. In this branch of mathematics, definitions are established for integrals and derivatives of arbitrary non-integer (even complex) order. It began in 1695 when Leibniz postulated his analysis of the derivative of order ½. Subsequently, it is developed primarily as a theoretical aspect of mathematics and was considered by some of the greatest names in mathematics, such as Euler, Lagrange, and Fourier. This branch of mathematics has seen a rapid development of interest in recent years, with applications in fractal [1], viscoelasticty [2], electrodynamics [3,4], optics [5,6], and thermodynamics [7]. The fractional calculus literature, which dates back to Leibniz, is rapidly expanding today [8,9,10,11,12,13,14]. Fractional derivatives, or more precisely, arbitrary order derivatives, are a generalization of classical calculus that have found applications in a variety of scientific and engineering fields [14,15,16,17,18,19,20]. There have been many attempt to include non-conservative forces in Lagrangian and Hamiltonian mechanics.
This mathematical field has been revived over time and is now used to study fractals, viscoelasticity, electrodynamics, optics, and thermodynamics [1,2,3,4,5,6,7]. The research on fractional calculus, which stretches back to Leibniz, is quickly growing today. Some of these applications are described in this section. The first is fractional calculus, which is used to understand the viscous interactions of fluids and solid structures. Reflection and transmission scattering operators for a slab of cancellous bone in the elastic frame are calculated using Blot's theory [19]. The approach of fractional derivatives in viscoelasticity concept is helpful since it allows for the formulation of constitutive equations for the elastic complex modulus of viscoelastic materials using only a few experimental measurements parameters. The fractional derivative method has also been utilized to investigate the impedances of many viscoelastic model [20]. Riewe [20, 21] formulated a version of the Euler-Lagrange equation for problems of calculus of variation with fractional derivatives. Recently, Diab et al [22] presented classical fields with fractional derivatives using the fractional Hamiltonian formulation. They obtained the fractional Hamilton's equations for two classical field examples. The formulation presented and the resulting equations are very similar to those appearing in classical field theory. Houas et al. [23] utilised MZ Sarikaya's fractional integral technique to develop new generalized fractional integral inequalities employing (k, s)-Riemann-Liouville integral operators. A few exceptional instances can be used to deduce classical and non-classical inequalities, such as the geometric series.
In another work, Alawaideh has recently found Euler-Lagrange fractional equations and Hamilton fractional equations for the Lee- wick field. A lagrangian density field

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is constructed using the Riemann-Liouville fractional derivative [24]. The key characteristics of the novel concepts presented in this manuscript are as follows.

- Complex scalar fields interacting with an electromagnetic field are rewritten with a fractional derivative to yield the Hamilton equations. This is the first time motion equations have been derived in terms of fractional derivatives using complex scalar fields interacting with electromagnetic fields and Hamilton's equation.

- The fractional order is used for the present formulation, making them more complex to solve in practice. As a result, we provide a one-of-a-kind and very successful technique.

- These formulations have been generalized such that they can be used with continuous first-order derivative systems. A generalized electrodynamics problem involving complex scalar fields interacting with an electromagnetic field is solved using this method.

The goal of this study is to develop fractional Hamiltonian equations for the combined scalar and electromagnetic fields using the Riemann-Liouville fractional derivative method.

The remaining of this paper is organized as follows: In Section 2, the definitions of fractional derivatives are discussed briefly. In Section 3 the fractional form of Euler-Lagrangian equation is presented. In Section 4, is devoted to the equations of motion in terms of Hamiltonian density in fractional form**.** In Section 5 one illustrative example is examined. Then in section 6 we obtain fractional combined scalar and electromagnetic equations using the Euler-Lagrange equations. Section 7 presents some fractional calculus application to scalar and electromagnetic fields. The work closes with some concluding remarks (Section 8).

2 Basic Definitions

In this section, we'll go over some basic mathematical definitions that you'll need for this work. The left Riemann-Liouville fractional derivative, or LRLFD, is defined as [25].

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)d\tau. \qquad (1)$$

The right Riemann- Liouville fractional derivative is defined as

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha+1} f(\tau)d\tau. \qquad (2)$$

where Γ denotes the Gamma function, and α is the order of the derivative such that $n-1<\alpha<n$. If α is an integer, these derivatives are defined in the usual sense, i.e.

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x) \qquad (3)$$

$$ {}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(t) \qquad \alpha = 1,2,.. \quad (4)$$

3 Fractions of Euler and Lagrange Interactions of a complex scalar field with the electromagnetic Lagrangian density equation

A continuous system with a lagrangian density expressed in terms of dynamical field variables, a generalized coordinate, and a derivative define as

$$ \mathcal{L} = \mathcal{L}\left[A_\mu, \phi, \phi^*, {}_aD_{x_\lambda}^\alpha A_\mu, {}_aD_{x_\lambda}^\alpha \phi, {}_aD_{x_\lambda}^\alpha \phi^*\right] \qquad (5)$$

Euler-Lagrange equation for such Lagrangian density in fractional form can be given as

$$ \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial\phi_\rho}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* \\ +\frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\lambda}^\alpha A_\mu}\delta\, {}_aD_{x_\lambda}^\alpha A_\mu \\ \frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\lambda}^\alpha \phi}\delta\, {}_aD_{x_\lambda}^\alpha \phi + \frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\lambda}^\alpha \phi^*}\delta\, {}_aD_{x_\lambda}^\alpha \phi^* \end{bmatrix} = 0 \qquad (6)$$

We can write the following using the variational principle:

$$ \delta S = \int \delta\mathcal{L}\ d^4x = 0 \qquad (7)$$

The variation of L can be obtained from Equation. (5) as follows:

$$ \delta\mathcal{L} = \begin{bmatrix} \frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial\phi_\rho}\delta\phi_\rho + \frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* \\ +\frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\lambda}^\alpha A_\mu}\delta\, {}_aD_{x_\lambda}^\alpha A_\mu + \frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\lambda}^\alpha \phi}\delta\, {}_aD_{x_\lambda}^\alpha \phi \\ +\frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\lambda}^\alpha \phi^*}\delta\, {}_aD_{x_\lambda}^\alpha \phi^* \end{bmatrix} d^3x \qquad (8)$$

By substituting Eq. (8) into Eq. (7) and using the commutation relation indicated below, we get:

$$ \begin{bmatrix} \delta\, {}_aD_{x_\lambda}^\alpha A_\mu = {}_aD_{x_\lambda}^\alpha \delta A_\mu \\ \delta\, {}_aD_{x_\lambda}^\alpha \phi = {}_aD_{x_\lambda}^\alpha \delta\phi \\ \delta\, {}_aD_{x_\lambda}^\alpha \phi^* = {}_aD_{x_\lambda}^\alpha \delta\phi^* \end{bmatrix} \qquad (9)$$

we get,

$$\int \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \phi^*}\delta\phi^* \\ + \underbrace{\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda} A_\mu}\ {}_aD^\alpha_{x_\lambda}\delta A_\mu}_{fourth} \\ + \underbrace{\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}\phi}\ {}_aD^\alpha_{x_\lambda}\delta\phi +}_{fifth} \\ \underbrace{\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}{}^\beta\phi^*}\ {}_aD^\alpha_{x_\lambda}\delta\phi^*}_{sixth} \end{bmatrix} d^4x = 0 \qquad (10)$$

Integrating the indicated terms in Eq. (10) with respect to time by parts yields the following:

$$\int \begin{bmatrix} \left[\frac{\partial \mathcal{L}}{\partial A_\mu} - {}_aD^\alpha_{x_\lambda}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}A_\mu}\right]\delta A_\mu \\ + \left[\frac{\partial \mathcal{L}}{\partial \phi} - {}_aD^\alpha_{x_\lambda}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}\phi}\right]\delta\phi \\ + \left[\frac{\partial \mathcal{L}}{\partial \phi^*} - {}_aD^\alpha_{x_\lambda}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}\phi^*}\right]\delta\phi^* \end{bmatrix} d^4x = 0$$

qA fractional Euler-Lagrange equation for such Lagrangian density is as follows:

$$\begin{bmatrix} \left[\frac{\partial \mathcal{L}}{\partial A_\mu} - {}_aD^\alpha_{x_\lambda}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}A_\mu}\right] + \\ \left[\frac{\partial \mathcal{L}}{\partial \phi} - {}_aD^\alpha_{x_\lambda}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}\phi}\right] \\ + \left[\frac{\partial \mathcal{L}}{\partial \phi^*} - {}_aD^\alpha_{x_\lambda}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_\lambda}\phi^*}\right] \end{bmatrix} = 0 \qquad (11)$$

Expanding A_μ, x_λ in terms of $(A_0, A_i, A_j), and\ (t, x_i)$ respectively, the Eq.11 has the form

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial A_0} - {}_aD^\alpha_t\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_t A_0} - {}_aD^\alpha_{x_i}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}A_0} = 0 & 12.a \\ \frac{\partial \mathcal{L}}{\partial A_i} - {}_aD^\alpha_t\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_t A_i} - {}_aD^\alpha_{x_i}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}A_i} = 0 & 12.b \\ \frac{\partial \mathcal{L}}{\partial A_j} - {}_aD^\alpha_t\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_t A_j} - {}_aD^\alpha_{x_i}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}A_j} = 0 & 12.c \\ \frac{\partial \mathcal{L}}{\partial \phi} - {}_aD^\alpha_t\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_t \phi} - {}_aD^\alpha_{x_i}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}\phi} = 0 & 12.d \\ \frac{\partial \mathcal{L}}{\partial \phi^*} - {}_aD^\alpha_t\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_t \phi^*} - {}_aD^\alpha_{x_i}\frac{\partial \mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}\phi^*} = 0 & 12.f \end{bmatrix} \qquad (12)$$

for α=1, ${}_aD^\alpha_{x_\lambda} = \partial_\lambda$ using α= 1, , we can rewrite Eq (11), become:

$$\left[\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\lambda\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\mu)}\right] + \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\lambda\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)}\right] + \left[\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\lambda\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi^*)}\right] = 0 \qquad (13)$$

4 Equations of Motion in terms of Hamiltonian Formulation

We begin our approach by assuming that the Lagrangian density is a function of field amplitude ϕ and that its fractional derivatives with regard to space and time are as follows:

$$\mathcal{L} = \mathcal{L}\begin{bmatrix} \phi, {}_aD^\alpha_t\phi, {}_aD^\alpha_{x_j}\phi, \varphi^*, {}_aD^\alpha_t\phi^*, {}_aD^\alpha_{x_j}\phi^*, A^0, A^i \\ , A^j, {}_aD^\alpha_t A^j, {}_aD^\alpha_t A^i, {}_aD^\alpha_t A^0, {}_aD^\alpha_{x_i}A^j \\ , {}_aD^\alpha_{x^j}A^i, {}_aD^\alpha_{x^i}A^0, t \end{bmatrix} \qquad (14)$$

The generalized momenta are defined as follows[26]:

$$\begin{cases} \pi^1_{\alpha_{A^0}} = \frac{\partial \mathcal{L}}{\partial({}_aD^\alpha_t A^0)} \\ \pi^1_{\alpha_{A^i}} = \frac{\partial \mathcal{L}}{\partial({}_aD^\alpha_t A^i)} \\ \pi^1_{\alpha_{A^j}} = \frac{\partial \mathcal{L}}{\partial({}_aD^\alpha_t A^j)}, \quad , \\ \pi_\phi = \frac{\partial \mathcal{L}}{\partial({}_aD^\alpha_t \phi)} \\ \pi_{\phi^*} = \frac{\partial \mathcal{L}}{\partial({}_aD^\alpha_t \phi^*)} \end{cases} \qquad (15)$$

The Hamiltonian depends on the fractional time derivatives and is written as

$$H = \pi_\phi\, {}_aD^\alpha_t\phi + \pi^*_{\phi^*}\, {}_aD^\alpha_t\phi^* + \pi_{\alpha_{A^0}}\, {}_aD^\alpha_t A^0 + \pi_{\alpha_{A^i}} - {}_aD^\alpha_t A^i + \pi_{\alpha_{A^j}}\, {}_aD^\alpha_t A^j - L\begin{bmatrix} (\phi, {}_aD^\alpha_t\phi, {}_aD^\alpha_{x_j}\phi, \phi^*, {}_aD^\alpha_t\varphi^*, {}_aD^\alpha_{x_j}\phi^*,, {}_aD^\alpha_t A^i \\ A^0, A^i, A^j, {}_aD^\alpha_t A^j, {}_aD^\alpha_t A^i, {}_aD^\alpha_{x_i}A^j, {}_aD^\alpha_{x^j}A^i, \\ {}_aD^\alpha_{x^i}A^0, t \end{bmatrix} \qquad (16)$$

Take the total of the differentials on both sides.

$$dh = \begin{bmatrix} \pi d({}_aD_t^\alpha\phi) + {}_aD_t^\alpha\phi d(\pi) + \pi^* d({}_aD_t^\alpha\phi^*) \\ + {}_aD_t^\alpha\phi^* d(\pi^*) - \frac{\partial L}{\partial t}dt - \frac{\partial L}{\partial\phi}d\phi - \\ \frac{\partial L}{\partial({}_aD_t^\alpha\phi)}d({}_aD_t^\alpha\phi) - \frac{\partial L}{\partial\left({}_aD_{x_j}^\alpha\phi\right)}d\left({}_aD_{x_j}^\alpha\phi\right) \\ -\frac{\partial L}{\partial\left({}_aD_{x_j}^\alpha\phi^*\right)}d\left({}_aD_{x_j}^\alpha\phi^*\right) + d\pi_{\alpha_{Aj}}({}_aD_t^\alpha A^j) \\ -\frac{\partial L}{\partial({}_aD_t^\alpha\phi^*)}d({}_aD_t^\alpha\phi^*) - \frac{\partial L}{\partial\varphi^*}d\phi^* - \frac{\partial H}{\partial A^i}dA^i \\ -\frac{\partial H}{\partial\emptyset}d\emptyset - \frac{\partial L}{\partial\left({}_aD_{x^j}^\alpha A^i\right)}d\left({}_aD_{x^j}^\alpha A^i\right) \\ -d\pi_{\alpha_{A^i}}({}_aD_t^\alpha A^i) - \frac{\partial H}{\partial A^j}dA^j - \\ \frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^j\right)}d\left({}_aD_{x^i}^\alpha A^j\right) \\ -\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^0\right)}d\left({}_aD_{x^i}^\alpha A^0\right) \end{bmatrix} \quad (17)$$

But the Hamiltonian is function of the form

$$H=H\begin{bmatrix} \pi,\phi,\pi^*,\phi^*,{}_aD_{x_j}^\alpha\phi,{}_aD_{x_j}^\alpha\phi^*,A^0,A^i \\ ,A^j,t,\pi_{\alpha_{A^0}},\pi_{\alpha_{A^i}},\pi_{\alpha_{A^j}},{}_aD_{x^i}^\alpha A^0, \\ {}_aD_{x^i}^\alpha A^j,{}_aD_{x^j}^\alpha A^i \end{bmatrix} \quad (18)$$

As a result, the Hamiltonian's total differential takes the following shape:

$$dH = \begin{bmatrix} \frac{\partial H}{\partial\phi}d\phi + \frac{\partial H}{\partial\pi}d\pi + \frac{\partial H}{\partial\left({}_aD_{x_j}^\alpha\phi\right)}d\left({}_aD_{x_j}^\alpha\phi\right) + \\ \frac{\partial H}{\partial\phi^*}d\phi^* + \frac{\partial H}{\partial\pi^*}d\pi^* + \frac{\partial H}{\partial\left({}_aD_{x_j}^\alpha\phi^*\right)}d\left({}_aD_{x_j}^\alpha\phi^*\right) \\ +\frac{\partial H}{\partial\pi_{\alpha_{Aj}}}d\pi_{\alpha_{Aj}} + \frac{\partial H}{\partial\pi_{\alpha_{Ai}}}d\pi_{\alpha_{Ai}} + \frac{\partial H}{\partial A^j}dA^j \\ +\frac{\partial H}{\partial A^i}dA^i + \frac{\partial H}{\partial A^0}dA^0 + \frac{\partial H}{\partial t}dt + \\ \frac{\partial H}{\partial\left({}_aD_{x^i}^\alpha A^0\right)}d\left({}_aD_{x^i}^\alpha A^0\right) + \\ \frac{\partial H}{\partial\left({}_aD_{x^i}^\alpha A^i\right)}d\left({}_aD_{x^i}^\alpha A^i\right) + \frac{\partial H}{\partial({}_aD_{x^j}^\alpha A^j)}d\left({}_aD_{x^j}^\alpha A^j\right) \end{bmatrix} \quad (19)$$

By comparing Eqs. (17) and (18), we obtain Hamilton's equations of motion.

$$\begin{cases} \frac{\partial H}{\partial\pi_{\alpha_{Aj}}} = {}_aD_t^\alpha A^j \qquad \frac{\partial H}{\partial\pi_{\alpha_{A^i}}} = {}_aD_t^\alpha A^i \\ \frac{\partial H}{\partial\pi_{\alpha_{A^0}}} = {}_aD_t^\alpha A^0 \qquad \frac{\partial H}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t} \end{cases} \quad (20)$$

$$\begin{cases} \frac{\partial H}{\partial\left({}_aD_{x^i}^\alpha\emptyset\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\emptyset\right)} \\ \frac{\partial H}{\partial\left({}_aD_{x^i}^\alpha A^j\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^j\right)} . \\ \frac{\partial H}{\partial\left({}_aD_{x^j}^\alpha A^i\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^j}^\alpha A^i\right)} . \\ \frac{\partial H}{\partial\left({}_aD_{x^j}^\alpha\phi^*\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^j}^\alpha\phi^*\right)} . \\ \frac{\partial H}{\partial\left({}_aD_{x^j}^\alpha\phi\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^j}^\alpha\phi\right)} . \end{cases} \quad (21)$$

The result of calculating these derivatives is

$$\begin{cases} \frac{\partial H}{\partial\emptyset} = -\frac{\partial L}{\partial A^0} \\ \frac{\partial H}{\partial A^i} = -\frac{\partial L}{\partial A^i} \\ \frac{\partial H}{\partial A^j} = -\frac{\partial L}{\partial A^j} \\ \frac{\partial H}{\partial\phi} = -\frac{\partial L}{\partial\phi} \\ \frac{\partial H}{\partial\phi^*} = -\frac{\partial L}{\partial\phi^*} \end{cases} \quad (22)$$

We can rewrite these equations using the Euler-Lagrange formulation, which results in the following equations:

$$\begin{cases} \frac{\partial H}{\partial A^0} = -{}_aD_t^\alpha\frac{\partial L}{\partial({}_aD_t^\alpha A^0)} - {}_aD_{x^i}^\alpha\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^0\right)} \qquad 23a \\ \frac{\partial H}{\partial A^i} = -{}_aD_t^\alpha\frac{\partial L}{\partial({}_aD_t^\alpha A^i)} - {}_aD_{x^i}^\alpha\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^i\right)} \qquad 23b \\ \frac{\partial H}{\partial A^j} = -{}_aD_t^\alpha\frac{\partial L}{\partial({}_aD_t^\alpha A^j)} - {}_aD_{x^i}^\alpha\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha A^j\right)} \qquad 23c \\ \frac{\partial H}{\partial\phi} = -{}_aD_t^\alpha\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\phi\right)} - {}_aD_{x^i}^\alpha\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\phi\right)} \qquad 23d \\ \frac{\partial H}{\partial\phi^*} = -{}_aD_t^\alpha\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\phi^*\right)} - {}_aD_{x^i}^\alpha\frac{\partial L}{\partial\left({}_aD_{x^i}^\alpha\phi^*\right)} \qquad 23f \end{cases} \quad (23)$$

5 Illustrative Examples

We begin with the Lagrangian of the combined scalar and electromagnetic fields [27].

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(\left[(\partial_\mu + ieA_\mu\right]\phi\right)^*\left(\left[\partial_\mu + ie\,A_\mu\right]\phi\right) - \mu\phi^*\phi \quad (24)$$

Where $F^{\mu\nu}$ is a four-dimension antisymmetric second rank tensor and A^μ is a the four – vector potential. Rebuild the complex scalar field interacting with the electromagnetic Lagrangian density in Riemann – Liouville fractional form using these relations.

$$\begin{bmatrix} F_{\mu\nu} = {}_aD^\alpha_{x_\mu}A_\nu - {}_aD^\alpha_{x_\nu}A_\mu \\ F^{\mu\nu} = {}_aD^\alpha_{x^\mu}A^\nu - {}_aD^\alpha_{x^\nu}A^\mu \end{bmatrix} \quad (25)$$

$$\begin{bmatrix} \partial_\mu = {}_aD^\alpha_{x_\mu} = \left({}_aD^\alpha_t,\ {}_aD^\alpha_{x_i}\right) \\ \partial^\mu = {}_aD^\alpha_{x^\mu} = \left({}_aD^\alpha_t, -{}_aD^\alpha_{x^i}\right) \end{bmatrix} \quad (26)$$

$$F_{\mu\nu}F^{\mu\nu} = 2\left[{}_aD^\alpha_{x_\mu}A_\nu\ {}_aD^\alpha_{x^\mu}A^\nu - {}_aD^\alpha_{x_\mu}A_\nu\ {}_aD^\alpha_{x^\nu}A^\mu\right] \quad (27)$$

$$\begin{bmatrix} A^\alpha = \left(A^0, \vec{A}\right) \\ A_\alpha = \left(A_0, -\vec{A}\right) \end{bmatrix} \quad (28)$$

When μ, ν is expanded in terms of 0,i and 0,j, and the definition of left Riemann – Liouville fractional derivative is applied, the fractional electromagnetic lagrangian density formulation takes the form

$$\mathcal{L} = -\frac{1}{2}\begin{bmatrix} -{}_aD^\alpha_t A^j\,{}_aD^\alpha_t A^j + {}_aD^\alpha_t A^j\,{}_aD^\alpha_{x^j}A^0 \\ -{}_aD^\alpha_{x^i}A^0\,{}_aD^\alpha_{x^i}A^0 + {}_aD^\alpha_{x^i}A^0\,{}_aD^\alpha_t A^i \\ {}_aD^\alpha_{x^i}A^j\,{}_aD^\alpha_{x^i}A^j - {}_aD^\alpha_{x^i}A^j\,{}_aD^\alpha_{x^j}A^i \end{bmatrix} + \begin{bmatrix} {}_aD^\alpha_t\phi^*\,{}_aD^\alpha_t\phi + {}_aD^\alpha_{x_i}\phi^*\,{}_aD^\alpha_{x_i}\phi + \\ ie\emptyset\psi\,{}_aD^\alpha_t\phi^* + ieA_l\phi\,{}_aD^\alpha_{x_i}\phi^* - \\ ie\emptyset\phi^*\,{}_aD^\alpha_t\psi - ieA_l\phi^*\,{}_aD^\alpha_{x_i}\phi + \\ e^2\emptyset^2\phi^*\phi + e^2A_l^2\phi^*\phi - \mu\phi^*\phi \end{bmatrix} \quad (29)$$

If $\alpha = 1$ then Eq. (39) become

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(\left[(\partial_\mu + ieA_\mu\right]\phi\right)^*\left(\left[\partial_\mu + ie\,A_\mu\right]\phi\right) - \mathcal{L}\ \mu\phi^*\phi \quad (30)$$

This is the well-known complex scalar field that interacts with the electromagnetic equation.

6 The Euler-Lagrangian Equation in Fractional Form

Let us begin with a definition of fractional Lagrangian density and then use the generalization formula of the Euler – Lagrange equation (16) to produce equations of motion from a complex scalar field interacting with the electromagnetic Lagrangian density.

Take the first fields variable A^0 , then

$$\frac{\partial\mathcal{L}}{\partial A_0} - {}_aD^\alpha_t\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_t A_0} - {}_aD^\alpha_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}A_0} = 0 \quad (31)$$

$$\frac{\partial\mathcal{L}}{\partial A^0} = ie\phi\,{}_aD^\alpha_t\phi^* - ie\phi^*\,{}_aD^\alpha_t\phi - 2e^2A^0\,\phi^*\phi \quad (32)$$

$$\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_t A^0\right)} = 0 \quad (33)$$

$$\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x_j}A^0\right)} = \left(-{}_aD^\alpha_{x^i}A^0 - {}_aD^\alpha_t A^i\right) \quad (34)$$

Equation (16) is obtained by substituting equations (17, 18, and 19) for equation (16).

$$\begin{bmatrix} -{}_aD^\alpha_t\begin{pmatrix} ie\phi\,{}_aD^\alpha_t\phi^* - ie\phi^*\,{}_aD^\alpha_t\phi - \\ 2e^2A^0\,\phi^*\phi \end{pmatrix} \\ -{}_aD^\alpha_{x^i}\left(-{}_aD^\alpha_{x^i}A^0 - {}_aD^\alpha_t A^i\right) \end{bmatrix} = 0 \quad (35)$$

Now use the general formula (7) to obtain other equations of motion from the other fields' variables

$$\frac{\partial\mathcal{L}}{\partial A_j} - {}_aD^\alpha_t\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_t A_j} - {}_aD^\alpha_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}A_j} = 0 \quad (36)$$

$$\frac{\partial\mathcal{L}}{\partial A^j} = +ie\phi\,{}_aD^\alpha_{x_j}\phi^* - ie\phi^*\,{}_aD^\alpha_{x_j}\phi + 2e^2A_i\phi^*\phi \quad (37)$$

$$\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_t A^j\right)} = -\frac{1}{2}\left(-2\,{}_aD^\alpha_t A^j + {}_aD^\alpha_{x^i}\right) \quad (38)$$

$$\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x^i}A^j\right)} = -\frac{1}{2}\left(2\,{}_aD^\alpha_{x^i}A^j - {}_aD^\alpha_{x^j}A^i\right) \quad (39)$$

Substituting equations (37, 38, and 39) in equation (36) we get

$$\begin{bmatrix} \left(ie\phi\,{}_aD^\alpha_{x_j}\phi^* - ie\phi^*\,{}_aD^\alpha_{x_j}\phi + 2e^2A_i\phi^*\phi\right) \\ +\frac{1}{2}\,{}_aD^\alpha_t\left(2\,{}_aD^\alpha_t A^j + {}_aD^\alpha_{x^i}\right) - \\ \frac{1}{2}D^\alpha_{x^i}\left(2\,{}_aD^\alpha_{x^i}A^j - {}_aD^\alpha_{x^j}A^i\right) \end{bmatrix} = 0 \quad (40)$$

And

$$\frac{\partial\mathcal{L}}{\partial A_i} - {}_aD^\alpha_t\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_t A_i} - {}_aD^\alpha_{x_i}\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_i}A_i} = 0 \quad (41)$$

$$\frac{\partial\mathcal{L}}{\partial A^i} = 0 \quad (42)$$

$$\frac{\partial \mathcal{L}}{\partial({}_aD_t^\alpha A^i)} = -\frac{1}{2}\left({}_aD_{x^i}^\alpha A_0\right) \quad (43)$$

$$\frac{\partial \mathcal{L}}{\partial\left({}_aD_{x^j}^\alpha A^i\right)} = -\frac{1}{2}(-{}_aD_{x^j}^\alpha A^i) \quad (44)$$

Equation (45) is obtained by substituting equations (42, 43, and 44) for equation (41).

$$\frac{1}{2}\,{}_aD_t^\alpha\left({}_aD_{x^i}^\alpha A_0\right) + \frac{1}{2}\,{}_aD_{x^j}^\alpha({}_aD_{x^j}^\alpha A^i) \quad (45)$$

Applying Euler-Lagrange equation (Eq. (5)) with respect to ϕ , we get

$$\begin{bmatrix} ieA_{\mu\,a}D_t^\alpha \phi^* + ieA_{\mu\,a}D_{x_j}^\alpha \phi^* + \\ e^2A_\mu^2\varphi^* - \mu\,\phi^* \\ -{}_aD_t^\alpha({}_aD_t^\alpha \phi^* - ie\emptyset\phi^*) - \\ {}_aD_{x_i}^\alpha\left({}_aD_{x_j}^\alpha \varphi^* - ieA_j\varphi^*\right) \end{bmatrix} = 0 \quad (46)$$

Using the Euler-Lagrange equations Eq.(5) and calculating the derivative with respect to ϕ^*, we get the following equations of motion:

$$\begin{bmatrix} -ieA_{\mu\,a}D_t^\alpha \varphi - \\ ieA_{\mu\,a}D_{x_j}^\alpha \varphi \\ +\mu\,\varphi + e^2A_\mu^2\varphi \\ -{}_aD_t^\alpha({}_aD_t^\alpha \varphi + ie\emptyset\varphi) \\ -{}_aD_{x_i}^\alpha\left({}_aD_{x_j}^\alpha \varphi + ieA_j\varphi\right) \end{bmatrix} = 0 \quad (47)$$

Taking the derivative with respect to A_0 from Hamiltonian equation (23a), we get:

$$\begin{bmatrix} -{}_aD_t^\alpha\begin{pmatrix} ie\phi\,{}_aD_t^\alpha \phi^* - ie\phi^*\,{}_aD_t^\alpha \phi \\ -2e^2A^0\,\phi^*\phi \end{pmatrix} \\ -{}_aD_{x^i}^\alpha(-{}_aD_{x^i}^\alpha A^0 - {}_aD_t^\alpha A^i) \end{bmatrix} = 0 \quad (48)$$

The above equation is exactly the same as the equation that has been derived by (equation. (46)) in fractional form

Now take other fields variables A^i, A^j

$$\frac{\partial H}{\partial A^i} = -{}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial({}_aD_t^\alpha A^i)} - {}_aD_{x^i}^\alpha \frac{\partial \mathcal{L}}{\partial\left({}_aD_{x^i}^\alpha A^i\right)} \quad (49)$$

We get

$$\frac{1}{2}\,{}_aD_t^\alpha\left({}_aD_{x^i}^\alpha \emptyset\right) + \frac{1}{2}\,{}_aD_{x^j}^\alpha({}_aD_{x^j}^\alpha A^i) = 0 \quad (50)$$

And

$$\frac{\partial H}{\partial A^j} = -{}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial({}_aD_t^\alpha A^j)} - {}_aD_{x^i}^\alpha \frac{\partial \mathcal{L}}{\partial\left({}_aD_{x^i}^\alpha A^j\right)} \quad (51)$$

$$0 = \begin{bmatrix} \left(ie\varphi\,{}_aD_{x_j}^\alpha \phi^* - ie\phi^*\,{}_aD_{x_j}^\alpha \phi + 2e^2A_i\phi^*\phi\right) \\ -\frac{1}{2}\,{}_aD_t^\alpha\left(-2\,{}_aD_t^\alpha A^j + {}_aD_{x^i}^\alpha\right) \\ -\frac{1}{2}D_{x^i}^\alpha\left(2\,{}_aD_{x^i}^\alpha A^j - {}_aD_{x^j}^\alpha A^i\right) \end{bmatrix} \quad (52)$$

This is the same as the results obtained using Euler-Lagrange, see equation (46).

Using Hamiltonian equation (23d), by taking the derivative with respect to ϕ, we get

$$\begin{bmatrix} ieA_{\mu\,a}D_t^\alpha \phi^* + ieA_{\mu\,a}D_{x_j}^\alpha \phi^* + \\ e^2A_\mu^2\varphi^* - \mu\,\varphi^* \\ -{}_aD_t^\alpha({}_aD_t^\alpha \varphi^* - ie\emptyset\varphi^*) - \\ {}_aD_{x_i}^\alpha\left({}_aD_{x_j}^\alpha \varphi^* - ieA_j\varphi^*\right) \end{bmatrix} = 0 \quad (53)$$

By using (23f) the fractional equation of motion is given bellow

$$\begin{bmatrix} -ieA_{\mu\,a}D_t^\alpha \varphi - ieA_{\mu\,a}D_{x_j}^\alpha \varphi + \\ e^2A_\mu^2\varphi + \mu\,\varphi \\ -{}_aD_t^\alpha({}_aD_t^\alpha \varphi + ie\emptyset\varphi) - \\ {}_aD_{x_i}^\alpha\left({}_aD_{x_j}^\alpha \varphi + ieA_j\varphi\right) \end{bmatrix} = 0 \quad (54)$$

7 Application of Fractional Calculus

This section will look at how fractional calculus can be used to look at complex scalar fields and interacting fields. Here are a few examples of applications.

- This method can also be used to calculate the potential energy of a fractional order energy. Fractional calculus is used to compute the force by increasing the slope of the potential energy scalar field by a gradient factor. Fractional derivatives are used to calculate the results. Potential fields, often called scalar fields, are used to describe well-known forces like Newton's gravitational potential and the electrostatic potential.
- Such technique can also be used to see how the fractional-order derivative affects the shape and structure of interacting field equations derived from order fractional complex scalar fields.
- The approach of fractional derivatives in complex scalar fields as interacting fields has the advantage of allowing for the calculation of energy and distance scales. The fractional derivative method was also utilized to look at uncertainty relationships

in the relativistic realm and the necessity for many-particle descriptions.

8 Conclusions

The Hamilton equations as well as the Hamiltonian formulation of complex scalar fields interacting with electromagnetic field systems are investigated. Fractional Euler-Lagrange equations and fractional Hamilton's equations of motion yield the same outcomes for a given Lagrangian density. We developed Lagrangian and Hamiltonic formulations for complex scalar fields interacting fields by using fractional derivatives from the Riemann-Liouville and Hamilton equations. The classical results (combined scalar and electromagnetic equations) are obtained as a special instance of the fractional formulation.

Conflict of interest:
The authors declare that they have no conflict of interest.

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