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Relations for Moments of Generalized Order Statistics from Power Lomax Distribution

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Abstract: Kamps [9] introduced the concept of generalized order statistics, as a common approach to various ordered random schemes, such as, order statistics, record values, sequential order statistics, progressively type II censored order statistics, Pfeifers records etc. The study of recurrence relations between moments has been of special interest to researchers. In this paper, recurrence relations for single and product moments of generalized order statistics have been derived for Power Lomax distribution, proposed by Rady [21]. Further, results are deduced for order statistics and records. At the end, some characterization theorems of this distribution are also presented.

Keywords: Power Lomax Distribution; Generalized Order Statistics; Single Moments; Product Moments; Truncated Moments; Characterization.

1 Introduction

The Power Lomax (POLO) distribution was proposed and studied by Rady [21], as a new extension of the Lomax distribution. It provides a much more flexible model for life time data as compared to its predecessor Lomax distributions since it can accommodate both inverted bathtub as well as decreasing hazard rate function.

A random variable X is said to follow the Power Lomax (POLO) distribution if its pdf is of the form

$$f(x) = \alpha\beta\lambda^{\alpha}x^{\beta-1}(\lambda+x^{\beta})^{-\alpha-1}, \quad x>0, \alpha>0, \beta>0, \lambda>0. \tag{1}$$

and the corresponding survival function is

$$\bar{F}(x) = \lambda^{\alpha}(\lambda+x^{\beta})^{-\alpha}, \quad x>0, \alpha>0, \beta>0, \lambda>0, \tag{2}$$

where

$$\bar{F}(x) = 1-F(x).$$

In view of (1) and (2), we have

$$\bar{F}(x) = \frac{(\lambda+x^{\beta})x^{1-\beta}}{\alpha\beta}f(x). \tag{3}$$

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \geq 1$ be the parameters, such that

$$\gamma_i = k+n-i+\sum_{j=i}^{n-1} m_j \geq 0, \ \text{for} \ 1 \leq i \leq n-1.$$

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The random variables $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ are said to be generalized order statistics (*gos*) from an absolutely continuous distribution function $F()$ with the probability density function (pdf) $f()$, if their joint pdf is of the form

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}\big[1-F(x_i)\big]^{m_i}f(x_i)\Big)\big[1-F(x_n)\big]^{k-1}f(x_n) \tag{4}$$

on the cone $F^{-1}(0) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$.

If $m_i = m = 0;\ i = 1\ldots n-1,\ k = 1$, we obtain the joint pdf of the order statistics and for $m = -1,\ k \in N$, we get joint pdf of k^{th} record values.

Recurrence relations for moments of *gos* for various distributions have been investigated by several authors. For detailed survey, one may refer to Athar and Islam [2], Anwar *et al.* [5], Khan *et al.* [14], Athar *et al.* [3], Keseling [12], Kamps and Cramer [11], Khwaja *et al.* [17], Nayabuddin and Athar [19], Singh *et al.* [22] and references therein.

The characterization of probability distributions, through different approaches, has been considered in the literature. The method of characterization through recurrence relations between moments of order statistics was given by Kamps [10]. For additional information on the topic, one may refer to Khan and Khan [15], Athar and Nayabuddin [4], Khan and Zia [16] among others. Several characterization results through truncated moments can be seen in the works of Galambos and Kotz [6], Kotz and Shanbhag [18], Glänzel [7], Ahsanullah *et al.* [1] and the references cited there.

2 Single Moments

Here we may consider two cases:

Case I. $\gamma_i \neq \gamma_j, i,j = 1,2,...,n-1, i \neq j$.

In view of (4), the pdf of r^{th} *gos* $X(r,n,\tilde{m},k)$ is given as (Kamps and Cramer [11])

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1}f(x)\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}, \tag{5}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \qquad \gamma_i = k+n-i+\sum_{j=1}^{n-1}m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{(\gamma_j-\gamma_i)}, \quad 1 \le i \le r \le n.$$

Case II. $m_i = m, i = 1,2,...n-1$.

The pdf of r^{th} *gos* $X(r,n,m,k)$ is given as (Kamps [9])

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)), \tag{6}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i \quad , \gamma_i = k+(n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \ m \neq -1 \\[2ex] \log\Big(\dfrac{1}{1-x}\Big) & , \ m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \ x \in [0,1).$$

Theorem 2.1. Let Case I be satisfied. For the Power Lomax distribution as given in (1) and $n \in N, \tilde{m} \in \mathbb{R}, k > 0, 1 \leq r \leq n, j = 1,2,\ldots$

$$E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] = \frac{j}{\gamma_r \alpha\beta}\Big[E[X^j(r,n,\tilde{m},k)] + \lambda E[X^{j-\beta}(r,n,\tilde{m},k)]\Big]. \tag{7}$$

Proof. We have, by Athar and Islam [2],

$$E[\xi\{X(r,n,\tilde{m},k)\}] - E[\xi\{X(r-1,n,\tilde{m},k)\}] = C_{r-2}\int_{-\infty}^{\infty} \xi'(x)\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}dx.$$

Let $\xi(x) = x^j$, then

$$E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] = j\,C_{r-2}\int_{-\infty}^{\infty} x^{j-1}\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}dx.$$

In view of (3) , we have

$$E[X^j(r,n,\tilde{m},k)] - E[X^j(r-1,n,\tilde{m},k)] = \frac{j\,C_{r-1}}{\gamma_r\alpha\beta}\int_{0}^{\infty}(\lambda + x^{\beta})x^{1-\beta}\,x^{j-1}\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i-1}\,f(x)dx,$$

which after simplification yields (7).

Remark 2.1. Let $m_i = m$, $i = 1,2,...,n-1$, then the recurrence relation for single moments of *gos* for *Case II* is given by

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{j}{\gamma_r\alpha\beta}\Big[E[X^j(r,n,m,k)] + \lambda E[X^{j-\beta}(r,n,m,k)]\Big]. \tag{8}$$

Remark 2.2. Let $m_i = 0$, $i = 1,2,...,n-1$ and $k = 1$, then the recurrence relation for single moments of order statistics is

$$E(X^j_{r:n}) - E(X^j_{r-1:n}) = \frac{j}{(n-r+1)\alpha\beta}\Big[E(X^j_{r:n}) + \lambda E(X^{j-\beta}_{r:n})\Big].$$

Remark 2.3. For $m_i = -1$, $i = 1,2,...,n-1$, the recurrence relation for single moments of k^{th} record values will be

$$E(X^{(k)}_{U(r)})^j - E(X^{(k)}_{U(r-1)})^j = \frac{j}{k\alpha\beta}\Big[E(X^{(k)}_{U(r)})^j + \lambda E(X^{(k)}_{U(r)})^{j-\beta}\Big].$$

3 Product Moments

Case I. $\gamma_i \neq \gamma_j;\; i,j = 1,2,...,n-1, i \neq j.$

The joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \leq r < s \leq n$, is given as (Kamps and Cramer [11])

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\left[\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right] \times \frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)},\quad x < y, \tag{9}$$

where

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1\\ j\neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \leq i \leq s \leq n.$$

Case II. $m_i = m, i = 1,2,...n-1$.

The joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given as (Pawlas and Syznal [20])

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}(F(x))$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} f(x)f(y), \quad -\infty \le x < y \le \infty. \tag{10}$$

Theorem 3.1. Let Case I be satisfied. For the Power Lomax distribution as given in (1) and $n \in N, \tilde{m} \in \mathbb{R}, k > 0, 1 \le r < s \le n, i,j = 1,2,...$

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] - E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$= \frac{j}{\gamma_s\alpha\beta}\Big[E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] + \lambda E[X^i(r,n,\tilde{m},k)X^{j-\beta}(s,n,\tilde{m},k)]\Big]. \tag{11}$$

Proof. We have by Athar and Islam [2],

$$E[\xi\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}] - E[\xi\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}]$$

$$= C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}\frac{d}{dy}\xi(x,y)\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\frac{f(x)}{\bar{F}(x)}dy\,dx.$$

Let $\xi(x,y) = \xi_1(x)\xi_2(y) = x^i y^j$. Then in view of (3), we get

$$E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)] - E[X^i(r,n,\tilde{m},k)X^j(s-1,n,\tilde{m},k)]$$

$$= \frac{j\,C_{s-1}}{\gamma_s\alpha\beta}\int_0^{\infty}\int_x^{\infty}(\lambda+y^\beta)y^{1-\beta}\,x^i y^{j-1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}$$

$$\times\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dy\,dx,$$

which upon simplification leads to (11).

Remark 3.1. Let $m_i = m$, $i = 1,2,...,n-1$, then the recurrence relation for product moments of *gos* for *Case II* is given by

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$

$$= \frac{j}{\gamma_s\alpha\beta}\Big[E[X^i(r,n,m,k)X^j(s,n,m,k)] + \lambda E[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)]\Big]. \tag{12}$$

Remark 3.2. Let $m_i = 0$, $i = 1,2,...,n-1$ and $k = 1$, then the recurrence relation for product moments of order statistics is

$$E[X_{r:n}^i X_{s:n}^j] - E[X_{r:n}^i X_{s-1:n}^j] = \frac{j}{(n-s+1)\alpha\beta}\Big[E[X_{r:n}^i X_{s:n}^j] + \lambda E[X_{r:n}^i X_{s:n}^{j-\beta}]\Big].$$

Remark 3.3. For $m_i = -1$, $i = 1,2,...,n-1$, the recurrence relation for product moments of k^{th} record values will be

$$E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^j] - E[(X_{U(r)}^{(k)})^i(X_{U(s-1)}^{(k)})^j]$$

$$= \frac{j}{k\alpha\beta}\Big[E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^j] + \lambda E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^{j-\beta}]\Big].$$

Remark 3.4. At $i = 0$ in (11), we get the relation for single moment as obtained in (7).

4 Characterizations

This section contains characterization results for the given distribution through recurrence relations for single and product moments of *gos* as well as through conditional expectation.

Theorem 4.1. Fix a positive integer k and let j be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$\left(1-\frac{j}{\gamma_r\alpha\beta}\right)E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] \; + \; \frac{j\lambda}{\gamma_r\alpha\beta}E[X^{j-\beta}(r,n,m,k)]. \quad (13)$$

Proof. The necessary part follows from (8). On the other hand, if the relation in (13) is satisfied, then

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)]$$
$$= \frac{j}{\gamma_r\alpha\beta}\Big[E[X^j(r,n,m,k)] + \lambda E[X^{j-\beta}(r,n,m,k)]\Big].$$

Now on using Athar and Islam [2] for $\xi(x) = x^j$, we have

$$\frac{j}{\gamma_r}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))dx$$
$$= \frac{j}{\gamma_r\alpha\beta}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_{r-1}} g_m^{r-1}(F(x))\left\{xf(x)+\lambda x^{1-\beta}f(x)\right\}dx.$$

or

$$\frac{j}{\gamma_r\alpha\beta}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_{r-1}} g_m^{r-1}(F(x))\left\{\alpha\beta\bar{F}(x) - xf(x) - \lambda x^{1-\beta}f(x)\right\}dx = 0. \quad (14)$$

Applying the extension of $M\ddot{u}ntz - Sz\acute{a}sz$ theorem (see, for example, Hwang and Lin [8]) to (14), we get

$$f(x) = \frac{\alpha\beta x^{\beta-1}}{(\lambda+x^\beta)}\bar{F}(x),$$

which proves the theorem.

Theorem 4.2. Fix a positive integer k and let i and j be non-negative integers. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is

$$\left(1-\frac{j}{\gamma_s\alpha\beta}\right)E[X^i(r,n,m,k)X^j(s,n,m,k)] = E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$
$$+\frac{j\lambda}{\gamma_s\alpha\beta}E[X^i(r,n,m,k)X^{j-\beta}(r,n,m,k)]. \quad (15)$$

Proof. The necessary part follows from (12). Now, suppose that the relation in (15) is satisfied, then

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] - E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$
$$= \frac{j}{\gamma_s\alpha\beta}\Big[E[X^i(r,n,m,k)X^j(s,n,m,k)] + \lambda E[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)]\Big].$$

Now by using Athar and Islam [2], for $\xi(x,y) = x^i y^j$, we have

$$\frac{j}{\gamma_s}\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s}dydx$$

$$= \frac{j}{\gamma_s \alpha\beta} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty x^i y^{j-1} [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} \left\{ y f(y) + \lambda y^{1-\beta} f(y) \right\} dydx,$$

which implies

$$\frac{j}{\gamma_s \alpha\beta} \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_0^\infty \int_x^\infty x^i y^{j-1} [\bar{F}(x)]^m f(x) g_m^{r-1}(x) [h_m(F(y)) - h_m(F(x))]^{s-r-1} \times [\bar{F}(y)]^{\gamma_s - 1} \left\{ \alpha\beta \bar{F}(y) - y f(y) - \lambda y^{1-\beta} f(y) \right\} dydx = 0. \quad (16)$$

Applying the extension of *Müntz – Szász* theorem (see, for example, Hwang and Lin [8]) to (16), we get

$$f(y) = \frac{\alpha\beta y^{\beta-1}}{(\lambda + y^\beta)} \bar{F}(y).$$

Hence the theorem.

Theorem 4.3. Let $X(r,n,m,k)$, $r = 1,2,...n$ be the the r^{th} *gos* based on continuous df $F()$ and $E(X)$ exists. Then for two consecutive values r and $r+1$, such that $1 \leq r < r+1 \leq n$,

$$E\left[X^\beta(r+1,n,m,k) \middle| X(r,n,m,k) = x\right] = \frac{\alpha\gamma_{r+1}}{\alpha\gamma_{r+1} - 1} x^\beta + \frac{\lambda}{\alpha\gamma_{r+1} - 1} \quad (17)$$

if and only if

$$\bar{F}(x) = \left(\frac{\lambda}{\lambda + x^\beta}\right)^\alpha, \; x > 0; \alpha, \beta, \lambda > 0. \quad (18)$$

Proof. Khan and Alzaid [13] have shown that

$$E\left[h\big(X(s,n,m,k)\big) | X(r,n,m,k) = x\right] = a^* h(x) + b^* \quad (19)$$

if and only if

$$\bar{F}(x) = \left[a h(x) + b\right]^c \quad (20)$$

with $a^* = \prod_{j=r+1}^{s} \left(\frac{c\gamma_j}{1 + c\gamma_j}\right)$ and $b^* = -\frac{b}{a}(1 - a^*)$.

Comparing (18) with (20), we get

$$a = \tfrac{1}{\lambda}, \, b = 1, \, c = -\alpha, \, h(x) = x^\beta.$$

Thus, the theorem can be proved in view of (19).

Corollary 4.1. For the r^{th} order statistics $X_{r:n}$, $r = 1,2,...n$ and under the condition as stated under Theorem 4.3

$$E\left[X_{r+1:n}^\beta | X_{r:n} = x\right] = \frac{\alpha(n-r)x^\beta + \lambda}{\alpha(n-r) - 1}, \quad (21)$$

and consequently

$$E\left[X_{n:n}^\beta | X_{n-1:n} = x\right] = E\left[X^\beta | X \geq x\right] = \frac{\alpha}{\alpha - 1} x^\beta + \frac{\lambda}{\alpha - 1} \quad (22)$$

if and only if

$$\bar{F}(x) = \left(\frac{\lambda}{\lambda + x^\beta}\right)^\alpha, \; x > 0; \alpha, \beta, \lambda > 0. \quad (23)$$

It may be noted that similar characterization result can also be seen for adjacent records as

$$E\left[X_{U(n)}^\beta | X_{U(n-1)} = x\right] = E\left[X^\beta | X \geq x\right] = \frac{\alpha}{\alpha - 1} x^\beta + \frac{\lambda}{\alpha - 1}. \quad (24)$$

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