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Numerical Simulation for System of Time-Fractional Linear and Nonlinear Differential Equations

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Abstract: This paper is concerned with q-homotopy analysis transform technique to investigate system of differential equations (DE) of arbitrary order. The proposed technique describes the convergence range at large domain, by appropriate selection of initial approximation, auxiliary parameter and asymptotic parameter n ($n \geq 1$). The proposed technique provides infinitely many more options for solution series and converge rapidly compared to Homotopy Analysis Method (HAM) and Homotopy Perturbation Transform Algorithm (HPTA) in same term iterations. A comparative study of suggested scheme with exact, HAM and HPTA have been done and Maple package is used to enhance the power and efficiency of proposed technique.

Keywords: Laplace transform algorithm, q-HATM, System of time fractional homogeneous linear equations, System of time-fractional inhomogeneous nonlinear equations.

1 Introduction

A system of differential equations plays a very crucial role in modeling of problems appearing in numerous field of science and engineering. In recent years these types of equations are numerically solved such as predator-prey system and rabies model of arbitrary order [1], mathematical description of computer viruses associated with fractional calculus [2], SIR model describing the dengue fever disease [3], fractional vibration equation [4], a mathematical model occurring in the chemical systems [5], the convection-diffusion equations [6], etc. The theory of fractional calculus is modulation and extension of integer order; providing a very good tool to reveal the out-of-sight and genetic aspects of various material and processes. A logistic, remarkable and conceptual work in the field of fractional calculus and its utilities has been done in several studies [7,8,9,10,11].

The present work adopts q-Homotopy Analysis Transform Method (q-HATM), which is a mixed form of two efficient methods, q-HAM [12,13] and Laplace transform algorithm to compute system of time-fractional homogeneous and inhomogeneous DE. The q-HAM based on classical homotopy analysis method (HAM), a well-known term in topology [14]. The q-HAM is a modification of HAM by generalizing the embedding parameter occurring in HAM. The HAM is an analytic scheme and was initially given and employed by Liao [15,16,17]. In recent years, HAM is successfully applied in various linear, nonlinear and mathematical modeling arising in science, finance and engineering [18,19,20,21]. In recent times standard analytical schemes have also been mixed with Laplace transform to produce highly efficient techniques such as LDM [22], HPTM [23,24,25] and homotopy analysis transform scheme [26,27,28] to compute the solution of nonlinear equations occurring in scientific and engineering applications. It is open fact that the combination of classical analytical approaches such as q-HAM with Laplace transform yielding time saving results and less C.P.U. time to investigate nonlinear mathematical models having usability in science and engineering.The organization of this article is as constructed: In Section 2, some definitions of fractional integrals and derivatives are discussed. In Section 3, the q-HATM is proposed. In Part 4, numerical examples showing the efficiency of q-HATM are studied. Finally in the Part 5 the conclusions are discussed.

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2 Basic definitions

The fractional derivative of $\ell(t)$ due to Caputo is written as [29]:

$$D^{\alpha}\ell(t) = J^{a-\alpha}D^{a}\ell(t)$$

$$= \frac{1}{\Gamma(a-\alpha)}\int_0^t (t-\tau)^{a-\alpha-1}\ell^{a}(\tau)d\tau, \quad (1)$$

for $a-1<\alpha\leq a, a\in N, t>0, \ell\in C^{a}_{-1}$.
Lemma: If $a-1<\alpha\leq a, a\in N, \ell\in C^{a}_{\mu}, \mu\geq -1,$ then

$$D^{\alpha}J^{\alpha}\ell(t) = \ell(t) - \sum_{r=0}^{n-1}\ell^{(r)}(0^{+})\,\frac{t^{r}}{\Gamma(r+1)},\ t>0. \quad (2)$$

Definition 2. The Laplace transform of $\ell(t)$, represented by $\overline{\ell}(s)$ is written as

$$L\{\ell(t),s\} = \overline{\ell}(s) = \int_0^{\infty} e^{-st}\ell(t)\,dt. \quad (3)$$

If $a\in N,$ then Laplace transform is expressed as

$$L\left\{\frac{d^{a}}{dt^{a}};\,\ell;\,s\right\} = s^{n}\,\overline{\ell}(s) - \sum_{r=0}^{a-1}s^{a-r-1}\ell^{(r)}(0^{+}). \quad (4)$$

In the similar manner the Laplace transform of $D^{\alpha}_{t}\ell(t)$ is presented as [29,30]

$$L[D^{\alpha}_{t}\ell(t)] = s^{\alpha}L[\ell(t)] - \sum_{r=0}^{a-1}s^{\alpha-r-1}\ell^{(r)}(0^{+}), a-1<\alpha\leq a. \quad (5)$$

3 Analysis of method

In order to present the main process of q-HATM, we take a NFDE, which is written below:

$$D^{\alpha}_{t}\,\ell(x,t) + A\,\ell(x,t) + H\,\ell(x,t) = B(x,t),\ \ a-1<\alpha\leq a \quad (6)$$

In Eq. (6) $D^{\alpha}_{t}\ell(x,t)$ is representing the arbitrary order derivative of $\ell(x,t)$, A is denoting the linear differential operator, H is representing the nonlinear differential operator of general form and B is representing the source term.
On making use the Laplace transform on (6), we have

$$L\,[D^{\alpha}_{t}\,\ell] + L\,[A\,\ell] + L\,[H\,\ell] = L\,[B]. \quad (7)$$

By utilizing a result of the Laplace transform for fractional derivatives, it gives

$$s^{\alpha}L[\ell] - \sum_{k=0}^{a-1}s^{\alpha-k-1}\ell^{(k)}(x,0) + L\,[A\,\ell] + L[H\,\ell] = L[B]. \quad (8)$$

On simplification, we get

$$L\,[\ell] - \frac{1}{s^{\alpha}}\sum_{k=0}^{a-1}s^{\alpha-k-1}\ell^{(k)}(x,0) + \frac{1}{s^{\alpha}}\,[L\,[A\ell] + L[H\,\ell] - L[B]] = 0. \quad (9)$$

Let us record the nonlinear operator as follows

$$\mathscr{W}\,[\mu(x,t;q)] = L\,[\mu(x,t;q)] - \frac{1}{s^{\alpha}}\sum_{k=0}^{a-1}s^{\alpha-k-1}\mu^{(k)}(x,t;q)(0^{+})$$

$$+\frac{1}{s^{\alpha}}\,[L\,[A\mu(x,t;q)] + L[H\,\mu(x,t;q)] - L[B]], \quad (10)$$

In the above equation $q \in [0, 1/n]$ and $\mu(x, t\,; q)$ is a function of x, t and q. With the aid of the HAM, we present a homotopy as follows [15,16,17]

$$(1 - nq)\,L\,[\mu(x,t\,;q) - \ell_0(x,t)] = \hbar q\,\mathscr{W}\,[\mu(x,t;q\,)], \tag{11}$$

$$\mu(x,t\,;0) = \ell_0(x,t), \qquad \mu(x,t\,;\frac{1}{n}) = \ell(x,t), \tag{12}$$

respectively. It can be noticed that as q changes from 0 to $\frac{1}{n}$, the solution $\mu(x,t\,;q)$ changes from the initial approximation $\ell_0(x,t)$ to the solution $\ell(x,t)$. With the aid of the theory of Taylor series, we expand the function $\mu(x,t\,;q)$ in series form as

$$\mu(x,t\,;q) = \ell_0(x,t) + \sum_{m=1}^{\infty} \ell_m(x,t)\,q^m, \tag{13}$$

where

$$\ell_m(x,t) = \frac{1}{m\,!}\frac{\partial^m \mu(x,t\,;q)}{\partial q^m}\,|_{q=0}. \tag{14}$$

On suitable choice of various parameters and operators, the series (13) converges at $q = \frac{1}{n}$ then we get

$$\ell = \ell_0 + \sum_{m=1}^{\infty} \ell_m(\frac{1}{n})^m. \tag{15}$$

The above result must be one of the solutions of the considered nonlinear fractional problem. According Eq. (15), the governing equation can be derived from Eq. (11).
Let us define the vectors as

$$\vec{\ell}_m = \{\ell_0,\, \ell_1, ..., \,\ell_m\}. \tag{16}$$

Differentiating the Eq. (11) m-times w.r.t. q and then dividing them by $m!$ and finally putting $q = 0$, we get

$$L\,[\ell_m - k_m\ell_{m-1}] = \hbar\,\Re_m(\vec{\ell}_{m-1}). \tag{17}$$

Operating with the inverse Laplace transform, it yields

$$\ell_m = k_m\ell_{m-1} + \hbar L^{-1}[\Re_m(\vec{\ell}_{m-1})], \tag{18}$$

where

$$\Re_m(\vec{\ell}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{W}\,[\mu(x,t\,;q)]}{\partial q^{m-1}}\,|_{q=0}, \tag{19}$$

and

$$k_m = \begin{cases} 0, & m \le 1, \\ n, & m > 1. \end{cases} \tag{20}$$

Theorem 1. Let us assume that $E \times F \subset \mathbb{R} \times \mathbb{R}^+$ Banach space having a suitable norm $\|\cdot\|$ over which the series sequence of the q-HATM solution of nonlinear fractional problem (6) is defined, and the operator $\Omega(\ell_m) = \ell_{m+1}$ defining the series solution (15) holds the Lipschitzian conditions that is $\|\Omega(\ell_r^*) - \Omega(\ell_r)\| \le \varepsilon\,\|\ell_r^* - \ell_r\|$ for all $(x,t,r) \in E \times F \times \mathbb{N}$, then the q-HATM solution (15) is unique.

Proof. If ℓ and ℓ^* are the series solutions satisfying nonlinear fractional Eq. (6), then $\ell(x,t,n) = \sum_{m=0}^{\infty} \ell_m(1/n)^m$ with the initial approximation ℓ_0 and $\ell^*(x,t,n) = \sum_{m=0}^{\infty} \ell_m^*(1/n)^m$ also with initial approximation $\ell_0(x,t)$, so for the uniqueness we show that

$$\|\ell_m^* - \ell_m\| = 0, \;\; for\; m = 0, 1, 2, \cdots. \tag{21}$$

The result (21) is to be proved by the method of mathematical induction. For $m = 0, \ell_m^* = \ell_m = \ell_0$, so it is clear that the result (21) is true. Next, let the result be true for $0 \le r < m$ i.e.

$$\|\ell_r^* - \ell_r\| = 0. \tag{22}$$

Now, we have

$$\left\|\ell^*_{r+1}-\ell_{r+1}\right\| = \|\Omega(\ell^*_r)-\Omega(\ell_r)\|$$

$$\leq \varepsilon \|\ell^*_r-\ell_r\| = 0. \quad (23)$$

Hence the q-HATM solution (15) is unique.

4 Implementation of the method

In this part, we consider three different types of systems having partial differential equations of arbitrary order. The q-HATM have $q\in[0,\frac{1}{n}],(n\geq 1)$ is the embedding parameter. We choose the value of $\hbar$ with respect to arbitrary defined value of n from the absolute convergence range of $\hbar$-curve.

Example 1. In this system we examine the subsequent time-fractional homogeneous linear differential equations

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial v}{\partial x}-(u+v)=0, & 0<\alpha\leq 1, \\ \frac{\partial^\beta v}{\partial t^\beta}+\frac{\partial u}{\partial x}-(u+v)=0, & 0<\beta\leq 1, \end{cases} \quad (24)$$

along with

$$u(x,0)=u_0=\eta_1(x)=\sinh(x), v(x,0)=v_0=\eta_2(x)=\cosh(x). \quad (25)$$

In view of Eqs. (24) and (25), we express the nonlinear operator in the following manner

$$\begin{cases} \mathscr{W}^1[\mu_1,\mu_2] \\ =L[\mu_1)]-\left(1-\frac{k_m}{n}\right)\frac{1}{s}\eta_1(x)+s^{-\alpha}L[\frac{\partial\mu_2}{\partial x}-(\mu_1+\mu_2], \\ \mathscr{W}^2[\mu_1,\mu_2] \\ =L[\mu_2]-\left(1-\frac{k_m}{n}\right)\frac{1}{s}\eta_2(x)+s^{-\beta}L[\frac{\partial\mu_1}{\partial x}-(\mu_1+\mu_2], \end{cases} \quad (26)$$

and the Laplace operator as

$$\begin{cases} L\left[u_m-k_m u_{m-1}\right]=\hbar R_{1,m}[\vec{u}_{m-1},\vec{v}_{m-1}], \\ L\left[v_m-k_m v_{m-1}\right]=\hbar R_{2,m}[\vec{u}_{m-1},\vec{v}_{m-1}], \end{cases} \quad (27)$$

where

$$\begin{cases} R_{1,m}[\vec{u}_{m-1},\vec{v}_{m-1}]=L\{u_{m-1}\}-\left(1-\frac{k_m}{n}\right)\frac{1}{s}\sinh(x)+s^{-\alpha}L\left\{\frac{\partial v_{m-1}}{\partial x}-(u_{m-1}+v_{m-1})\right\}, \\ R_{2,m}[\vec{u}_{m-1},\vec{v}_{m-1}]=L\{v_{m-1}\}-\left(1-\frac{k_m}{n}\right)\frac{1}{s}\cosh(x)+s^{-\beta}L\left\{\frac{\partial u_{m-1}}{\partial x}-(u_{m-1}+v_{m-1})\right\}. \end{cases} \quad (28)$$

Obviously from equation (27), we get

$$\begin{cases} u_m=k_m u_{m-1}+\hbar L^{-1}\{R_{1,m}[\vec{u}_{m-1},\vec{v}_{m-1}]\}, \\ v_m=k_m v_{m-1}+\hbar L^{-1}\{R_{2,m}[\vec{u}_{m-1},\vec{v}_{m-1}]\}. \end{cases} \quad (29)$$

On solving system of Eqs. (29), we have

$$u_0=\sinh(x), v_0=\cosh(x),$$

$$u_1=\frac{-\hbar\cosh(x)t^\alpha}{\Gamma(\alpha+1)}, v_1=\frac{-\hbar\sinh(x)t^\beta}{\Gamma(\beta+1)},$$

$$u_2=\frac{-\hbar(\hbar+n)\cosh(x)t^\alpha}{\Gamma(\alpha+1)}+\hbar^2\sinh(x)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$-\hbar^2\cosh(x)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\hbar^2\cosh(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$v_2=\frac{-\hbar(\hbar+n)\sinh(x)t^{\beta}}{\Gamma(\beta+1)}+\hbar^2\cosh(x)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$-\hbar^2\sinh(x)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\hbar^2\sinh(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

$$\vdots \tag{30}$$

Thus the remaining terms can be evaluated. So, the q-HATM solution of the system (24) is

$$\begin{cases} u=u_0+\sum_{m=1}^{\infty}u_m(\frac{1}{n})^m, \\ v=v_0+\sum_{m=1}^{\infty}v_m(\frac{1}{n})^m. \end{cases} \tag{31}$$

If it is taken that $\alpha=1$, $\beta=1, n=1$ and $\hbar=-1$ then it is converted to the HPTA solution obtained by Khan et al. [31]. We can observed that $\sum_{m=0}^{N}u_m(\frac{1}{n})^m$ and $\sum_{m=0}^{N}v_m(\frac{1}{n})^m$ when $N\to\infty$, it converges to the closed form solutions $u=\sinh(x+t)$ and $v=\cosh(x+t)$ of system of equation (24). Figs. 1-2 represent the absolute error which reveal that the q-HATM solution tends to the exact solution rapidly. From Figs. 1-2, it is observed that our q-HATM solution converges very speedily to the closed form solutions in 4th-order iterations. The correctness of the q-HATM solution can be enhanced by increasing more components of the solution. It is also detected from Fig. 3-4 that as the values of α and β increase up, the values of u and v decrease down. Fig. 5 represents $\hbar-$curve for distinct order q-HATM approximation of system of fractional problems (24), the line segments parallel to u demonstrate the absolute convergence range of $\hbar$ with different values of α & β and the justifiable range of $\hbar$ for u is $-2.01\le\hbar<0$. Fig. 6 presents $\hbar-$curve at $x=0.5$, $t=0.02$ for non-identical order q-HATM approximation of system of equations (24), line segments parallel to v show the absolute convergence range of $\hbar$ with diverse values of α& β and the justifiable range of $\hbar$ for v is $-2.048\le\hbar<0$.

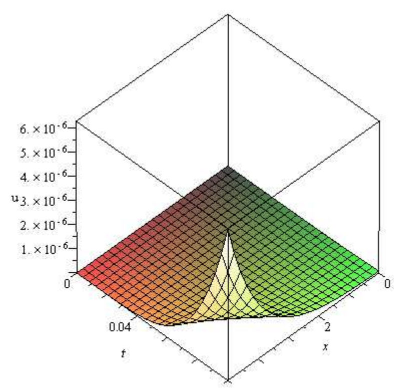

Fig. 1: Absolute error $E_4(u)=|u_{exa.}-u_{appr.}|$ for system of equations (24), when $\alpha=1$, $\beta=1$, $n=1$, and $\hbar=-1$.

Example 2. Next, we examine the subsequent system of inhomogeneous DE of fractional order

$$\begin{cases} \frac{\partial^{\alpha}u}{\partial t^{\alpha}}-\frac{\partial v}{\partial x}-(u-v)=2, & 0<\alpha\le1, \\ \frac{\partial^{\beta}v}{\partial t^{\beta}}+\frac{\partial u}{\partial x}-(u-v)=2, & 0<\beta\le1, \end{cases} \tag{32}$$

with

$$u(x,0)=u_0=\eta_1(x)=1+e^x, v(x,0)=v_0=\eta_2(x)=-1+e^x. \tag{33}$$

Solving the above system (32) and (33) by using q-HATM, we get

$$u_0=1+e^x, v_0=-1+e^x,$$

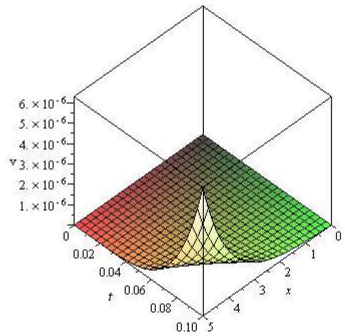

Fig. 2: Absolute error $E_4(v) = |v_{exa.} - v_{appr.}|$ for system of equations (24), when $\alpha = 1$, $\beta = 1$, $n = 1$, and $\hbar = -1$.

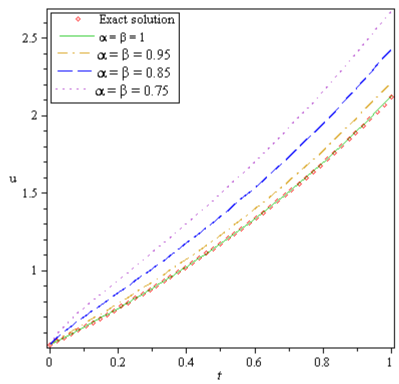

Fig. 3: Comparative nature of 4th-order q-HATM solution with exact solution of system of equations (24) at $x = 0.5$ with different order of derivatives.

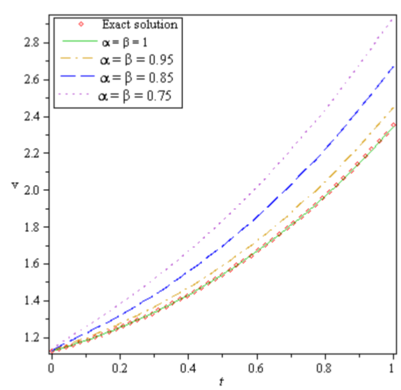

Fig. 4: Comparative behavior of 4th-order q-HATM solution with exact solution of system of equations (24) at $x = 0.5$ with different order of derivatives.

$$u_1 = \frac{-\hbar e^x t^\alpha}{\Gamma(\alpha+1)}, v_1 = \frac{\hbar e^x t^\beta}{\Gamma(\beta+1)},$$

$$u_2 = \frac{-\hbar(\hbar+n)e^x t^\alpha}{\Gamma(\alpha+1)} + \frac{\hbar^2 e^x t^{2\alpha}}{\Gamma(2\alpha+1)}, v_2 = \frac{\hbar(\hbar+n)e^x t^\beta}{\Gamma(\beta+1)} + \frac{\hbar^2 e^x t^{2\beta}}{\Gamma(2\beta+1)},$$

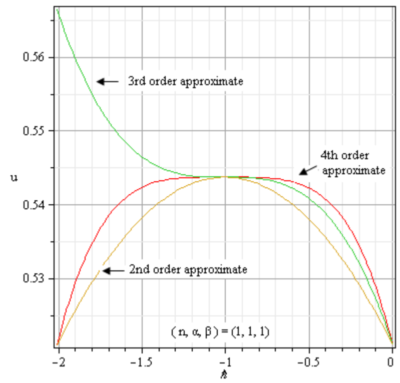

Fig. 5: $\hbar-$curve for diverse order q-HATM approximation for system of equations (24).

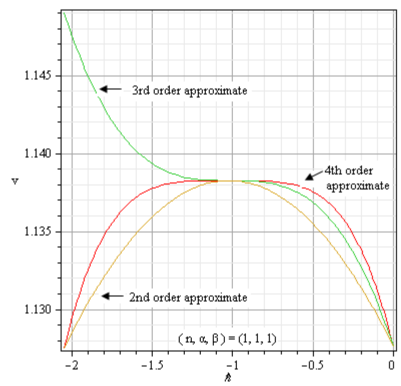

Fig. 6: $\hbar-$curve for diverse order q-HATM approximation of system of equations (24).

$$u_3=\frac{-\hbar(\hbar+n)^2e^xt^\alpha}{\Gamma(\alpha+1)}+2\hbar^2(\hbar+n)e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\hbar^3e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$v_3=\frac{\hbar(\hbar+n)^2e^xt^\beta}{\Gamma(\beta+1)}+2\hbar^2(\hbar+n)e^x\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\hbar^3e^x\frac{t^{3\beta}}{\Gamma(3\beta+1)}, \quad (34)$$

$$\vdots$$

and so on.
Making use of the similar process the further iterations can be evaluated. Thus, the q-HATM solution of the system (32) is presented as

$$\begin{cases} u=u_0+\sum_{m=1}^{\infty}u_m(\frac{1}{n})^m, \\ v=v_0+\sum_{m=1}^{\infty}v_m(\frac{1}{n})^m. \end{cases} \quad (35)$$

When $\alpha=1$, $\beta=1, n=1$ and $\hbar=-1$ then it is converted to the HPTA solution obtained by Khan et al. [31], we can observe that the solutions $\sum_{m=0}^{N}u_m(\frac{1}{n})^m$ and $\sum_{m=0}^{N}v_m(\frac{1}{n})^m$ when $N\to\infty$ tends to the exact solution $u=1+e^{x+t}$ and $v=-1+e^{x-t}$ for system of fractional differential equations (32). Figs. 7-8 depict the absolute error which reveal that the q-HATM solution converges to the exact solution very fast by using only third or fourth terms of q-HATM solution. It is to be noted from Fig. 9 that as the values of α and β are enhanced, the value of u decreases. From Fig. 10, we can see that as the values of α and β enhance, the values of v increase but after some time its character is contrary to previous result. Fig. 11 show $\hbar-$curve at $x=0.5, t=0.02$ for q-HATM approximation of system fractional equations (32), horizontal-line segments display the absolute convergence range of $\hbar$ and the justifiable range of $\hbar$ for u is $-2.02 \leq \hbar < 0$. Fig. 12 depicts $\hbar-$curve at $x = 0.5$, $t = 0.02$ for q-HATM approximation of system (32), horizontal-line segments display the absolute convergence range of $\hbar$ and the justifiable range of $\hbar$ for v is $-1.98 \leq \hbar < 0$.

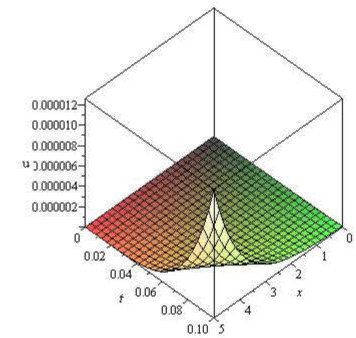

Fig. 7: Absolute error $E_4(u) = |u_{exa.} - u_{appr.}|$ for system of equations (32), when $n = 1, \hbar = -1, \alpha = 1$ and $\beta = 1$.

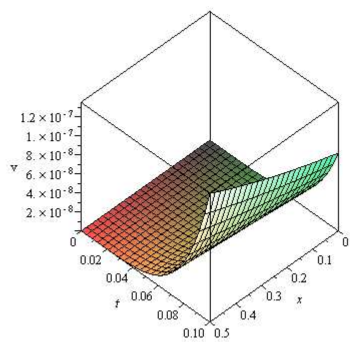

Fig. 8: Absolute error $E_4(v) = |v_{exa.} - v_{appr.}|$ for system (32), when $n = 1, \hbar = -1, \alpha = 1$ and $\beta = 1$.

Example 3. Lastly, we study the following time-fractional inhomogeneous nonlinear differential equations

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} v + u = 1, & 0 < \alpha \leq 1, \\ \frac{\partial^\beta v}{\partial t^\beta} - u \frac{\partial v}{\partial x} - v = 1, & 0 < \beta \leq 1, \end{cases} \tag{36}$$

with

$$u(x,0) = u_0 = \eta_1(x) = e^x, v(x,0) = v_0 = \eta_2(x) = e^{-x}. \tag{37}$$

Solving the above system (36) and (37) by using q-HATM, we get

$$u_0 = e^x, v_0 = e^{-x},$$

$$u_1 = \frac{\hbar e^x t^\alpha}{\Gamma(\alpha+1)}, v_1 = \frac{-\hbar e^x t^\beta}{\Gamma(\beta+1)}, u_2 = \frac{\hbar(\hbar+n) e^x t^\alpha}{\Gamma(\alpha+1)} - \frac{\hbar^2 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\hbar^2 (e^x +) t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$v_2 = \frac{-\hbar(\hbar+n) e^{-x} t^\beta}{\Gamma(\beta+1)} + \frac{\hbar^2 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\hbar^2 (e^{-x} - 1) t^{2\beta}}{\Gamma(2\beta+1)}, \tag{38}$$

$$\vdots$$

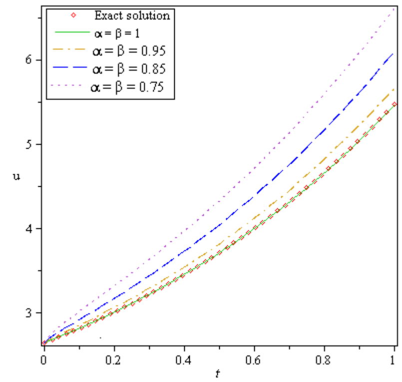

Fig. 9: Comparative nature of 4th-order q-HATM solution with exact solution of system of equations (32) at $x = 0.5$ v/s with different-order derivative.

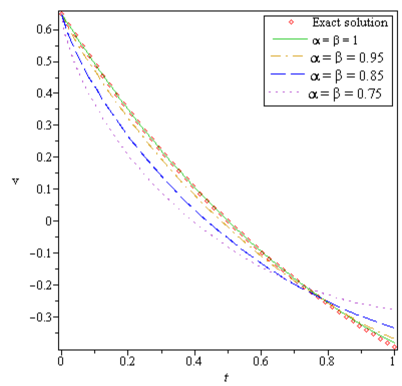

Fig. 10: Comparative nature of 4th-order q-HATM solution with exact solution of system of equations (32) at $x = 0.5$ with different-order derivative.

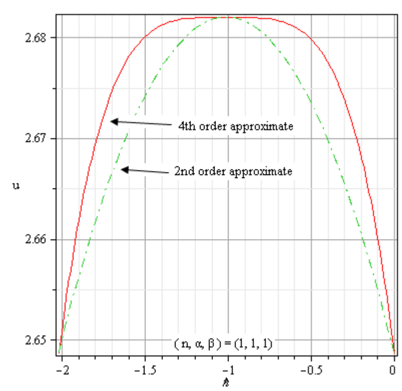

Fig. 11: $\hbar-$curve at $x = 0.5, t = 0.02$ for q-HATM approximation of system of equations (32).

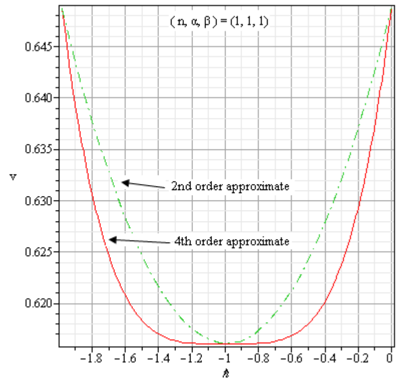

Fig. 12: $\hbar-$curve at $x = 0.5, t = 0.02$ for q-HATM approximation of system of equations (32).

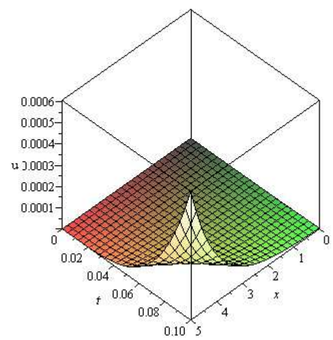

Fig. 13: Absolute error $E_3(u) = |u_{exa.} - u_{appr.}|$ for system of equations (36), if $n = 1, \hbar = -1, \alpha = 1$ and $\beta = 1$.

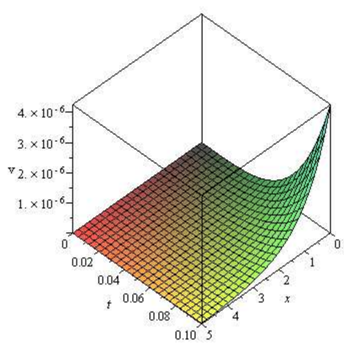

Fig. 14: Absolute error $E_3(v) = |v_{exa.} - v_{appr.}|$ for system of equations (36), if $n = 1, \hbar = -1, \alpha = 1$ and $\beta = 1$.

Thus the rest of the iterative terms can be evaluated. So, the q-HATM solution of the system (36) is written as

$$\begin{cases} u = u_0 + \sum_{m=1}^{\infty} u_m (\frac{1}{n})^m, \\ v = v_0 + \sum_{m=1}^{\infty} v_m (\frac{1}{n})^m. \end{cases} \tag{39}$$

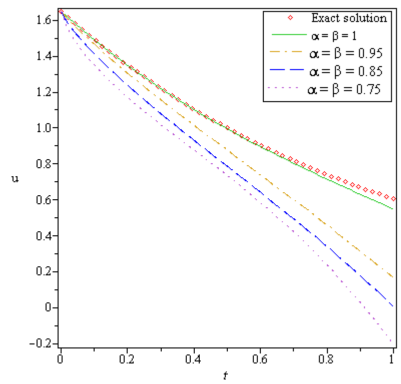

Fig. 15: Comparative study of 3rd-order q-HATM ($n = 1$) solution with exact solution of system (36) with different order of derivative.

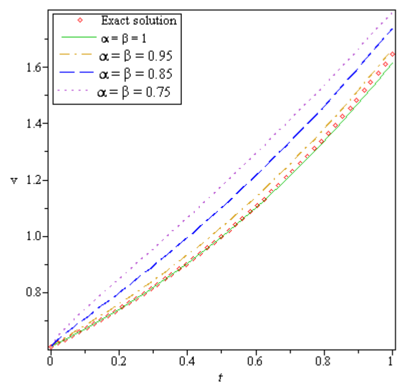

Fig. 16: Comparative behavior of 3rd-order q-HATM ($n = 1$) solution with exact solution of system (36) with different order of derivative.

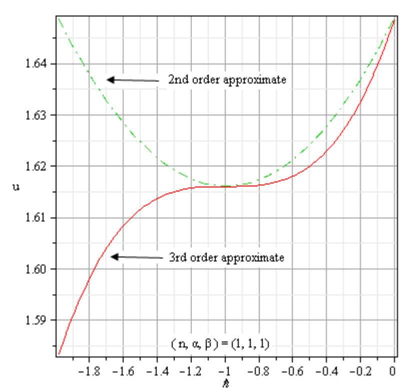

Fig. 17: $\hbar-$curve for diverse order q-HATM approximation of system (36).

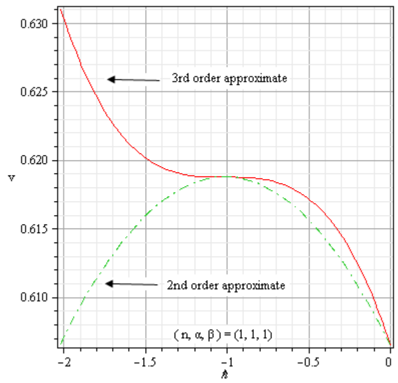

Fig. 18: $\hbar-$curve for distinct order q-HATM approximation of system (36).

Therefore, the solution for the system (36), when $\alpha = 1$, $\beta = 1$, $\hbar = -1$ and $n = 1$ is converted to the solution of HPTA [31] and presented as

$$\begin{cases} u = \lim\limits_{m\to\infty} u_m(x,t) = e^x - e^x t - \frac{t^2}{2} + e^x\frac{t^2}{2} + \frac{t^2}{2} - e^x\frac{t^3}{6} + \frac{t^3}{6} - \frac{t^3}{6} + e^x\frac{t^3}{6} \\ \quad + \frac{t^3}{3} - e^x\frac{t^3}{6} - \frac{t^3}{3}\cdots, \\ v = \lim\limits_{m\to\infty} v_m(x,t) = e^{-x} + e^{-x}t + \frac{t^2}{2} + e^{-x}\frac{t^2}{2} - \frac{t^2}{2} + e^{-x}\frac{t^3}{6} + \frac{t^3}{6} - \frac{t^3}{6} \\ \quad + e^{-x}\frac{t^3}{6} - \frac{t^3}{3} - e^{-x}\frac{t^3}{6} + \frac{t^3}{3}\cdots. \end{cases} \tag{40}$$

It is obvious that the self-canceling ‘noise’ terms and keeping the non-noise terms in system of Eqs. (40) yield the exact solution $u = e^{x-t}$and $v = e^{-x+t}$ it can be simply verified and formally proved by Khan et al. [31]. Figs. 13-14 present the absolute error which display that the q-HATM solution tends to the exact solution very fast. It can be noted from Fig. 15 that as the values of α and β rise, the values of u rises. From Fig. 16 we can note that as the values of α and β rise, the values of v decreases. Fig. 17 represents $\hbar-$ curve for different-order q-HATM approximation of system (36), horizontal line segments display the absolute convergence range of $\hbar$ and the justifiable range of $\hbar$ for u is $-1.98 \leq \hbar < 0$. Fig. 18 presents $\hbar-$ curve for differ-order q-HATM approximation of system (36), line segment parallel to v displays the absolute convergence range of $\hbar$ and the justifiable range of $\hbar$ for v is $-2.02 \leq \hbar < 0$.

5 Conclusions

In this work, the q-HATM is utilized for obtaining the solution of system of time-fractional linear and nonlinear equations. The acceptable convergence range of q-HATM solution represents the horizontal-line segments in $\hbar$-curves, such range is increased to their valid range, if we include the series solution of systems $\sum_{m=0}^{N} u_m(x,t)(\frac{1}{n})^m$ and $\sum_{m=0}^{N} v_m(x,t)(\frac{1}{n})^m$ when $N \to \infty$. Positivism of proposed method to HPTA is demonstrated by comparing the absolute errors and straightforward solution procedures that are free from He’s polynomials and assumption of small/large physical parameters.

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