Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/050103

Extension of Incomplete Gamma, Beta and Hypergeometric Functions

Mehmet Ali Özarslan and Ceren Ustaoglu*

Department of Mathematics, Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, Turkey

Received: 2 May 2018, Revised: 8 Jun. 2018, Accepted: 12 Jun. 2018
Published online: 1 Jan. 2019

Abstract: Recently, some extensions of the generalised gamma, beta, Gauss hypergeometric and confluent hypergeometric functions have been introduced. In this paper, we introduce generalisations of incomplete gamma, beta, Gauss, confluent and Appell's hypergeometric functions. Some integral representations, Mellin transforms, transformation formulae, differentiation formulae, difference formulae and fractional calculus formulae are obtained for these functions.

Keywords: Generalisations of incomplete gamma and beta functions, generalisations of incomplete Gauss and confluent hypergeometric function, generalisations of incomplete Appell's hypergeometric function, Mellin transforms, fractional calculus formulae.

1 Introduction

In recent years, some extensions of the well-known special functions have been considered by several authors [2], [3], [4], [5], [6], [7], [8]. The following extension of gamma function was introduced by Chaudhry and Zubair [2], in 1994

$$\Gamma_p(x) := \int_0^\infty t^{x-1} \exp\left[-t - \frac{p}{t}\right] dt, \; Re(p) > 0. \tag{1}$$

The following extension of beta function has been defined by Chaudhry [3], in 1997

$$B_p(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt, \;\; Re(p) > 0, \; Re(x) > 0, \; Re(y) > 0. \tag{2}$$

Afterwards, Chaudhry [9] introduced to extended hypergeometric and confluent hypergeometric functions as follows:

$$F_p(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \;\; p \geq 0; \; Re(c) > Re(b) > 0, \tag{3}$$

$$\phi_p(b;c;z) := \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \;\; p \geq 0; \; Re(c) > Re(b) > 0, \tag{4}$$

where $(\lambda)_\nu$ is the Pochhammer given by

$$(\lambda)_0 \equiv 1 \;\text{ and }\; (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

and the Euler type integral representation is given

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[\frac{-p}{t(1-t)}\right] dt,$$
$$p > 0; \; p = 0 \text{ and } |\arg(1-z)| < \pi < p; \; Re(c) > Re(b) > 0.$$

* Corresponding author e-mail: mehmetali.ozarslan@emu.edu.tr

Several properties for these functions are investigated such as Mellin transforms of $F_p(a,b;c;z)$, transformation formulae, differentiation properties, recurrence relations, series and asymptotic formulae. The extended (or generalised) Gauss hypergeometric function has been an active research topic in recent years [10], [11], [12], [13], [14].

Recently, the incomplete Pochhammer symbols are defined with the help of incomplete gamma functions as follows [15]:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)}, \quad \lambda,\nu \in \mathbb{C};\ x \geqq 0,$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)}, \quad \lambda,\nu \in \mathbb{C};\ x \geqq 0.$$

With the help of these symbols, the authors gave the incomplete hypergeometric functions and investigate their properties [15]. Some other properties of these functions have been investigated in [16], [17], [18], [19].

Incomplete Pochhammer ratios were introduced by using the incomplete beta function as follows [20]:

$$[b,c;y]_n := \frac{B_y(b+n,c-b)}{B(b,c-b)}$$

and

$$\{b,c;y\}_n := \frac{B_{1-y}(c-b,b+n)}{B(b,c-b)}$$

where $0 \le y < 1$. Since the incomplete beta function $B_y(x,z)$ is defined by

$$B_y(x,z) := \int_0^y t^{x-1}(1-t)^{z-1}dt, \ \ Re(x) > Re(z) > 0,\ 0 \le y < 1.$$

Then, incomplete Gauss, confluent and Appell's hypergeometric functions are defined in terms of these incomplete Pochhammer ratios and integration formulae, derivative formulae, transformation formulae and recurrence relations for these functions are obtained [20].

The incomplete confluent hypergeometric functions were defined by [20]

$${}_1F_1([\alpha,\beta;y];z) := \sum_{n=0}^{\infty}[\alpha,\beta;y]_n\frac{z^n}{n!}, \ \ 0 \le y < 1,$$

and

$${}_1F_1(\{\alpha,\beta;y\};z) := \sum_{n=0}^{\infty}\{\alpha,\beta;y\}_n\frac{z^n}{n!}, \ \ 0 \le y < 1.$$

The integral representations are given by [20]

$${}_1F_1([\alpha,\beta;y],z) = \frac{y^\alpha}{B(\alpha,\beta-\alpha)}\int_0^1 u^{\alpha-1}(1-uy)^{\beta-\alpha-1}e^{zuy}du \quad (Re(\alpha) > Re(\beta) > 0), \tag{5}$$

and

$${}_1F_1(\{\alpha,\beta;y\},z) = \frac{(1-y)^{\beta-\alpha}}{B(\alpha,\beta-\alpha)}\int_0^1 u^{\beta-\alpha-1}(1-u(1-y))^{\alpha-1}e^{(1-u(1-y))z}du \quad (Re(\alpha) > Re(\beta) > 0).$$

The structure of this paper is as follows: Generalisations of incomplete gamma and Euler's beta function are elaborated in Section 2. Later, we obtain distinct integration formulae and several properties of generalised incomplete Euler's beta function. Furthermore, connections of generalised incomplete gamma and beta functions are also examined. The Generalised Incomplete Gauss Hypergeometric Function (GIGHF), Generalised Incomplete Confluent Hypergeometric Function (GICHF) and Generalised Incomplete Appell's Hypergeometric Function (GIAHF). The study proceeds by obtaining some integral representations of these functions. Mellin's transform representation of the GIGHF and GICHF are also investigated. Differentiation and transformation formulae of the above mentioned functions are presented. In addition, fractional calculus formulae for GIGHF can be given in terms of the GIAHF.

2 Generalisations of incomplete gamma and Euler's beta function

This section dwells on the following generalisations of incomplete gamma and beta functions

$$\Gamma_p^{(\alpha,\beta;y)}(x) \ : \ = \int_0^\infty t^{x-1} \ {}_1F_1\left([\alpha,\beta;y];-t-\frac{p}{t}\right)dt, \tag{6}$$
$$(Re(\alpha)>0,\ Re(\beta)>0,\ Re(p)>0,\ Re(x)>0,\ 0\le y<1)$$

and

$$B_p^{(\alpha,\beta;y)}(x,z) \ : \ = \int_0^1 t^{x-1}(1-t)^{z-1} \ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt, \tag{7}$$
$$(Re(\alpha)>0,\ Re(\beta)>0,\ Re(p)>0,\ Re(x)>0,\ Re(z)>0,\ 0\le y<1)$$

respectively.

*Remark.*In the case $p=0$, (6) reduces to

$$\Gamma^{(\alpha,\beta;y)}(x) := \int_0^\infty t^{x-1} \ {}_1F_1\left([\alpha,\beta;y];-t\right)dt. \tag{8}$$

*Remark.*In the case $p=0$, (7) reduces to

$$B_p^{(\alpha,\beta;y)}(x,z) = B(x,z).$$

*Remark.*In the case $\lim_{y\to 1}\Gamma_p^{(\alpha,\beta;y)}(x)$ is reduced to the extension of gamma function $\Gamma_p^{(\alpha,\beta)}(x)$ which was defined in [1] (similarly, when $\lim_{y\to 1}B_p^{(\alpha,\beta;y)}(x,z)$ is reduced to the extension of beta function $B_p^{(\alpha,\beta;y)}(x,z)$ which was defined in [1]).

The study is continuous by obtaining the integral representations of the functions given in the above definition.

Theorem 1.*The following integral representation holds true for the generalised incomplete gamma function:*

$$\Gamma_p^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)}\int_0^1 \Gamma_{\mu^2y^2p}(s)\mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1}d\mu,$$

where $\Gamma_p(s)$ is given in (1).

*Proof.*Using (5), we have

$$\Gamma_p^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha}}{B(\alpha,\beta-\alpha)}\int_0^\infty\int_0^1 t^{s-1}e^{-uyt-\frac{uyp}{t}}u^{\alpha-1}(1-uy)^{\beta-\alpha-1}dudt.$$

Let us now use the mapping $v=uyt,\ \mu=u$, which is bijective except possibly at the boundary and which maps $(0,\infty)\times(0,1)$ onto itself. The Jacobian is $J=\frac{1}{\mu y}$, so

$$\Gamma_p^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)}\int_0^\infty\int_0^1 v^{s-1}e^{-v-\frac{\mu^2y^2p}{v}}dv\mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1}d\mu.$$

The order of integration can be interchanged from the uniform convergence of the integral. It gives us

$$\begin{aligned}\Gamma_p^{(\alpha,\beta;y)}(s) &= \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)}\int_0^1\left[\int_0^\infty v^{s-1}e^{-v-\frac{\mu^2y^2p}{v}}dv\right]\mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1}d\mu \\ &= \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)}\int_0^1 \Gamma_{\mu^2y^2p}(s)\mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1}d\mu.\end{aligned}$$

Hence the proof is completed.

*Remark.*In the case $p=0$, we have

$$\Gamma^{(\alpha,\beta;y)}(s) = \frac{y^{\alpha-s}}{B(\alpha,\beta-\alpha)}\int_0^1 \Gamma(s)\mu^{\alpha-s-1}(1-\mu y)^{\beta-\alpha-1}d\mu = \frac{y^{\alpha-s}\Gamma(s)B_y(\alpha-s,\beta-\alpha)}{B(\alpha,\beta-\alpha)}. \tag{9}$$

This part of the study concentrates on the integral representation of $B_p^{(\alpha,\beta;y)}(x,z)$ given in terms of the extended beta function.

Theorem 2. *For the generalised incomplete Euler's beta function, we have*

$$B_p^{(\alpha,\beta;y)}(x,z)=\frac{y^\alpha}{B(\alpha,\beta-\alpha)}\int_0^1 B_{uyp}(x,z)u^{\alpha-1}(1-uy)^{\beta-\alpha-1}du,$$

where $B_p(x,z)$ is given in (2).

Proof. Using (5) yields

$$B_p^{(\alpha,\beta;y)}(x,z)=\frac{y^\alpha}{B(\alpha,\beta-\alpha)}\int_0^1\int_0^1 t^{x-1}(1-t)^{z-1}\exp\left(\frac{-uyp}{t(1-t)}\right)u^{\alpha-1}(1-uy)^{\beta-\alpha-1}dudt.$$

The order of integration can be interchanged from the uniform convergence of the integrals. It gives us

$$B_p^{(\alpha,\beta;y)}(x,z)=\frac{y^\alpha}{B(\alpha,\beta-\alpha)}\int_0^1\left[\int_0^1 t^{x-1}(1-t)^{z-1}\exp\left(\frac{-uyp}{t(1-t)}\right)dt\right]u^{\alpha-1}(1-uy)^{\beta-\alpha-1}du.$$

In view of (2), we get

$$B_p^{(\alpha,\beta;y)}(x,z)=\frac{y^\alpha}{B(\alpha,\beta-\alpha)}\int_0^1 B_{uyp}(x,z)u^{\alpha-1}(1-uy)^{\beta-\alpha-1}du.$$

Hence the proof is completed.

In the following theorem, we compute the Mellin transform for the function $B_p^{(\alpha,\beta;y)}(x,z)$, an expression which involves ordinary beta function and $\Gamma^{(\alpha,\beta;y)}(s)$.

Theorem 3. *The following Mellin transform representation holds true for the generalised incomplete beta function:*

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta;y)}(x,z)dp=B(x+s,z+s)\Gamma^{(\alpha,\beta;y)}(s), \tag{10}$$

where $Re(s)>0,\ Re(x+s)>0,\ Re(z+s)>0,\ Re(p)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ 0\le y<1.$

Proof. Multiplying (7) by p^{s-1} and apply $\int_0^\infty dp$

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta;y)}(x,z)dp=\int_0^\infty p^{s-1}\int_0^1 t^{x-1}(1-t)^{z-1}\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dtdp. \tag{11}$$

The order of integration in (11) can be interchanged from the uniform convergence of the integral. Therefore, we have

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta;y)}(x,z)dp=\int_0^1 t^{x-1}(1-t)^{z-1}\int_0^\infty p^{s-1}\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dpdt. \tag{12}$$

Let us now use the mapping $v=\frac{p}{t(1-t)},\ \mu=t$ in (12), which is bijective except possibly at the boundary and which maps $(0,\infty)\times(0,1)$ onto itself. We have

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta;y)}(x,z)dp=\int_0^1\mu^{(x+s)-1}(1-\mu)^{(z+s)-1}d\mu\int_0^\infty v^{s-1}\,{}_1F_1([\alpha,\beta;y];-v)dv.$$

Substituting from (8) gives

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta;y)}(x,z)dp=B(x+s,z+s)\Gamma^{(\alpha,\beta;y)}(s),$$

and whence the result is.

Remark. Setting $s=1$ and using the fact that $\Gamma^{(\alpha,\beta;y)}(1)=\frac{y^{\alpha-1}B_y(\alpha-1,\beta-\alpha)}{B(\alpha,\beta-\alpha)}$ in (10), we get

$$\int_0^\infty B_p^{(\alpha,\beta;y)}(x,z)dp=B(x+1,z+1)\frac{y^{\alpha-1}B_y(\alpha-1,\beta-\alpha)}{B(\alpha,\beta-\alpha)}.$$

Theorem 4. *Mellin transform representation of the generalised incomplete gamma function is given by*

$$\mathfrak{M}\left\{\Gamma_p^{(\alpha,\beta;y)}(x):s\right\} = \frac{\Gamma(s)\Gamma(x+s)B_y(\alpha-2s-x,\beta-\alpha)}{B(\alpha,\beta-\alpha)}. \tag{13}$$

Remark. Setting $s=1$ in (13), we have

$$\int_0^\infty \Gamma_p^{(\alpha,\beta;y)}(x)\,dp = \frac{\Gamma(x+1)B_y(\alpha-x-2,\beta-\alpha)}{B(\alpha,\beta-\alpha)}.$$

Theorem 5. *The following integral representations hold true for the generalised incomplete beta function:*

$$B_p^{(\alpha,\beta;y)}(x,z) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\sin^{2z-1}\theta\ {}_1F_1([\alpha,\beta;y];-p\sec^2\theta-p\csc^2\theta)d\theta, \tag{14}$$

$$B_p^{(\alpha,\beta;y)}(x,z) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+z}}\ {}_1F_1\left([\alpha,\beta;y];-2p-p\left(u+\frac{1}{u}\right)\right)du. \tag{15}$$

Proof. The proofs of (14) and (15) are obtained from (7) with the substitution $t=\cos^2\theta$ and $t=\frac{u}{1+u}$, respectively.

Theorem 6. *The following functional relation holds true for the generalised incomplete beta function:*

$$B_p^{(\alpha,\beta;y)}(x,z+1)+B_p^{(\alpha,\beta;y)}(x+1,z) = B_p^{(\alpha,\beta;y)}(x,z).$$

Proof. Direct calculations yield,

$$\begin{aligned}
B_p^{(\alpha,\beta;y)}(x,z+1)+B_p^{(\alpha,\beta;y)}(x+1,z) &= \int_0^1 t^{x-1}(1-t)^z\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\
&\quad+\int_0^1 t^{x}(1-t)^{z-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\
&= \int_0^1 \left[t^{x-1}(1-t)^z+t^x(1-t)^{z-1}\right]\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\
&= \int_0^1 t^{x-1}(1-t)^{z-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt \\
&= B_p^{(\alpha,\beta;y)}(x,z).
\end{aligned}$$

Hence the proof is completed.

Theorem 7. *The following integral representation holds true for the product of two generalised incomplete gamma function:*

$$\begin{aligned}
\Gamma_p^{(\alpha,\beta;y)}(x)\Gamma_p^{(\alpha,\beta;y)}(z) &= 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+z)-1}\cos^{2x-1}\theta\sin^{2z-1}\theta \\
&\times\ {}_1F_1\left([\alpha,\beta;y];-r\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right)\ {}_1F_1\left([\alpha,\beta;y];-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)drd\theta.
\end{aligned}$$

Proof. Setting $\eta=\sqrt{t}$ in (6), we have

$$\Gamma_p^{(\alpha,\beta;y)}(x) = 2\int_0^\infty \eta^{2x-1}\ {}_1F_1\left([\alpha,\beta;y];-\eta^2-\frac{p}{\eta^2}\right)d\eta.$$

Therefore

$$\Gamma_p^{(\alpha,\beta;y)}(x)\Gamma_p^{(\alpha,\beta;y)}(z) = 4\int_0^\infty\int_0^\infty \eta^{2x-1}\xi^{2z-1}\ {}_1F_1\left([\alpha,\beta;y];-\eta^2-\frac{p}{\eta^2}\right)\ {}_1F_1\left([\alpha,\beta;y];-\xi^2-\frac{p}{\xi^2}\right)d\eta d\xi.$$

Setting $\eta = r\cos\theta$ and $\xi = r\sin\theta$ in the above equality,

$$\Gamma_p^{(\alpha,\beta;y)}(x)\Gamma_p^{(\alpha,\beta;y)}(z) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(x+z)-1}\cos^{2x-1}\theta\sin^{2z-1}\theta$$
$$\times\ {}_1F_1\left([\alpha,\beta;y];-r\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right)\ {}_1F_1\left([\alpha,\beta;y];-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)drd\theta,$$

and whence the result is.

Theorem 8. *The following summation relation holds true for the generalised incomplete beta function:*

$$B_p^{(\alpha,\beta;y)}(x,1-z) = \sum_{n=0}^{\infty}\frac{(z)^n}{n!}B_p^{(\alpha,\beta;y)}(x+n,1),\quad Re(p)>0,\ 0\le y<1.$$

Proof. Equation (7) gives us

$$B_p^{(\alpha,\beta;y)}(x,1-z) = \int_0^1 t^{x-1}(1-t)^{-z}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.$$

Use the following expansion

$$(1-t)^{-z} = \sum_{n=0}^{\infty}(z)_n\frac{t^n}{n!},\quad |t|<1,$$

to have

$$B_p^{(\alpha,\beta;y)}(x,1-z) = \int_0^1\sum_{n=0}^{\infty}\frac{(z)_n}{n!}t^{x+n-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.$$

Hence, by changing the order of integration and summation and after that using (7), we obtain

$$B_p^{(\alpha,\beta;y)}(x,1-z) = \sum_{n=0}^{\infty}\frac{(z)_n}{n!}\int_0^1 t^{x+n-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt,$$
$$= \sum_{n=0}^{\infty}\frac{(z)_n}{n!}B_p^{(\alpha,\beta;y)}(x+n,1),$$

and whence the result is.

In addition, we introduce another extension of the generalised incomplete gamma and beta functions

$$\Gamma_p^{(\alpha,\beta)}[x;y] := \int_0^y t^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{16}$$

$$\Gamma_p^{(\alpha,\beta)}\{x;y\} := \int_y^{\infty} t^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{17}$$

$$B_p^{(\alpha,\beta)}[x,z;y] := \int_0^y t^{x-1}(1-t)^{z-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,\ 0\le y<1,\ Re(x)>0, Re(z)>0, \tag{18}$$

and

$$B_p^{(\alpha,\beta)}\{x,z;y\} := \int_0^{1-y} t^{z-1}(1-t)^{x-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,\ 0\le y<1,\ Re(x)>0, Re(z)>0, \tag{19}$$

respectively. For these functions, we have the following decomposition formulae

$$\Gamma_p^{(\alpha,\beta)}[x;y]+\Gamma_p^{(\alpha,\beta)}\{x;y\} = \Gamma_p^{(\alpha,\beta)}(x)$$

and

$$B_p^{(\alpha,\beta)}[x,z;y]+B_p^{(\alpha,\beta)}\{x,z;y\} = B_p^{(\alpha,\beta)}(x,z).$$

The integral representations of these functions are given in the following theorem.

Theorem 9. *The following integral representation holds true:*

$$\Gamma_p^{(\alpha,\beta)}[x;y] = y^x \int_0^1 u^{x-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right) du. \tag{20}$$

Remark. Using the integral representation of ${}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)$

$${}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right) = \frac{1}{B(\alpha,\beta-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1} \exp\left(\frac{-pt}{uy(1-uy)}\right) dt, \tag{21}$$

in (20), we get

$$\Gamma_p^{(\alpha,\beta)}[x;y] = \frac{y^x}{B(\alpha,\beta-\alpha)} \int_0^1 \int_0^1 u^{x-1} t^{\alpha-1}(1-t)^{\beta-\alpha-1} \exp\left(\frac{-pt}{uy(1-uy)}\right) dudt.$$

Theorem 10. *The following integral representation holds true:*

$$B_p^{(\alpha,\beta)}[x,z;y] = y^x \int_0^1 u^{x-1}(1-uy)^{z-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right) du \tag{22}$$

and

$$B_p^{(\alpha,\beta)}\{x,z;y\} = y^{z-1} \int_0^1 u^{z-1}(1-u(1-y))^{x-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{u(1-y)(1-u(1-y))}\right) du. \tag{23}$$

Remark. Using (21) in (22) and (23), we get

$$B_p^{(\alpha,\beta)}[x,z;y] = \frac{y^x}{B(\alpha,\beta-\alpha)} \int_0^1 \int_0^1 u^{x-1}(1-uy)^{z-1} t^{\alpha-1}(1-t)^{\beta-\alpha-1} \exp\left(\frac{-pt}{uy(1-uy)}\right) dudt \tag{24}$$

and

$$B_p^{(\alpha,\beta)}\{x,z;y\} = \frac{y^{z-1}}{B(\alpha,\beta-\alpha)} \int_0^1 \int_0^1 u^{z-1}(1-u(1-y))^{x-1} t^{\alpha-1}(1-t)^{\beta-\alpha-1} \exp\left(\frac{-pt}{u(1-y)(1-u(1-y))}\right) dudt,$$

respectively.

3 Generalised incomplete hypergeometric functions

In this part, we apply (7) to generalise the GIGHF and GICHF as follows:

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{25}$$

and

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z) := \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \tag{26}$$

respectively. Furthermore, using (18) we define another extension of the GIGHF and GIAHF as follows:

$$F_p^{\alpha,\beta}(a,[b,c;y];z) \ : = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{\alpha,\beta}[b+n,c-b;y]}{B(b,c-b)} \frac{z^n}{n!}, \tag{27}$$
$$Re(c) > Re(b) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0,$$

and

$$F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p) \ : = \sum_{n,m=0}^{\infty} (\rho)_{n+m} \frac{B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]}{B(\nu,\gamma-\nu)} \frac{B_p(\lambda+m,\mu-\lambda)}{B(\lambda,\mu-\lambda)} \frac{x^n}{n!} \frac{z^m}{m!}, \tag{28}$$
$$Re(\gamma) > Re(\nu) > 0,\ Re(\mu) > Re(\lambda) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0.$$

3.1 Integral representations

Theorem 11. *For the GIGHF, we have the following integral representation:*

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{y^{\alpha}}{B(\alpha,\beta-\alpha)}\int_0^1 F_{uyp}(a,b;c;z)\,u^{\alpha-1}(1-uy)^{\beta-\alpha-1}\,du,$$
$$Re(\beta)>Re(\alpha)>0,\ Re(c)>Re(b)>0,\ |\arg(1-u)|<1,$$

where $F_p(a,b;c;z)$ is given in (3).

Proof. Since

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}$$

we have from the Theorem 2

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{y^{\alpha}}{B(\alpha,\beta-\alpha)}\sum_{n=0}^{\infty}\frac{(a)_n}{B(b,c-b)}\int_0^1 B_{uyp}(b+n,c-b)u^{\alpha-1}(1-uy)^{\beta-\alpha-1}du\frac{z^n}{n!}.$$

Focusing on the uniform convergence of the series involved and the absolute convergence of the integral, interchanging the order of series and the integral, we have

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{y^{\alpha}}{B(\alpha,\beta-\alpha)}\int_0^1\left\{\sum_{n=0}^{\infty}(a)_n\frac{B_{uyp}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\}u^{\alpha-1}(1-uy)^{\beta-\alpha-1}du,$$

and whence the result is.

Remark. To obtain the following integral representation using (3)

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{y^{\alpha}}{B(\alpha,\beta-\alpha)B(b,c-b)}$$
$$\times\int_0^1\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp\left(\frac{-uyp}{t(1-t)}\right)(1-zt)^{-a}u^{\alpha-1}(1-uy)^{\beta-\alpha-1}dtdu.$$

Theorem 12. *For the GICHF, we have the following integral representation:*

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)=\frac{y^{\alpha}}{B(\alpha,\beta-\alpha)}\int_0^1\phi_{uyp}(b;c;z)u^{\alpha-1}(1-uy)^{\beta-\alpha-1}du.$$

Remark. To obtain the following integral representation using (4)

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)=\frac{y^{\alpha}}{B(\alpha,\beta-\alpha)B(b,c-b)}$$
$$\times\int_0^1\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp\left(\frac{-uyp}{t(1-t)}+zt\right)u^{\alpha-1}(1-uy)^{\beta-\alpha-1}dtdu.$$

Theorem 13. *The GIGHF can be represented by an integral as follows:*

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}$$
$$\times\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt, \tag{29}$$
$$Re(p)>0;\ 0\le y<1;\ Re(c)>Re(b)>0;$$
$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^{\infty}u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a}\,{}_1F_1\left([\alpha,\beta;y];-2p-p\left(u+\frac{1}{u}\right)\right)du,$$
$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}}\sin^{2b-1}v\cos^{2c-2b-1}v(1-z\sin^2 v)\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{\sin^2 v\cos^2 v}\right)dv.$$

*Proof.*Using the definitions

$$\begin{aligned} F_p^{(\alpha,\beta;y)}(a,b;c;z) &= \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \\ &= \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)\frac{z^n}{n!}dt \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}dt \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt. \end{aligned}$$

Substituting $u=\frac{t}{1-t}$ in (29), we get

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^{\infty}u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a}\ {}_1F_1\left([\alpha,\beta;y];-2p-p\left(u+\frac{1}{u}\right)\right)du.$$

Otherwise, setting $t=\sin^2 v$ in (29), we get

$$F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}}\sin^{2b-1}v\cos^{2c-2b-1}v(1-z\sin^2 v)\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{\sin^2 v\cos^2 v}\right)dv,$$

and whence the result is.

Theorem 14.*For the GIGHF, we have the following integral representation:*

$$F_p^{\alpha,\beta}(a,[b,c;y];z)=\frac{y^b}{B(b,c-b)}\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-uyz)^{-a}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)du, \quad (30)$$
$$Re(c)>Re(b)>0,\ Re(\alpha)>0,\ Re(\beta)>0,\ |\arg(1-z)|<\pi.$$

*Proof.*Replacing the generalised incomplete beta function $B_p^{\alpha,\beta}[b+n,c-b;y]$ by its integral representation given by (18) and interchanging the order of summation and integral which is permissible under the conditions given in the hypothesis of the theorem, we find

$$F_p^{\alpha,\beta}(a,[b,c;y];z)=\frac{1}{B(b,c-b)}\int_0^y t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,$$

which can be written as follows:

$$F_p^{\alpha,\beta}(a,[b,c;y];z)=\frac{y^b}{B(b,c-b)}\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-uyz)^{-a}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)du.$$

*Remark.*To obtain the following integral representation using (24)

$$F_p^{\alpha,\beta}(a,[b,c;y];z)=\frac{y^b}{B(b,c-b)}\int_0^1\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-uyz)^{-a}t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left(\frac{-pt}{uy(1-uy)}\right)dudt.$$

By a similar argument, it is possible to find an integral representation of the GICHF using the equation (7).

Theorem 15.*The GICHF can be represented by an integral as follows:*

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt,$$

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1(1-u)^{b-1}u^{c-b-1}e^{z(1-u)}\ {}_1F_1\left([\alpha,\beta;y];\frac{-p}{u(1-u)}\right)du,$$

$p\geq 0$ and $Re(c)>Re(b)>0$.

Theorem 16. *The GIAHF can be represented by an integral as follows:*

$$F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p) = \frac{y^{\nu}}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\int_0^1\int_0^1 u^{\nu-1}(1-uy)^{\gamma-\nu-1} \times {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)s^{\lambda-1}(1-s)^{\mu-\lambda-1}(1-xuy-zs)^{-\rho}\,dsdu. \quad (31)$$

Proof. Replacing the generalised incomplete beta and extended beta functions by their integral representations (18) and (2), we get

$$F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p) = \frac{1}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\sum_{n,m=0}^{\infty}(\rho)_{n+m}\frac{x^n}{n!}\frac{z^m}{m!} \times \int_0^y\int_0^1 t^{\nu+n-1}(1-t)^{\gamma-\nu-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)s^{\lambda+m-1}(1-s)^{\mu-\lambda-1}\exp\left(\frac{-p}{s(1-s)}\right)dsdt,$$

We are interchanging the order of summation and integration focusing on the uniform convergence of the series involved, we get

$$\begin{aligned} F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p) &= \frac{1}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\int_0^y\int_0^1 t^{\nu-1}(1-t)^{\gamma-\nu-1} \\ &\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)s^{\lambda-1}(1-s)^{\mu-\lambda-1}\exp\left(\frac{-p}{s(1-s)}\right)(1-xt-zs)^{-\rho}\,dsdt \\ &= \frac{y^{\nu}}{B(\nu,\gamma-\nu)B(\lambda,\mu-\lambda)}\int_0^1\int_0^1 u^{\nu-1}(1-uy)^{\gamma-\nu-1} \\ &\times {}_1F_1\left(\alpha;\beta;\frac{-p}{uy(1-uy)}\right)s^{\lambda-1}(1-s)^{\mu-\lambda-1}(1-xuy-zs)^{-\rho}\,dsdu, \end{aligned}$$

and whence the result is.

3.2 Differentiation and difference formulae

In this subsection, we obtain the differentiation formulae of GIGHF and GICHF with respect to the variable z.

Theorem 17. *The GIGHF has a differentiation formulae which can be written as follows:*

$$\frac{d^n}{dz^n}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\} = \frac{(b)_n(a)_n}{(c)_n}F_p^{(\alpha,\beta;y)}(a+n,b+n;c+n;z).$$

Proof. To obtain the differentiation formulae taking derivative of $F_p^{(\alpha,\beta;y)}(a,b;c;z)$ with respect to z, by using the formulae $B(b,c-b)=\frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1}=a(a+1)_n$,

$$\begin{aligned} \frac{d}{dz}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\} &= \frac{d}{dz}\left\{\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\} \\ &= \sum_{n=1}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;y)}(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}. \end{aligned}$$

Replacing $n\to n+1$, we get

$$\begin{aligned} \frac{d}{dz}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\} &= \frac{ba}{c}\sum_{n=0}^{\infty}(a+1)_n\frac{B_p^{(\alpha,\beta;y)}(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!} \\ &= \frac{ba}{c}F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z). \end{aligned} \quad (32)$$

Recursive application of this procedure gives us the general form:

$$\frac{d^n}{dz^n}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\}=\frac{(b)_n(a)_n}{(c)_n}F_p^{(\alpha,\beta;y)}(a+n,b+n;c+n;z),$$

and whence the result is.

Theorem 18. *The GICHF has a differentiation formulae which can be written as follows:*

$$\frac{d^n}{dz^n}\left\{{}_1F_1^{((\alpha,\beta;y);p)}(b;c;z)\right\}=\frac{(b)_n}{(c)_n}\,{}_1F_1^{((\alpha,\beta;y);p)}(b+n;c+n;z).$$

Theorem 19. *For the GIGHF, we have the following recurrence relation:*

$$\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{bz}{c}F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z).$$

Proof. We obtain the following formulae from the equation (29)

$$\begin{aligned}\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z)&=F_p^{(\alpha,\beta;y)}(a+1,b;c;z)-F_p^{(\alpha,\beta;y)}(a,b;c;z)\\&=\frac{z}{B(b,c-b)}\int_0^1 t^b(1-t)^{c-b-1}(1-zt)^{-a-1}\\&\quad\times{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.\end{aligned}\tag{33}$$

Then, changing the arguments in the GIGHF by raising each parameter by one, we also find from (29) that

$$\begin{aligned}F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z)&=\frac{1}{B(b+1,c-b)}\int_0^1 t^b(1-t)^{c-b-1}(1-zt)^{-a-1}\\&\quad\times{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.\end{aligned}\tag{34}$$

Finally, by comparing (33) and (34), we have

$$\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z)=\frac{bz}{c}F_p^{(\alpha,\beta;y)}(a+1,b+1;c+1;z),\tag{35}$$

which is our first recurrence relation for the GIGHF.

Remark. Further, we obtain the following differential difference equation by using the differentiation formulae (32):

$$\frac{d}{dz}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z)\right\}=\frac{a}{z}\Delta_a F_p^{(\alpha,\beta;y)}(a,b;c;z),$$

where, just as in (33), Δ_a is the shift operator with respect to a.

3.3 Mellin Transform Representation

In this subsection, expressions for the Mellin transforms of the GIGHF and GICHF are given.

Theorem 20. *The GIGHF has a Mellin transform which can be written as follows:*

$$\mathfrak{M}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z):s\right\}=\frac{\Gamma^{(\alpha,\beta;y)}(s)\,B(b+s,c-b+s)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z).\tag{36}$$

*Proof.*Multiply on both sides of (29) by p^{s-1} and apply $\int_0^\infty dp$. Then we have,

$$\begin{aligned}\mathfrak{M}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z):s\right\} &= \int_0^\infty p^{s-1}F_p^{(\alpha,\beta;y)}(a,b;c;z)\,dp \\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \\ &\quad\times\left[\int_0^\infty p^{s-1}\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dp\right]dt\end{aligned} \tag{37}$$

Setting $p=ut(1-t)$ in (37),

$$\begin{aligned}\int_0^\infty p^{s-1}\,{}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dp &= \int_0^\infty u^{s-1}t^s(1-t)^s\,{}_1F_1([\alpha,\beta;y];-u)\,du \\ &= t^s(1-t)^s\int_0^\infty u^{s-1}\,{}_1F_1([\alpha,\beta;y];-u)\,du \\ &= t^s(1-t)^s\Gamma^{(\alpha,\beta;y)}(s).\end{aligned}$$

In this way, we obtain

$$\begin{aligned}\mathfrak{M}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z):s\right\} &= \frac{1}{B(b,c-b)}\int_0^1 t^{b+s-1}(1-t)^{c-b+s-1}(1-zt)^{-a}\Gamma^{(\alpha,\beta;y)}(s)dt \\ &= \frac{\Gamma^{(\alpha,\beta;y)}(s)B(b+s,c-b+s)}{B(b,c-b)} \\ &\quad\times\frac{1}{B(b+s,c-b+s)}\int_0^1 t^{b+s-1}(1-t)^{c+2s-(b+s)-1}(1-zt)^{-a}\,dt \\ &= \frac{\Gamma^{(\alpha,\beta;y)}(s)B(b+s,c-b+s)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z).\end{aligned}$$

*Remark.*Setting $s=1$ in (36), we have

$$\mathfrak{M}\left\{F_p^{(\alpha,\beta;y)}(a,b;c;z):1\right\} = \frac{b(c-b)}{c(c+1)}\Gamma^{(\alpha,\beta;y)}(1)\,{}_2F_1(a,b+1;c+2;z).$$

Theorem 21.*The GICHF has a Mellin transform which can written as follows:*

$$\mathfrak{M}\left\{{}_1F_1^{((\alpha,\beta;y);p)}(b;c;z):s\right\} = \frac{\Gamma^{(\alpha,\beta;y)}(s)B(b+s,c-b+s)}{B(b,c-b)}\,{}_1F_1(b+s;c+2s;z). \tag{38}$$

*Remark.*Setting $s=1$ in (38), we have

$$\mathfrak{M}\left\{{}_1F_1^{((\alpha,\beta;y);p)}(b;c;z):1\right\} = \frac{b(c-b)}{c(c+1)}\Gamma^{(\alpha,\beta;y)}(1)\,{}_1F_1(b+1;c+2;z).$$

3.4 Transformation formulae

Theorem 22.*The GIGHF satisfies the following functional equation:*

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) = (1-z)^{-a}F_p^{(\alpha,\beta;y)}\left(a,c-b;c;\frac{z}{z-1}\right), \quad |\arg(1-z)|<\pi.$$

*Proof.*Using the identity

$$[1-z(1-t)]^{-a} = (1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a}$$

and substituting $1-t$ for t in (29), we obtain

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) = \frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 (1-t)^{b-1}t^{c-b-1}\left(1-\frac{z}{z-1}t\right)^{-a} {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt.$$

Hence,

$$F_p^{(\alpha,\beta;y)}(a,b;c;z) = (1-z)^{-a}F_p^{(\alpha,\beta;y)}\left(a,c-b;c;\frac{z}{z-1}\right).$$

*Remark.*By substituting $1-\frac{1}{z}$ for z in Theorem 22, the following functional equation can be obtained easily

$$F_p^{(\alpha,\beta;y)}\left(a,b;c;1-\frac{1}{z}\right) = z^a F_p^{(\alpha,\beta;y)}(a,c-b;c;1-z),\quad |\arg(z)|<\pi.$$

Furthermore, substituting $\frac{z}{1+z}$ for z in Theorem 22 gives the following functional equation:

$$F_p^{(\alpha,\beta;y)}\left(a,b;c;\frac{z}{1+z}\right) = (1+z)^a F_p^{(\alpha,\beta;y)}(a,c-b;c;-z),\quad |\arg(1+z)<\pi|.$$

Theorem 23.*The GICHF satisfies the following functional equation:*

$${}_1F_1^{((\alpha,\beta;y);p)}(b;c;z) = e^z\, {}_1F_1^{((\alpha,\beta;y);p)}(c-b;c;z).$$

*Remark.*To obtain the following connection between GIGHF and beta functions, setting $z=1$ in (29)

$$\begin{aligned}F_p^{(\alpha,\beta;y)}(a,b;c;1) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}\, {}_1F_1\left([\alpha,\beta;y];\frac{-p}{t(1-t)}\right)dt\\ &= \frac{B_p^{(\alpha,\beta;y)}(b,c-a-b)}{B(b,c-b)}.\end{aligned}$$

3.5 *Fractional calculus formulae*

This subsection identifies extended Riemann-Liouville fractional derivative of the GIGHF showing the generalisation of the incomplete Appell's hypergeometric function. The extended Riemann-Liouville fractional derivative operator is defined by [21]

$$D_z^{\mu,p}\{f(z)\} = \frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt,\ Re(\mu)<0,\ Re(p)>0.$$

It is well known [21] that

$$D_z^{\mu,p}\left\{z^\lambda\right\} = \frac{B_p(\lambda+1,-\mu)}{\Gamma(-\mu)}z^{\lambda-\mu},\ Re(\lambda)>-1,\ Re(\mu)<0. \tag{39}$$

Theorem 24.*For $Re(\mu)>Re(\lambda)>0,\ Re(\rho)>0,\ Re(\nu)>0,\ Re(\gamma)>0;\ \left|\frac{x}{1-z}\right|<1$ and $|x|+|z|<1$, we have*

$$D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\rho}F_p^{\alpha,\beta}\left(\rho,[\nu,\gamma;y],\frac{x}{1-z}\right)\right\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p).$$

*Proof.*Using (39) and (31), we get

$$
\begin{aligned}
& D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\rho}F_p^{\alpha,\beta}\left(\rho,[\nu,\gamma;y],\frac{x}{1-z}\right)\right\} \\
&= D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\rho}\sum_{n=0}^{\infty}(\rho)_n\frac{B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]}{B(\nu,\gamma-\nu)}\frac{\left(\frac{x}{1-z}\right)^n}{n!}\right\} \\
&= \frac{1}{B(\nu,\gamma-\nu)}D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\rho)_n B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]\frac{x^n}{n!}(1-z)^{-\rho-n}\right\} \\
&= \frac{1}{B(\nu,\gamma-\nu)}\sum_{n,m=0}^{\infty}B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]\frac{x^n}{n!}\frac{(\rho)_n(\rho+n)_m}{m!}D_z^{\lambda-\mu,p}\{z^{\lambda-1+m}\} \\
&= \frac{1}{B(\nu,\gamma-\nu)}\sum_{n,m=0}^{\infty}B_p^{\alpha,\beta}[\nu+n,\gamma-\nu;y]\frac{x^n}{n!}\frac{(\rho)_{n+m}}{m!}\frac{B_p(\lambda+m,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu+m-1} \\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_{2,y}^{\alpha,\beta}(\rho,\nu,\lambda;\gamma,\mu;x,z;p).
\end{aligned}
$$

Hence the proof is completed.

4 Conclusion

Very recently, the incomplete Pochhammer ratios [20] were introduced in terms of the incomplete beta functions. With the help of these Pochhammer ratios we have defined the incomplete Gauss, confluent and Appell's hypergeometric functions. We have investigated several properties of them such as integral representation, differentiation formulae, transformation formulae etc. In our present work, the generalisation of incomplete gamma and beta functions are identified by means of the incomplete confluent hypergeometric functions. Also, we obtain integration formulae, Mellin transform, functional and summation relation for these functions. Furthermore, Generalised Incomplete Gauss Hypergeometric Functions (GIGHF), Generalised Incomplete Confluent Hypergeometric Function (GICHF) and Generalised Incomplete Appell's Hypergeometric Function (GIAHF) are defined by means of generalised incomplete beta functions. Additionally, several properties of these functions are given. Finally, we use the extended R-L fractional derivative operator to obtain the images of the generalised incomplete Gauss hypergeometric function.

References

[1] E. Özergin, M. A. Özarslan and A. Altın, Extension of gamma, beta and hypergeometric functions, *J. Comput. Appl. Math.* **235**, 4601-4610 (2011).
[2] M. A. Chaudhry and S. M. Zubair, generalised incomplete gamma functions with applications, *J. Comput. Appl. Math.* **55**, 99–124 (1994).
[3] M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, Extension of Euler's function, *J. Comput. Appl. Math.* **78**, 19–32 (1997).
[4] M. A. Chaudhry and S. M. Zubair, On the decomposition of generalised incomplete gamma functions with applications to Fourier transforms, *J. Comput. Appl. Math.* **59**, 253-284 (1995).
[5] M. A. Chaudhry, N. M. Temme and E. J. M. Velling, Asymptotic and closed form of a generalised incomplete gamma function, *J. Comput. Appl. Math.* **67**, 371–379 (1996).
[6] M. A. Chaudhry and S. M. Zubair, Extended incomplete gamma functions with applications, *J. Math. Anal. Appl.* **274**, 725–745 (2002).
[7] A. R. Miller, Reduction of generalised incomplete gamma function, related Kampe de Feriet functions and incomplete Weber integrals, *Rocky Mountain J. Math.* **3**, 703–714 (2000).
[8] F. Al-Musallam and S. L. Kalla, Further results on a generalised gamma function occurring in a diffraction theory, *Int. Trans. Spec. Funct.* **7**(3-4), 175–190 (1998).
[9] M. A. Chaudhry, A. Qadir, H. M. Srivastava and R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, *Appl. Math. Comput.* **159**, 589–602 (2004).
[10] Shy-Der Lin, H. M. Srivastava and Mu-Ming Wong, Some applications of Srivastava's theorem involving a certain family of generalised and extended hypergeometric polynomials, *Filomat* **29**, 1811-1819 (2015).
[11] R. Srivastava, Some classes of generating functions associated with a certain family of extended and generalised hypergeometric functions, *Appl. Math. Comput.* **243**, 132–137 (2014).

[12] R. Srivastava and Nak Eun Cho, Some extended Pochhammer symbols and their applications involving generalised hypergeometric polynomials, *Appl. Math. Comput.* **234**, 277–285 (2014).
[13] H. M. Srivastava, A. Çetinkaya and O. I. Kıymaz, A certain generalised Pochhammer symbol and its applications to hypergeometric functions, *Appl. Math. Comput.* **226**, 484–491 (2014).
[14] R. Srivastava, Some generalisations of Pochhammer's symbol and their associated families of hypergeometric functions and hypergeometric Polynomials, *Appl. Math. Inf. Sci.* **7**, 2195-2206 (2013).
[15] H. M. Srivastava, M. A. Chaudhry and R. P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, *Int. Trans. Spec. Funct.* **23**, 659-683 (2012).
[16] H. M. Srivastava, R. K. Parmar and P. Chopra, Some families of generalised complete and incomplete elliptic-type integrals, *J. Nonlin. Sci. Appl.* **10**, 1162-1182 (2017).
[17] R. Srivastava, R. Agarwal and S. Jain, A Family of the incomplete hypergeometric functions and associated integral transform and fractional derivative formulae, *Filomat* **3**, 125-140 (2017).
[18] R. Srivastava, Some properties of a family of incomplete hypergeometric functions, *Russian J. Math. Phys.* **20**, 121-128 (2013).
[19] R. Srivastava and N. Eun Cho, Generating functions for a certain class of incomplete hypergeometric polynomials, *Appl. Math. Comput.* **219**, 3219–3225 (2012).
[20] M. A. Özarslan and C. Ustaoglu, Incomplete Riemann-Liouville fractional derivative operators and incomplete hypergeometric functions, Submitted.
[21] M. A. Özarslan and E. Özergin, Some generating relations for extended hypergeometric functions via generalised fractional derivative operator, *Math. Comput.Model.* **52**, 1825-1833 (2010).