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Fractional Modelling and the Leibniz (L-Fractional) derivative as Viscoelastic Respondents in Polymer Biomaterials

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Abstract: Leibniz fractional (L-Fractional) derivative is used to model viscoelastic mechanical systems. Since this derivative has important physical and mathematical meaning, it would be interesting to compare the theoretical with experimental data. Specifically the relaxation behaviour of the Zener fractional viscoelastic model is verified by experiment. The experimental results of the viscoelastic relaxation behaviour in a polymer mesh used for the surgical treatment of female urinary incontinence are used in order to check the applicability of fractional modelling in these systems. Data from relaxation experiments are used in combination with theoretical analysis to prove the Zener-model fractional analysis concept.

Keywords: Leibniz L-fractional derivative, fractional analysis, Zener viscoelastic model, relaxation, experimental data, theoretical results.

1 Introduction

Fractional Calculus is a novel mathematical concept with many applications in physics: particle physics, optics and corrosion, mechanics of materials, electromagnetics, electrochemistry, hydrodynamics, quantum mechanics,rheology, viscoelasticity etc. Especially in mechanics we have many studies that introduce fractional strain, Lazopoulos et. al [1,2] Drapaca et al. [3], Di Paola et al. [4] , Carpinteri et al. [5], Atanackovic et al. [6], Agrawal [7], etc. There were many attempts to introduce fractional calculus into viscoelasticity, especially from Atanackovic et. al. [8] and Mainardi et al. [9]. To be more specific, Bagley and Torvik [10,11] and Koeller [12] introduced fractional calculus in viscoelasticity while Atanackovic [8,13,14,15,16] and Mainardi et al. [9,17,18] have expanded the idea in many variational problems. Of course, many other scientists have applied viscoelasticity in the frame of fractional calculus, such as: Meral et.al [19], Muller et al. [20,21], Sabatier et al. [22], Adolfsson et al.[23]). It is also interesting to underline that many articles concerning fractional viscoelasticity are connected to biomedical applications (Craiem et.al [24], Djorjevic et.al [25], Doehring et.al [26], Magin et. Al [27]). Lazopoulos et al. used the Leibnitz Derivative in their work on viscoelasticity [28]. This mathematical novelty has many advantages along with mathematical and physical meaning: The according differential is defined and, in contrast to other fractional derivatives, physical dimensions are not altered. In this article the viscoelastic behaviour of the Zener model is revisited using Leibniz fractional time derivatives. Comparison of the proposed model to experimental data is discussed. As a model system, a polypropylene-filament mesh is used in the female urinary incontinence treatment. Experimental data from stress-relaxation testing were reported in a previously published work [32]. Furthermore,there is a discussion of the behaviour of the proposed model concerning its relaxation and compared to the existing experimental data. The article starts with a presentation of the Leibnitz derivative, and continues with the derivation of fractional Zener model for relaxation. Finally the model is compared with experimental data.

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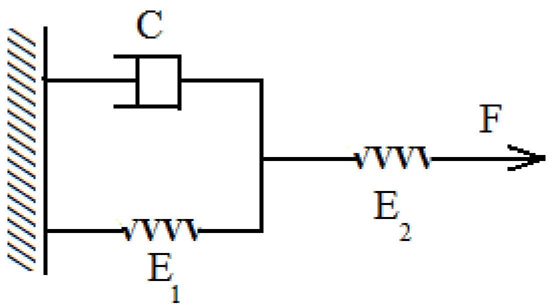

Fig. 1: The Zener model (E_1, E_2, C).

2 Fractional Calculus:the Leibniz(-L)derivative.

Using the Adda [29] definition of a fractional differential and due to Lazopoulos et.al [2,31] the proposed L-fractional (in honour of Leibniz) derivative ${}_a^L D_x^a f(x)$ is defined by :

$$d^{\alpha} f(x) ={}_a^L D_x^a f(x) d^a x \tag{1}$$

where $d^a f(x)$ is the fractional differential of function f(x) and $d^a x$ the fractional differential of x. The Leibniz L-fractional derivative is then defined as the ratio of the corresponding Caputo derivatives (See Lazopoulos et al.[2,31]):

$${}_a^L D_x^a f(x) = \frac{{}_a^c D_x^a f(x)}{{}_a^c D_x^a f(x)} ds \tag{2}$$

It is proven in Lazopoulos et al.[2,31] that among the various derivatives only the L-derivative has any geometrical or physical meaning. Therefore,Leibniz derivative is defined by(Lazopoulos & Lazopoulos [2,30, and 31]):

$${}_a^L D_x^a f(x) = (1-\alpha).(x-a)^{(a-1)} \int_a^x \frac{f'(s)}{(x-s)^a} ds \tag{3}$$

for the left Leibniz Fractional derivative, while for the right Leibniz fractional derivative it is defined by,

$${}_a^L D_x^a f(x) = (1-a).(b-x)^{(a-1)} \int_x^b \frac{f'(s)}{(s-x)^a} ds. \tag{4}$$

3 Theoretical Viscoelastic model.

The Zener model is a structure composed of 2 springs E1 and E2 and a dashpot C, as seen in Fig.1. This model is well described and applied in the international literature and its solution as presented in [32] is:

$$\sigma(t) = \frac{E_1.E_2}{E_1+E_2}.\varepsilon_0.(1-e^{(-\frac{t}{\tau})}) + \sigma_0.e^{(-\frac{t}{\tau})} \tag{5}$$

where E_1 and E_2 are the elastic constraints of the springs, t the time, $\sigma_(t)$ the relaxation stress, ε_0 the initial strain and $\sigma_(0)$ the stress for $t=0$ and τ the time constant for which the following holds:

$$\tau = \frac{C}{E_1+E_2} \tag{6}$$

In Eq. (6) C is the viscosity constant of the dashpot. As far as the fractional Zener model is concerned, Leibnitz L-Fractional model has true physical meaning since it defines a differential and at the same time does not alter physical dimensions. Therefore it is most suited for expressing the L-Fractional Zener model. Following the steps of Lazopoulos et.al in fractional viscoelasticity [28] we have:

$$[1+\frac{C}{E_1+E_2}{}_0^L D_t^a]\sigma_t = \frac{E_2}{E_1+E_2}[E_1+C_0^L D_t^a]\varepsilon(t). \tag{7}$$

As far as the relaxation behaviour of the Fractional Zener Viscoelastic model is concerned, the constant strain $\varepsilon(t) = \varepsilon$ is considered. Then Eq. (7) becomes:

$$C_0^L D_t^a \frac{\sigma(t)}{\varepsilon} + (E_1 + E_2)\frac{\sigma(t)}{\varepsilon} = E_1 E_2 \tag{8}$$

For the relaxation modulus $y(t) = G(t) = \frac{\sigma_t}{\varepsilon}$, the Eq.(8) above takes the form:

$$C_0^L D_t^a y(t) + (E_1 + E_2)y(t) = E_1 E_2. \tag{9}$$

Looking for solution of the type

$$y(t) = \sum_{k=0}^{\infty} y_k t^k, \tag{10}$$

and substituting in Eq.(9), we get,

$$\sum_{k=0}^{\infty} C y_{k+1} \frac{\Gamma(2-a)\Gamma(k+2)}{\Gamma(k+2-a)} t^k + \sum_{k=0}^{\inf} (E_1 + E_2) y_k t^k = E_1 E_2. \tag{11}$$

Since Eq.(11) is valid for any t , it is an identity. Hence,

$$y_1 = -\frac{E_1 + E_2}{C} y_0 + \frac{E_1 E_2}{C}. \tag{12}$$

with

$$y_{k+1} = -(\frac{E_1 + E_2}{C}) \frac{\Gamma(m+2-a)}{\Gamma(2-a)\Gamma(k+2)} y_k, \quad \forall\, k \geq 2. \tag{13}$$

Those relations yield,

$$y_k = (-\frac{E_1 + E_2}{C\Gamma(2-a)})^{k-1} \prod_{m=1}^{k-1} \frac{\Gamma(m+2-a)}{\Gamma(m+2)} y_1, \quad \forall\, k \geq 2. \tag{14}$$

4 Experimental procedures

The biomaterial used as a model for this study is thoroughly examined in stress-relaxation experiments which are reported in full detail in a previous article [32]. Summarizing, a polypropylene- based commercially available mesh, used for the treatment of urinary incontinence in females, was studied in dry isothermal conditions (37^oC) and after immersion in a ringers solution. Experimental isothermal stress-relaxation testing is carried out at a strain level of $\varepsilon_0 = 5\%$. For the purposes of the fractional modeling process experimental stress-relaxation data of one of the specimens stored for 20 days in a ringers solution (Case1) and one of the pristine meshes (Case 2) are used respectively. The results of the fractional modeling are compared against those from the conventional analytical Zeners model equation.

5 Comparison of the theoretical with experimental data

The solution to the theoretical model of relaxation viscoelasticity for Zener model is given by the formula described in Eq. (5) [32]. It is evident from paragraph 3 that the solution has the form:

$$y_t = \sum_{k=0}^{\infty} y_k t^k. \tag{15}$$

where we must find the coefficients y_k. In case of relaxation, where strain remains constant, the following occurs:

$$y_k = (-\frac{E_1 + E_2}{C\Gamma(2-a)})^{k-1} \prod_{m=1}^{k-1} \frac{\Gamma(m+2-a)}{\Gamma(m+2)} y_1 \;\; \forall\, k \geq 2 \;\; and \;\; y_1 = -\frac{(E_1 + E_2)}{C} y_0 + \frac{E_1 E_2}{C}. \tag{16}$$

Table 1: Values of constants of the Zener model in the Ringer Solution case.

$E1_2$(MPa)	E_2(MPa)	C(MPa.min)
1043.38	0.22815	9742.43

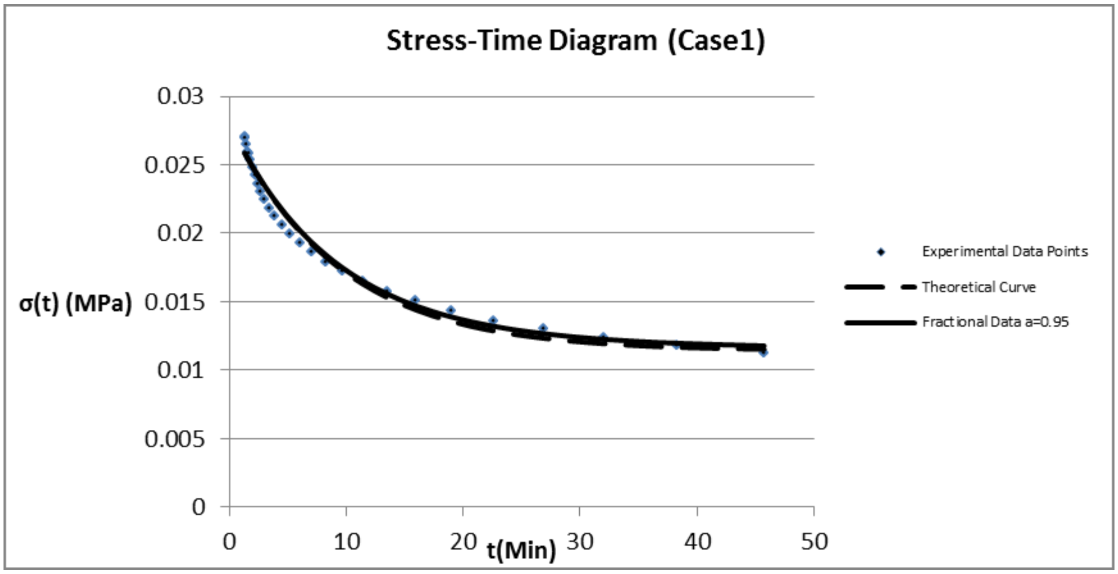

Fig. 2: The stress-time diagram for case 1(Ringer Solution).

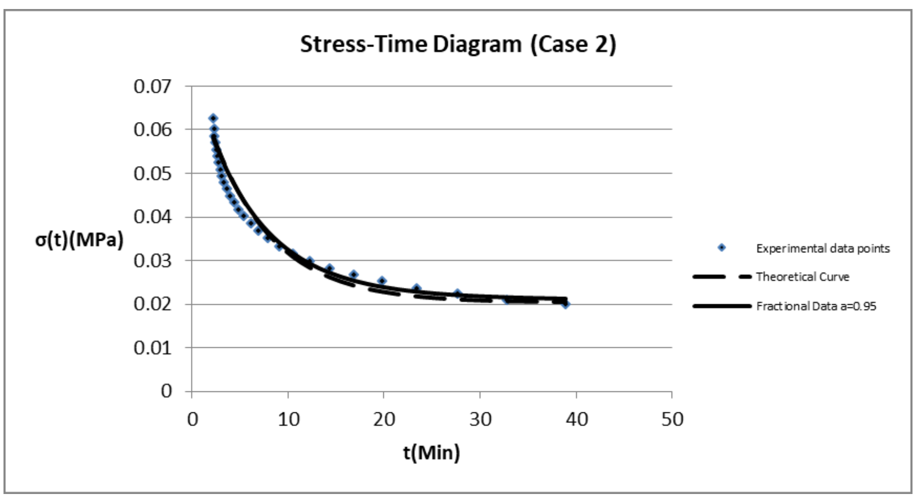

Fig. 3: The stress-time diagram for case 2 (Pristine sample).

With the help of these formulas we are going to examine two cases of relaxation viscoelasticity, as described in paragraph 4. The first case, that of Ringer Solution has the following input data: The initial deformation ε_0 is 5%. The diagram of the stress-time is shown in the figure 2. In this diagram the relaxation stress-time is shown. There are the experimental data points (Scattered points without line), the theoretical curve, which is given by Eq.5, and finally the curve which occurs by the solution of the fractional Zener model. It is obvious from the diagram that the fitting of the fractional data is best: Although the differences are not great, these differences are significant when pictured on the diagram. The diagram shows a completely different picture when the fractional approximation is shown. This proves that fractional analysis gives a much more accurate picture of the phenomenon. On the other hand, while studying case 2 we can conclude that fractional analysis is better. The input data is given from Table 2.

Table 2: Values of constants of the Zener model in the Pristine Solution case.

$E1_2$(MPa)	E_2(MPa)	C(MPa.min)
1235.25	0.40977	7949.81

More specifically, we can see from Figure 3. From this figure we can observe that the fractional data curve gives a much better picture of the phenomenon that the theoretical curve. It is obvious in this second case that fractional analysis has a better performance than theoretical analysis.

6 Conclusions

Our study indicates that fractional analysis in the viscoelastic Zener model is more accurate and effective. The picture that is presented from cases 1 and 2 is so clear that it makes us wonder whether the fractional model is accurate and not the classical model. Is nature best described by fractional derivatives? This is a question that could only be answered by thorough investigation of the phenomena.

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