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Incomplete Fractional Calculus

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Abstract: Here we shall define certain integral operators involving the incomplete hypergeometric function due to Srivastava Chaudhry and Agarwal. The considered generalized fractional integration and differentiation operators contain the incomplete hypergeometric function as a kernel. Some illustrative examples are presented to reveal the effectiveness and conveniences of the method.

Keywords: Incomplete hypergeometric function, incomplete fractional order integral and derivatives, Riemann-Liouville fractional integration operator, Wright function, $\overline{H}$-function, Mittag-Leffler function, Bessel function.

1 Introduction

Various types of fractional operator equation play very important role not only in mathematics but also in physics, control system, dynamical systems and engineering. Recently, it has grabbed the interest of many researchers. There are several authoritative accounts of fractional calculus and fractional differential equations. Over the years many mathematicians, using their own notation and approach, have found various definitions that fit the idea of non-integer order integral or derivative. Numerous papers, including [1–11] as well as [12–15]. The present study aims to introduce fractional calculus as a new tool using incomplete hypergeometric function. Some differential properties, integral representations and special cases are presented.

Incomplete hypergeometric function:

Definitions of these incomplete hypergeometric functions were based on some generalizations of the Pochhammer symbol by means of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$. The incomplete gamma type functions $\gamma(s,x)$ and $\Gamma(s,x)$ [16–19] are certain generalization of classical gamma function $\Gamma(z)$ arising in the solution of physical problems and are also of a great importance in various branches of mathematical analysis.

$$\gamma(s,x) = \int_0^x e^{-t} t^{s-1} dt, \quad s > 0 \tag{1}$$

called the incomplete gamma function. This function most commonly arises in probability theory, particularly the applications involving the Chi-Square distribution. It is customary to be introduced

$$\Gamma(s,x) = \int_x^\infty e^{-t} t^{s-1} dt, \quad s > 0 \tag{2}$$

which is known as the complementary incomplete gamma function. Thus it follows that

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s). \tag{3}$$

Because of the close relationship between these two functions, using $\gamma(s,x)$ and $\Gamma(s,x)$ in practice is simply a matter of convenience.

In 1950, theory of incomplete Gamma functions, as a part of the theory of confluent hypergeometric functions,has received its first systematic exposition by Tricomi [20]. Recently Srivastava et al. ([21], see also [22]) introduced the incomplete

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Pochhammer symbols by means of the incomplete gamma functions (1) and (2), and defined incomplete hypergeometric functions

$$ {}_p\gamma_q \left[\begin{array}{c} (a_1,x),a_2,a_3,...,a_p \\ b_1,b_2,b_3,...,b_q \end{array} |z \right] = \sum_{k=0}^{\infty} \frac{(a_1;x)_k(a_2)_k...(a_p)_k}{(b_1)_k(b_2)_k...(b_q)_k} \frac{z^k}{k!}, \tag{4} $$

$$ {}_p\Gamma_q \left[\begin{array}{c} [a_1,x],a_2,a_3,...,a_p \\ b_1,b_2,b_3,...,b_q \end{array} |z \right] = \sum_{k=0}^{\infty} \frac{[a_1;x]_k(a_2)_k...(a_p)_k}{(b_1)_k(b_2)_k...(b_q)_k} \frac{z^k}{k!} \tag{5} $$

where $(a;x)_k$ and $[a;x]_k$ are incomplete Pochhammer symbols and defined as

$$ (\lambda;x)_\nu = \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda,\nu \in C; x \geq 0) \tag{6} $$

and

$$ [\lambda;x]_\nu = \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda,\nu \in C; x \geq 0), \tag{7} $$

these incomplete pochhammer symbols (6) and (7) satisfy the following decomposition formula

$$ (\lambda;x)_\nu + [\lambda;x]_\nu = (\lambda)_\nu \quad (\lambda,\nu \in C; x \geq 0). \tag{8} $$

$$ |(\lambda;x)_\nu| \leq |(\lambda)_\nu| \quad \text{and} \quad |[\lambda;x]_\nu| \leq |(\lambda)_\nu| \quad (x \geq 0; \lambda,\nu \in C), \tag{9} $$

provided that defining infinite series in each case is absolutely convergent.

Incomplete Fractional Order Integrals and derivatives:

In 1978 Saigo [23] defined the generalized fractional calculus operators associated with Gauss hypergeometric function in the kernel and derived their special cases. These operators serve the study of certain boundary problems arising in applied sciences. Here, we define certain integral operators involving the incomplete hypergeometric function.

Let $\alpha,\beta,\gamma \in C$, and let $x \in \Re_+$, the incomplete fractional integral and incomplete fractional derivative of a function $f(x)$ on $\Re_+$ are defined in the following forms:

Definition 1.

$$ \left({}_\Gamma I_{0,+}^{\alpha,\beta,\gamma} f \right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2\Gamma_1 \left((\alpha+\beta,x), -\gamma; \alpha; 1-\frac{t}{x} \right) f(t)dt, \Re(\alpha) > 0. \tag{10} $$

$$ = \frac{d^n}{dx^n} \left({}_\Gamma I_{0,+}^{\alpha+n,\beta-n,\gamma-n} f \right)(x), 0 < \Re(\alpha)+n \leq 1 (n \in N_0). \tag{11} $$

Definition 2.

$$ \left({}_\Gamma I_{-}^{\alpha,\beta,\gamma} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2\Gamma_1 \left((\alpha+\beta,x), -\gamma; \alpha; 1-\frac{x}{t} \right) f(t)dt, \Re(\alpha) > 0. \tag{12} $$

$$ = (-1)^n \frac{d^n}{dx^n} \left({}_\Gamma I_{-}^{\alpha+n,\beta-n,\gamma} f \right)(x), 0 < \Re(\alpha)+n \leq 1 (n \in N_0). \tag{13} $$

Definition 3.

$$ \left({}_\Gamma D_{0,+}^{\alpha,\beta,\gamma} f \right)(x) = \left({}_\Gamma I_{0,+}^{-\alpha,-\beta,\alpha+\gamma} f \right)(x) $$

$$ = \left(\frac{d}{dx} \right)^n \left({}_\Gamma I_{0,+}^{-\alpha+n,-\beta-n,\alpha+\gamma-n} f \right)(x), \quad \Re(\alpha) > 0; n = [\Re(\alpha)]+1 \tag{14} $$

Definition 4.

$$ \left({}_\Gamma D_{-}^{\alpha,\beta,\gamma} f \right)(x) = \left({}_\Gamma I_{-}^{-\alpha,-\beta,\alpha+\gamma} f \right)(x) $$

$$ = (-1)^n \left(\frac{d}{dx} \right)^n \left({}_\Gamma I_{-}^{-\alpha+n,-\beta-n,\alpha+\gamma} f \right)(x), \quad \Re(\alpha) > 0; n = [\Re(\alpha)]+1. \tag{15} $$

For the justification of equations (11), (13), (14) and (15)

Theorem 1.*If $\alpha,\beta,\gamma \in C$ and let $x \in \Re_+$, incomplete fractional derivative of function $f(x)$ on $\Re_+$ holds the relation*

$$\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}f\right)(x) = \frac{d}{dx}\left({}_{\Gamma}I_{0,x}^{\alpha+1,\beta-1,\gamma-1}f\right)(x) \tag{16}$$

*Proof.*From equation (10)

$$\left({}_{\Gamma}I_{0,x}^{\alpha+1,\beta-1,\gamma-1}f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}\ {}_2\Gamma_1\left((\alpha+\beta,x),-\gamma+1;\alpha+1;1-\frac{t}{x}\right)f(t)dt.$$

$$\frac{d}{dx}(I_{0,x}^{\alpha+1,\beta-1,\gamma-1}f)(x) = \frac{d}{dx}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}\ {}_2\Gamma_1\left((\alpha+\beta,x),-\gamma+1;\alpha+1;1-\frac{t}{x}\right)f(t)dt$$

$$= \frac{1}{\Gamma(\alpha+1)}\int_0^x \frac{\partial}{\partial x}(x^{-\alpha-\beta}(x-t)^{\alpha})\ {}_2\Gamma_1\left((\alpha+\beta,x),-\gamma+1;\alpha+1;1-\frac{t}{x}\right)f(t)dt \tag{17}$$

Now, with formula ([24], p. 156)

$$\frac{d(x^n(1-x)^nMy^{(n)})}{dx} = (\alpha+n-1)(\beta+n-1)x^{n-1}(1-x)^{n-1}My^{(n-1)}, \tag{18}$$

where $M = x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma}$. We can express

$$\frac{d}{dz}\left(z^{\gamma-1}(1-z)^{\alpha-\gamma+1}\ {}_2\Gamma_1((\alpha,x),\beta;\gamma;z)\right) = (\gamma-1)z^{\gamma-2}(1-z)^{\alpha-\gamma}\ {}_2\Gamma_1((\alpha,x),\beta-1;\gamma-1;z). \tag{19}$$

Then equation (17) becomes

$$\frac{d}{dx}({}_{\Gamma}I_{0,x}^{\alpha+1,\beta-1,\gamma-1}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)f(t)dt.$$

$$= \left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}f\right)(x) \tag{20}$$

Theorem 2.*If $\alpha,\beta,\gamma \in C$ and let $x \in \Re_+$, incomplete fractional derivative of function $f(x)$ on $\Re_+$ holds the relation*

$$\left({}_{\Gamma}I_{-}^{\alpha,\beta,\gamma}f\right)(x) = (-1)\frac{d}{dx}\left({}_{\Gamma}I_{-}^{\alpha+1,\beta-1,\gamma}f\right)(x) \tag{21}$$

*Proof.*With equation (12)

$$\frac{d}{dx}({}_{\Gamma}I_{-}^{\alpha+1,\beta-1,\gamma}f)(x) = \frac{d}{dx}\frac{1}{\Gamma(\alpha+1)}\int_x^{\infty}(t-x)^{\alpha}t^{-\alpha-\beta}\ {}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha+1;1-\frac{x}{t}\right)f(t)dt$$

$$= \frac{1}{\Gamma(\alpha+1)}\int_x^{\infty}\frac{\partial}{\partial x}((t-x)^{\alpha}t^{-\alpha-\beta})\ {}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha+1;1-\frac{x}{t}\right)f(t)dt.$$

Using differentiation formula

$$\frac{d^n}{dz^n}[z^{c-1}{}_2\Gamma_1((a,x),b;c;z)] = (c-1)_n z^{c-n-1}{}_2\Gamma_1((a,x),b;c-n;z) \tag{22}$$

We achieve

$$(-1)\frac{d}{dx}\left({}_{\Gamma}I_{-}^{\alpha+1,\beta-1,\gamma}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{x}{t}\right)f(t)dt.$$

$$= \left({}_{\Gamma}I_{-}^{\alpha,\beta,\gamma}f\right)(x). \tag{23}$$

Theorem 3. *If $\alpha,\beta,\gamma \in C$ and let $x \in \Re_+$, incomplete fractional derivative of function $f(x)$ on $\Re_+$ holds the relation*

$$\left({}_\Gamma D_{0,x}^{\alpha,\beta,\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^n \left({}_\Gamma I_{0,x}^{-\alpha+n,-\beta-n,\alpha+\gamma-n} f\right)(x)$$

$$= \left({}_\Gamma I_{0,x}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x). \tag{24}$$

Proof. It is similar to the previous theorem 1.

Theorem 4. *If $\alpha,\beta,\gamma \in C$ and let $x \in \Re_+$, incomplete fractional derivative of function $f(x)$ on $\Re_+$ holds the relation*

$$\left({}_\Gamma D_{-}^{\alpha,\beta,\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^n \left({}_\Gamma I_{-}^{-\alpha+n,-\beta-n,\alpha+\gamma} f\right)(x)$$

$$= \left({}_\Gamma I_{-}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x). \tag{25}$$

Proof. It is similar to the previous theorem 2.

Theorem 5. *If $\alpha,\beta,\gamma,\rho \in C, \Re(\alpha) > 0$, and $\Re(\rho) > max[0, \Re(\beta-\gamma)]$ and condition (9) exist, then*

$$({}_\Gamma I_{0,x}^{\alpha,\beta,\gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\alpha+\rho+\gamma)} x^{\rho-\beta-1} - \frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)} {}_2F_2\left[\begin{matrix}\alpha+\rho+\gamma, \alpha+\beta \\ \alpha+\rho, \alpha+\beta+1\end{matrix}; -x\right]. \tag{26}$$

Proof. With (10) and (26), we get

$$\left({}_\Gamma I_{0,x}^{\alpha,\beta,\gamma} t^{\rho-1}\right) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2\Gamma_1\left((\alpha+\beta,x), -\gamma; \alpha; \left(1-\frac{t}{x}\right)\right) f(t)dt.$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha+\beta;x)_k(-\gamma)_k}{(\alpha)_k k!} \int_0^x (x-t)^{\alpha-1} \left(1-\frac{t}{x}\right)^k t^{\rho-1} dt.$$

Consider $\frac{t}{x} = u$

$$= \frac{x^{\rho-\beta-1}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha+\beta;x)_k(-\gamma)_k}{(\alpha)_k k!} \int_0^1 (1-u)^{\alpha+k-1} u^{\rho-1} du$$

$$= x^{\rho-\beta-1} \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)} {}_2\Gamma_1\left[\begin{matrix}(\alpha+\beta,x), -\gamma \\ \alpha+\rho\end{matrix}; 1\right]. \tag{27}$$

Now, using the special value of incomplete hypergeometric function ([21], p. 667, Equ. (3.20))

$${}_2\Gamma_1\left[\begin{matrix}(a,x), b \\ c\end{matrix}; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - \frac{x^a}{\Gamma(a+1)} {}_2F_2\left[\begin{matrix}c-b, a \\ c, a+1\end{matrix}; -x\right] \tag{28}$$

in (27), we get

$$= x^{\rho-\beta-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\rho+\gamma+\alpha)} - \frac{x^{\rho+\alpha-1}\Gamma(\rho)}{\Gamma(\alpha+\rho)\Gamma(\alpha+\beta+1)} {}_2F_2\left[\begin{matrix}\alpha+\gamma+\rho, \alpha+\beta \\ \alpha+\rho, \alpha+\beta+1\end{matrix}; -x\right]. \tag{29}$$

Which is required result.
Furthermore, with the formula ([18], p. 104)

$$\frac{1}{\Gamma(a)} \int_0^x t^{a-1} {}_1F_1\left[\begin{matrix}c-b \\ c\end{matrix}; -t\right] = \frac{x^a}{\Gamma(a+1)} {}_2F_2\left[\begin{matrix}c-b, a \\ c, a+1\end{matrix}; -x\right]. \tag{30}$$

Equation (26) becomes

$$\left({}_\Gamma I_{0,x}^{\alpha,\beta,\gamma} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\alpha+\rho+\gamma)} x^{\rho-\beta-1} - \frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}$$

$$\times\left[\frac{\Gamma(\alpha+\beta+1)}{x^{\alpha+\beta}}\frac{1}{\Gamma(\alpha+\beta)}\int_0^x t^{\alpha+\beta-1}{}_1F_1\left(\begin{matrix}\alpha+\rho+\gamma\\ \alpha+\rho\end{matrix};-t\right)dt\right]$$

$$\left({}_\Gamma I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\alpha+\rho+\gamma)}x^{\rho-\beta-1}-\frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}$$

$$\times\left[\frac{(\alpha+\beta)\Gamma(\alpha+\beta)}{x^{\alpha+\beta}}\frac{1}{\Gamma(\alpha+\beta)}\int_0^x t^{\alpha+\beta-1}{}_1F_1\left(\begin{matrix}\alpha+\rho+\gamma\\ \alpha+\rho\end{matrix};-t\right)dt\right]$$

$$\left({}_\Gamma I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\alpha+\rho+\gamma)}x^{\rho-\beta-1}-\frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}$$

$$\times\left[\frac{(\alpha+\beta)}{x^{\alpha+\beta}}\int_0^x t^{\alpha+\beta-1}{}_1F_1\left(\begin{matrix}\alpha+\rho+\gamma\\ \alpha+\rho\end{matrix};-t\right)dt\right]. \tag{31}$$

Now, If we set $\beta=-\alpha$ in (31), equation(31) as Riemann-Liouville fractional calculus operator

$$\left({}_\Gamma I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\beta)}{\Gamma(\rho-\beta)\Gamma(\alpha+\rho+\gamma)}x^{\rho-\beta-1}-\frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}$$

$$\times\left[\frac{(\alpha-\alpha)}{x^{\alpha+\beta}}\int_0^x t^{\alpha+\beta-1}{}_1F_1\left(\begin{matrix}\alpha+\rho+\gamma\\ \alpha+\rho\end{matrix};-t\right)dt\right]$$

$$\left({}_\Gamma I_{0,x}^{\alpha,-\alpha,\gamma}t^{\rho-1}\right)(x)=\left(I_{0,x}^{\alpha}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)}x^{\rho+\alpha-1},\quad \Re(\rho)>0. \tag{32}$$

If we take $\beta=0$ in (26), we get

$$({}_\Gamma I_{0,x}^{\alpha,0,\gamma}t^{\rho-1})(x)=\frac{\Gamma(\rho+\gamma)}{\Gamma(\alpha+\rho+\gamma)}x^{\rho-1}-\frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+1)}\frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}{}_2F_2\left[\begin{matrix}\alpha+\rho+\gamma,\alpha\\ \alpha+\rho,\alpha+1\end{matrix};-x\right]$$

$$=x^{\rho-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)}{}_2\Gamma_1\left(\begin{matrix}(\alpha,x),-\gamma\\ \alpha+\rho\end{matrix};1\right)$$

$$=\left({}_\Gamma E_{0,x}^{\alpha,\gamma}t^{\rho-1}\right)(x),\Re(\alpha)>0. \tag{33}$$

Using the same technique, we have

$$({}_\gamma I_{0,x}^{\alpha,0,\gamma}t^{\rho-1})(x)=x^{\rho-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)}{}_2\gamma_1\left(\begin{matrix}(\alpha,x),-\gamma\\ \alpha+\rho\end{matrix};1\right)$$

$$=\left({}_\gamma E_{0,x}^{\alpha,\gamma}t^{\rho-1}\right)(x),\Re(\alpha)>0. \tag{34}$$

Here, classical Erdélyi-Kober operator obtained when the decomposition formula (8) is used with incomplete Erdélyi-Kober operators (33) and (34).

Theorem 6. *If $\alpha,\beta,\gamma,\rho\in C,\Re(\alpha)>0$, and $\Re(\rho)>max[\Re(-\beta),\Re(-\gamma)]$, then*

$$({}_\Gamma I_{-}^{\alpha,\beta,\gamma}t^{\rho-1})(x)=\frac{\Gamma(1+\beta-\rho)\Gamma(1+\gamma-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\beta+\gamma-\rho)}x^{\rho-\beta-1}-\frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(1+\beta-\rho)}{\Gamma(1+\alpha+\beta-\rho)}$$

$$\times{}_2F_2\left[\begin{matrix}1+\alpha+\beta+\gamma-\rho,\alpha+\beta\\ 1+\alpha+\beta-\rho,\alpha+\beta+1\end{matrix};-x\right]. \tag{35}$$

Proof. Using (12) and (35), we get

$$\left({}_\Gamma I_{-}^{\alpha,\beta,\gamma}t^{\rho-1}\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;\left(1-\frac{x}{t}\right)\right)t^{\rho-1}dt.$$

$$=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{(\alpha+\beta;x)_k(-\gamma)_k}{(\alpha)_k k!}\int_x^\infty t^{\rho-\beta-2}\left(1-\frac{x}{t}\right)^{\alpha+k-1}dt.$$

Consider $\frac{x}{t} = u$

$$= \frac{1}{\Gamma(\alpha)} x^{\rho-\beta-1} \sum_{k=0}^{\infty} \frac{(\alpha+\beta;x)_k(-\gamma)_k}{(\alpha)_k k!} \int_0^1 u^{\beta-\rho} (1-u)^{\alpha+k-1} du.$$

$$= x^{\rho-\beta-1} \frac{\Gamma(1-\rho+\beta)}{\Gamma(\alpha+\beta-\rho+1)} {}_2\Gamma_1 \left[\begin{matrix} (\alpha+\beta,x),-\gamma \\ \alpha+\beta-\rho+1 \end{matrix} ;1 \right]. \quad (36)$$

Finally using (28) in (36), we achieve the desired result (35).
If we take $\beta = -\alpha$ in (35), then (30) as Weyl operator

$$\left({}_{\Gamma}I_{-}^{\alpha,-\alpha,\gamma} t^{\rho-1} \right)(x) = I_{-}^{\alpha} = W_{x,\infty} t^{\rho-1} = x^{\rho+\alpha+1} \frac{\Gamma(1-\alpha-\rho)}{\Gamma(1-\rho)}, \quad (37)$$

where $\Re(\alpha) > 0, \Re(1-\rho) < -\Re(\alpha)$.
If we set $\beta = 0$ in (35), we get

$$({}_{\Gamma}I_{-}^{\alpha,0,\gamma} t^{\rho-1})(x) = \frac{\Gamma(1+\gamma-\rho)}{\Gamma(1+\alpha-\rho+\gamma)} x^{\rho-1} - \frac{x^{\alpha+\rho-1}}{\Gamma(\alpha+1)} \frac{\Gamma(1-\rho)}{\Gamma(1+\alpha-\rho)}$$

$$\times {}_2F_2 \left[\begin{matrix} 1+\alpha-\rho+\gamma,\alpha \\ 1+\alpha-\rho,\alpha+1 \end{matrix} ;-x \right]. \quad (38)$$

$$= x^{\rho-1} \frac{\Gamma(1-\rho)}{\Gamma(1+\alpha-\rho)} {}_2\Gamma_1 \left[\begin{matrix} (\alpha,x),-\gamma \\ 1+\alpha-\rho \end{matrix} ;1 \right] \quad (39)$$

$$= \left({}_{\Gamma}K_{x,\infty}^{\alpha,\gamma} t^{\rho-1} \right)(x), \Re(\alpha) > 0. \quad (40)$$

Using the same technique, we have

$$({}_{\gamma}I_{-}^{\alpha,0,\gamma} t^{\rho-1})(x) = x^{\rho-1} \frac{\Gamma(1-\rho)}{\Gamma(1+\alpha-\rho)} {}_2\gamma_1 \left[\begin{matrix} (\alpha,x),-\gamma \\ 1+\alpha-\rho \end{matrix} ;1 \right]$$

$$= \left({}_{\gamma}K_{x,\infty}^{\alpha,\gamma} t^{\rho-1} \right)(x), \Re(\alpha) > 0. \quad (41)$$

Here, classical Erdélyi-Kober operator obtained when the decomposition formula (8) is used with incomplete Erdélyi-Kober operators (40) and (41).

2 Applications

2.1 Wright function

The generalized hypergeometric Wright function introduced by Wright [25, 26] is called the generalized hypergeometric function see ([16], section 4.1) defined by

$${}_p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i+\beta_i k)}{\prod_{j=1}^{q} \Gamma(\alpha_j+\beta_j k)} \frac{z^k}{k!} \quad (42)$$

where

$$\alpha_i, \beta_j \in R = (-\infty,\infty), (\alpha_i, \beta_j \neq 0,\ i = 1,2,...,p\ \&\ j = 1,2,...,q).$$

Here $z \in C$ is the set of complex numbers and $\Gamma(z)$ is Eulers Gamma function ([16], sec 1.1) condition for the existence of equation (42) together with its representation in terms of Mellin Barnes integral and of H-function where established in [27]. In particular Wright hypergeometric function is an entire function

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \quad (43)$$

Theorem 7. *If* $\alpha,\beta,\gamma,\rho\in C,\Re(\alpha)>0,\Re(\rho)>max[0,\Re(\beta-\gamma)]$ *and condition (43) exist, then*

$$({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}{}_p\Psi_q(\mu t^{\sigma}))(x)=x^{\rho-\beta-1}{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a,b)_{1,p},(\rho,\sigma),(\rho-\beta+\gamma,\sigma)\\(a,b)_{1,q},(\rho-\beta,\sigma),(\rho+\alpha+\gamma,\sigma)\end{matrix};\mu x^{\sigma}\right]-\frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_n}{(\alpha+\beta+1)_n}\frac{(-x)^n}{n!}{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(a,b)_{1,p},(\rho,\sigma),(\alpha+\gamma+\rho+n,\sigma)\\(a,b)_{1,q},(\rho+\alpha+n,\sigma),(\rho+\alpha+\gamma,\sigma)\end{matrix};\mu x^{\sigma}\right].\qquad(44)$$

Proof. With (10) and (42)

$$\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}{}_p\Psi_q(\mu t^{\sigma})\right)(x)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{\mu^k}{k!}$$

$$\times\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)t^{\rho+\sigma k-1}dt.$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{\mu^k}{k!}\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho+\sigma k-1}\right)(x).\qquad(45)$$

Now, using(26) in (45), we get

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{\mu^k}{k!}$$

$$\times\left[\frac{\Gamma(\rho+\sigma k)\Gamma(\rho-\beta+\gamma+\sigma k)}{\Gamma(\rho-\beta+\sigma k)\Gamma(\rho+\alpha+\gamma+\sigma k)}x^{\rho+\sigma k-\beta-1}-\frac{x^{\rho+\alpha+\sigma k-1}}{\Gamma(1+\alpha+\beta)}\frac{\Gamma(\rho+\sigma k)}{\Gamma(\rho+\alpha+\sigma k)}{}_2F_2\left(\begin{matrix}\alpha+\gamma+\rho+\sigma k,\alpha+\beta\\\alpha+\rho+\sigma k,\alpha+\beta+1\end{matrix};-x\right)\right]$$

$$=x^{\rho-\beta-1}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{\Gamma(\rho+\sigma k)\Gamma(\rho-\beta+\gamma+\sigma k)}{\Gamma(\rho-\beta+\sigma k)\Gamma(\rho+\alpha+\gamma+\sigma k)}\frac{(\mu x^{\sigma})^k}{k!}-\frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_n}{(\alpha+\beta+1)_n}\frac{(-x)^n}{n!}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+b_ik)}{\prod_{j=1}^{q}\Gamma(a_j+b_jk)}\frac{\Gamma(\rho+\sigma k)\Gamma(\alpha+\gamma+\rho+n+\sigma k)}{\Gamma(\rho+\alpha+\gamma+\sigma k)\Gamma(\rho+\alpha+n+\sigma k)}\frac{(\mu x^{\sigma})^k}{k!}.\qquad(46)$$

Which is a required result.

2.2 Mittag-Leffler function

In the mathematical literature, it is well-known that the Mittag-Leffler function [28] due to Swedish mathematician Gosta Mittag-Leffler [29] in 1903

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\qquad(47)$$

where z is a complex variable defined as the Mittag-Leffler function of order alpha. The Mittag Leffler function is a direct generalization of the exponential function $e^z=\frac{z^n}{\Gamma(n+1)}$ and admits a first generalization given by two parameter Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},(\alpha,\beta\in C;\Re(\alpha)>0,\Re(\beta)>0),\qquad(48)$$

which is known as Wiman's function or generalized Mittag-Leffler function [30] as $E_{\alpha,1}(z)=E_{\alpha}(z)$.
In 1971, Prabhakar [31] introduced the Mittag-Leffler type function $E_{\alpha,\beta}^{\gamma}(z)$ defined by

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha k+\beta)}\frac{z^n}{n!}\qquad(49)$$

where α,β and γ are complex number, $\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0$.

Theorem 8. *If $\alpha,\beta,\gamma,\rho \in C, \Re(\alpha)>0\ \Re(\rho)>max[0,\Re(\beta-\gamma)]$ and condition (43) exist, then*

$$({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}E_{a,b}^{c}(\mu t^{\sigma}))(x)=\frac{x^{\rho-\beta-1}}{\Gamma(c)}{}_3\Psi_3\left[\begin{matrix}(c,1),(\rho,\sigma)(\rho-\beta+\gamma,\sigma)\\(b,a),(\rho-\beta,\sigma),(\rho+\alpha+\gamma,\sigma)\end{matrix}\mu x^{\sigma}\right]-\frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}\frac{1}{\Gamma(c)}$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_n}{(\alpha+\beta+1)_n}\frac{(-x)^n}{n!}{}_3\Psi_3\left[\begin{matrix}(c,1),(\rho,\sigma),(\rho+\alpha+\gamma+n,\sigma)\\(b,a),(\rho+\alpha+\gamma,\sigma),(\rho+\alpha+n,\sigma)\end{matrix};\mu x^{\sigma}\right] \tag{50}$$

Proof. Using (10) and (50), we have

$$\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}E_{a,b}^{c}(\mu t^{\sigma})\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma,\alpha;1-\frac{t}{x}\right)t^{\rho-1}E_{a,b}^{c}(\mu t^{\sigma})dt.$$

$$=\sum_{k=0}^{\infty}\frac{(c)_k}{\Gamma(ak+b)}\frac{\mu^k}{k!}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma,\alpha;1-\frac{t}{x}\right)t^{\rho+\sigma k-1}dt.$$

$$=\sum_{k=0}^{\infty}\frac{(c)_k}{\Gamma(ak+b)}\frac{\mu^k}{k!}\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho+\sigma k-1}\right)(x) \tag{51}$$

Finally, using (26) in (51) we arrive at

$$=\frac{x^{\rho-\beta-1}}{\Gamma(c)}\sum_{k=0}^{\infty}\frac{\Gamma(c+k)\Gamma(\rho+\sigma k)\Gamma(\rho-\beta+\gamma+\sigma k)}{\Gamma(ak+b)\Gamma(\rho-\beta+\sigma k)\Gamma(\rho+\gamma+\alpha+\sigma k)}\frac{(\mu x^{\sigma})^k}{k!}-\frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}\frac{1}{\Gamma(c)}$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_n}{(\alpha+\beta+1)_n}\frac{(-x)^n}{n!}\sum_{k=0}^{\infty}\frac{\Gamma(c+k)\Gamma(\rho+\sigma k)\Gamma(\rho+\alpha+\gamma+n+\sigma k)}{\Gamma(ak+b)\Gamma(\rho+\alpha+\gamma+\sigma k)\Gamma(\rho+\alpha+n+\sigma k)}\frac{(\mu x^{\sigma})^k}{k!}. \tag{52}$$

Which is a required result.

2.3 $\overline{H}$-function

The $\overline{H}$-function introduced by Inayat-Hussain [32] in terms of Mellin Barnes type contour integral is defined by

$$\overline{H}(z)=\overline{H}_{p,q}^{m,n}=\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(\alpha_j,A_j;a_j)_{1,n},(\alpha_j,A_j)_{n+1,p}\\(\beta_j,B_j)_{1,m},(\beta_j,B_j;b_j)_{m+1,q}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\chi(s)z^s ds, \tag{53}$$

where

$$\chi(s)=\frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_js)\prod_{j=1}^{n}\{\Gamma(1-\alpha_j+A_js)\}^{a_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-\beta_j+B_js)\}^{b_j}\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_js)} \tag{54}$$

which contains fractional powers of some of the gamma functions $L=L_{i\infty}$ is a contour starting at the point $\tau-i\infty$, terminating at the point $\tau+i\infty$ with $\tau\in\Re=(-\infty,\infty)$. Here, z may be real or complex but unequal to zero and an empty product is interpreted as unity; m, n, p, q are integers such that $1\le m\le q, 0\le n\le p; A_j>0\,(j=1,\ldots,p), B_j>0\,(j=1,\ldots,q)$ and $a_j\,(j=1,\ldots,p)$ and $b_j\,(j=1,\ldots,q)$ are the complex numbers. The exponents $\alpha_j(j=1,2,\ldots,n)$ and $\beta_j(j=m+1,\ldots,q)$ take on non integer values.

Moreover, from Inayat-Hussain [32], it follows that

$$\overline{H}_{p,q}^{m,n}[z]=o\left(|z|^{\zeta^*}\right)\text{ for small z, where}\zeta^*=\min_{1\le j\le m}\left[\Re\left(\frac{b_j}{B_j}\right)\right] \tag{55}$$

and

$$\overline{H}_{p,q}^{m,n}[z]=o\left(|z|^{\xi^*}\right)\text{ for large z, where}\xi^*=\max_{1\le j\le n}\Re\left[\alpha_j\left(\frac{a_j-1}{A_j}\right)\right]. \tag{56}$$

When the exponents $\alpha_j = \beta_j = 1 \forall\, i$ and j, the $\overline{H}$-function reduced to the familiar Fox's H-function defined by Fox [33], and see also [34, 35].
Buschman and Srivastava ([36], p. 4708) have shown that the sufficient condition for absolute convergence of the contour integral (53) is given by

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{p} |A_j| > 0. \tag{57}$$

This condition evidently provides exponential decay of the integer in (53), and the region of absolute convergence in (53) is

$$|\arg z| < \frac{1}{2}\pi\Omega. \tag{58}$$

Theorem 9. *If* $\alpha,\beta,\gamma,\rho \in C, \Re(\alpha) > 0$ *and* $\Re(\rho) > max[0, \Re(\beta-\gamma)]$ *and condition (58) exist, then*

$$\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho-1}\overline{H}_{p,q}^{m,n}(\mu t^{\sigma})\right)\right)(x)$$

$$= x^{\rho-\beta-1}\overline{H}_{p,q+2}^{m,n+2}\left[\begin{matrix}(a_i,A_i,r_i)_{1,n},(a_i,A_i)_{n+1,p},(1-\rho,\sigma,1),(1-\rho+\beta-\gamma,\sigma,1)\\(b_j,\beta_j)_{1,m},(b_j,B_j,h_j)_{m+1,q},(1-\rho+\beta,\sigma,1)(1-\rho-\alpha-\gamma,\sigma,1)\end{matrix};\mu x^{\sigma}\right] - \frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}$$

$$\times \sum_{k=0}^{\infty}\frac{(\alpha+\beta)_k}{(\alpha+\beta+1)_k}\frac{(-x)^k}{k!}\overline{H}_{p,q+2}^{m,n+2}\left[\begin{matrix}(a_i,A_i,r_i)_{1,n},(a_i,A_i)_{n+1.p},(1-\rho,\sigma,1),(1-\rho-\alpha-\gamma-k,\sigma,1)\\(b_j,B_j)_{1,m},(b_j,B_j,h_j)_{m+1,q},(1-\rho-\alpha-\gamma,\sigma,1),(1-\rho-\alpha-k,\sigma,1)\end{matrix};\mu x^{\sigma}\right]. \tag{59}$$

Proof. Using (10) and (53), we have

$$\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho-1}\overline{H}_{p,q}^{m,n}(\mu t^{\sigma})\right)\right) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)t^{\rho-1}\overline{H}_{p,q}^{m,n}(\mu t^{\sigma})dt.$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)t^{\rho-1}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\chi(s)(\mu t^{\sigma})^s ds\right)dt.$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\chi(s)\mu^s\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho+\sigma s-1}\right)(x)ds. \tag{60}$$

Using (26) in (60), we get

$$= x^{\rho-\beta-1}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\chi(s)\frac{\Gamma(\rho+\sigma s)\Gamma(\rho-\beta+\gamma+\sigma s)}{\Gamma(\rho-\beta+\sigma s)\Gamma(\rho+\gamma+\alpha+\sigma s)}(\mu x^{\sigma})^s ds - \frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}$$

$$\times \sum_{k=0}^{\infty}\frac{(\alpha+\beta)_k}{(\alpha+\beta+1)_k}\frac{(-x)^k}{k!}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\chi(s)\frac{\Gamma(\rho+\sigma s)\Gamma(\rho+\alpha+\gamma+k+\sigma s)}{\Gamma(\rho+\alpha+\gamma+\sigma s)\Gamma(\rho+\alpha+k+\sigma s)}(\mu x^{\sigma})^s ds. \tag{61}$$

Which is a required result.

2.4 Bessel function

Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. They arise in the study of free vibrations of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur in electromagnetic theory and other areas of physics and engineering. Bessel function is denoted by $J_n(x)$ and defined by [18, 19]

$$J_n(x) = \frac{(\frac{x}{2})^n}{\Gamma(1+n)}{}_0F_1\left(-;1+n;\frac{-x^2}{4}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+n}}{2^{2k+n}k!\Gamma(1+n+k)}, \qquad -\infty < x < \infty. \tag{62}$$

Theorem 10. *If $\alpha,\beta,\gamma,\rho \in C, \Re(\alpha)>0$, $\Re(\rho)>max[0,\Re(\beta-\gamma)]$ and condition (43) exist, then*

$$\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}\left(t^{\rho-1}J_n(t)\right)\right)(x)=\frac{x^{\rho+n-\beta-1}}{(2)^n}{}_2\Psi_3\left[\begin{matrix}(\rho+n,2),(\rho+\gamma-\beta+n,2)\\(1+n,1),(\rho-\beta+n,2),(\rho+\gamma+\alpha+n,2)\end{matrix};\frac{-x^2}{4}\right]$$

$$-\left(\frac{x}{2}\right)^n\frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_m}{(\alpha+\beta+1)_m}\frac{(-x)^m}{m!}{}_2\Psi_3\left[\begin{matrix}(\rho+n,2),(\rho+\alpha+\gamma+n+m,2)\\(1+n,1),(\rho+\alpha+\gamma+n,2),(\rho+\alpha+n+m,2)\end{matrix};\frac{-x^2}{4}\right]. \quad (63)$$

Proof. With (10) and (62), we have

$$\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho-1}J_n(t)\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)t^{\rho-1}J_n(t)dt.$$

$$=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)t^{\rho-1}\frac{(\frac{t}{2})^n}{\Gamma(1+n)}{}_0F_1\left(-;1+n;\frac{-t^2}{4}\right)\,dt.$$

$$=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)t^{\rho-1}\frac{(\frac{t}{2})^n}{\Gamma(1+n)}\sum_{k=0}^{\infty}\frac{1}{(1+n)_k}\left(\frac{-t^2}{4}\right)^k\frac{1}{k!}dt.$$

$$=\left(\frac{1}{2}\right)^n\sum_{k=0}^{\infty}\frac{1}{\Gamma(1+n+k)}\left(\frac{-1}{4}\right)^k\frac{1}{k!}\left({}_{\Gamma}I_{0,x}^{\alpha,\beta,\gamma}t^{\rho+n+2k-1}\right)(x) \quad (64)$$

Using (26) in (64), we get

$$=x^{\rho+n-\beta-1}\left(\frac{1}{2}\right)^n\sum_{k=0}^{\infty}\frac{\Gamma(\rho+n+2k)\Gamma(\rho+\gamma-\beta+n+2k)}{\Gamma(1+n+k)\Gamma(\rho-\beta+n+2k)\Gamma(\rho+\gamma+\alpha+n+2k)}\left(\frac{-x^2}{4}\right)^k\frac{1}{k!}$$

$$-\left(\frac{x}{2}\right)^n\frac{x^{\rho+\alpha-1}}{\Gamma(\alpha+\beta+1)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_m}{(\alpha+\beta+1)_m}\frac{(-x)^m}{m!}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\rho+n+2k)\Gamma(\rho+\alpha+\gamma+n+m+2k)}{\Gamma(1+n+k)\Gamma(\rho+\alpha+\gamma+n+2k)\Gamma(\rho+\alpha+n+m+2k)}\left(\frac{-x^2}{4}\right)^k\frac{1}{k!}. \quad (65)$$

Which is a required result.

2.5 Millen transform of incomplete fractional calculus operator

The Mellin transform of a function $f(x)$, denoted by $f^*(s)$, is defined by [16]

$$f^*(s)=m[f(x);s]=\int_0^{\infty}x^{s-1}f(x)ds,x>0. \quad (66)$$

Theorem 11. *If $\Re(\alpha)>0$ and $\Re(s)<1+min[0,\Re(\eta-\beta)]$, the following formula holds for $f(x)\in L_p(0,\infty)$ with $1\le p\le 2$ or $f(x)\in M_p(0,\infty)$ with $p>2$,*

$$m\left[x_{\Gamma}^{\beta}I_{0,x}^{\alpha,\beta,\gamma}f\right](x)=\frac{\Gamma(1-s)\Gamma(\gamma-\beta+1-s)}{\Gamma(1-s-\beta)\Gamma(1+\alpha+\gamma-s)}m[f(x)]-\frac{\Gamma(1-s)}{\Gamma(\alpha+\beta+1)\Gamma(1+\alpha-s)}$$

$$\times{}_2F_2\left[\begin{matrix}1+\alpha-s+\gamma,\alpha+\beta\\1+\alpha-s,\alpha+\beta+1\end{matrix};-x\right]m\left[x^{\alpha+\beta}f(x)\right] \quad (67)$$

*Proof.*Using (10) and (66) in (67)

$$m\left[x_{\Gamma}^{\beta}I_{0,x}^{\alpha,\beta,\gamma}f\right](x)=\int_0^{\infty}x^{s-1}\left(x^{\beta}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^{x}(x-t)^{\alpha-1}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)f(t)dt\right)dx$$

$$=\int_0^{\infty}f(t)dt\left(\frac{1}{\Gamma(\alpha)}\int_t^{\infty}(x-t)^{\alpha-1}x^{-\alpha-\beta}{}_2\Gamma_1\left((\alpha+\beta,x),-\gamma;\alpha;1-\frac{t}{x}\right)x^{\beta+s-1}\right)dx.$$

$$=\int_0^{\infty}f(t)dt\left({}_{\Gamma}I_{-}^{\alpha,\beta,\gamma}x^{\beta+s-1}\right)(t) \tag{68}$$

Using (35) in (68), we arrive at

$$=\frac{\Gamma(1-s)\Gamma(1+\gamma-\beta-s)}{\Gamma(1-\beta-s)\Gamma(1+\alpha+\gamma-s)}\int_0^{\infty}t^{s-1}f(t)dt-\frac{\Gamma(1-s)}{\Gamma(\alpha+\beta+1)\Gamma(1+\alpha-s)}$$

$$\times{}_2F_2\left[\begin{matrix}1+\alpha+\gamma-s,\alpha+\beta\\1+\alpha-s,\alpha+\beta+1\end{matrix};-t\right].$$

$$\int_0^{\infty}t^{\alpha+\beta+s-1}f(t)dt. \tag{69}$$

Which is a required result.

Conflict of Interest

The authors declare that they have no conflict of interest.

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