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On the Spectrum of Difference Operator Δ_{ab} Over the Sequence Spaces ℓ_p and bv_p, $(1 < p < \infty)$

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Abstract: The main purpose of the present paper is to determine the spectrum of the difference operator Δ_{ab} over the sequence spaces ℓ_p and $bv_p, (1 < p < \infty)$. For any two sequences $a = (a_k)$ and $b = (b_k)$ of distinct, non zero real numbers and satisfying certain conditions, the difference operator Δ_{ab} is defined by $(\Delta_{ab}x)_k = a_k x_k + b_{k-1}x_{k-1}$, where $(x_k) \in \ell_p$ or bv_p and $x_{-1} = 0$. Finally, we obtain the spectrum, point spectrum, residual spectrum and the continuous spectrum of the difference operator Δ_{ab} over ℓ_p and bv_p.

Keywords: Difference operator Δ_{ab}; Spectrum of an operator; Sequence spaces ℓ_p and bv_p

1 Introduction

As it is well known, the spectrum of an operator generalizes the notion of eigen values of the corresponding matrix so, the study of spectrum and fine spectrum for various operators plays a significant role in the field of analysis. Several authors have contributed a lot to spectrum and fine spectrum of bounded linear operators over different basic sequences. For example: The fine spectrum of the Cesàro operator on the sequence space ℓ_p for $1 < p < \infty$ has been studied by Gonzalez [1]. The fine spectrum of the integer power of the Cesàro operator over c was examined by Wenger [2] and then Rhoades [3] generalized this result to the weighted mean method. Reade [4] studied the spectrum of the Cesàro operator over the sequence space c_0. Okutoyi [5] computed the spectrum of the Cesàro operator over the sequence space bv. The fine spectra of the Cesàro operator over the sequence spaces c_0 and bv_p have been determined by Akhmedov and Başar [6,7]. Akhmedov and Başar [8,9] have studied the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p and bv_p where $1 < p < \infty$. Altay and Başar [10,11] have determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, for $0 < p < 1$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv was investigated by Kayaduman and Furkan [12]. Srivastava and Kumar [13, 14] have examined the fine spectrum of the generalized difference operator Δ_ν over the sequence spaces c_0 and ℓ_1. Akhmedov and Shabrawy [15] determined the fine spectrum of the operator $\Delta_{a,b}$ over the the sequence space c. Recently, Panigrahi and Srivastava [16] and Dutta and Baliarsingh [17] have studied the spectrum and fine spectrum of second order difference operator Δ^2_{uv} and Δ^2 on the sequence space c_0, respectively. In a generalization to most of the difference operators, the fine spectrum of the generalized difference operator Δ^r_ν, $r \in \mathbb{N}$ over sequence space ℓ_1, c_0 and ℓ_p have been studied by Dutta and Baliarsingh [18,19,21], respectively. Quite recently, certain linear bounded operators via difference sequence spaces of different order have been introduced. Also, their inverses, topological properties, duals, matrix transformations and spectral characterizations have been studied in detail (see [23-31]).

Let ω be the set of all sequences of real or complex numbers. Any subspace of ω is called a sequence space and by ℓ_∞, c and c_0, we denote the spaces of all bounded, convergent and null sequences, respectively. These are Banach spaces with the sup norm

$$\|x\| = \sup_k |x_k|.$$

Also, by ℓ_1, ℓ_p and bv_p, we denote the spaces of all absolutely summable, $p-$summable and $p-$bounded variation series, respectively.

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The sequence spaces l_p and bv_p are defined by

$$\ell_p = \left\{ x \in \omega : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\},$$
$$bv_p = \left\{ x \in \omega : \sum_{k=1}^{\infty} |x_k - x_{k-1}|^p < \infty \right\}.$$

Let $a = (a_k)$ and $b = (b_k)$ be two sequences of non zero distinct real numbers satisfying

(i) $\lim_{k\to\infty} a_k = \alpha$,
(ii) $\lim_{k\to\infty} b_k = \beta \neq 0$,
(iii) $|\alpha - a_k| < |\beta|$ for each $k \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, the set of non negative integers.

We define the difference operator $\Delta_{ab} : \ell_p \to \ell_p$ by $\Delta_{ab}(x) = (\Delta_{ab}x)_k$, where

$$(\Delta_{ab}x)_k = a_k x_k + b_{k-1}x_{k-1}, \tag{1}$$

with $x_{-1} = 0$ where $x \in \ell_p$ and $k \in \mathbb{N}_0$. Throughout we use the convention that any term with a negative subscript is equal to zero. It is easy to verify that the operator Δ_{ab} can be represented by the matrix (a_{nk}) for all $n,k \in \mathbb{N}_0$ where

$$a_{nk} = \begin{cases} a_k, & k = n, \\ b_k, & k = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$\Delta_{ab} = (a_{nk}) = \begin{pmatrix} a_0 & 0 & 0 & 0 & \ldots \\ b_0 & a_1 & 0 & 0 & \ldots \\ 0 & b_1 & a_2 & 0 & \ldots \\ 0 & 0 & b_2 & a_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, we have the following generalizations:

(i) If $a_k = r$ and $b_k = s \neq 0$ for all $k \in \mathbb{N}_0$, then Δ_{ab} generalizes the difference operator $B(r,s)$ considered by Altay and Başar [20] and Furkan et al. [22], i.e.,

$$B(r,s) = \begin{pmatrix} r & 0 & 0 & 0 & \ldots \\ s & r & 0 & 0 & \ldots \\ 0 & s & r & 0 & \ldots \\ 0 & 0 & s & r & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(ii) If $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}_0$, then Δ_{ab} generalizes the difference operator Δ, considered by Altay and Başar [10,11], i.e.,

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & \ldots \\ 0 & -1 & 1 & 0 & \ldots \\ 0 & 0 & -1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(iii) If $a_k = v_k$, $b_k = v_{k-1}$ and $(a_k),(b_k)$ are either constant or strictly decreasing sequences, then Δ_{ab} generalizes the difference operator Δ_v, considered by Srivastava and Kumar [13,14], where

$$\Delta_v = \begin{pmatrix} v_0 & 0 & 0 & 0 & \ldots \\ -v_0 & v_1 & 0 & 0 & \ldots \\ 0 & -v_1 & v_2 & 0 & \ldots \\ 0 & 0 & -v_2 & v_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

2 Preliminaries and definitions

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $\mathscr{R}(T)$, we denote the range of T, i.e,

$$\mathscr{R}(T) = \{y \in Y : y = Tx\,;\, x \in X\}.$$

By $B(X)$, we denote the set all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \{\mathbf{0}\}$ be a normed linear space over the complex field and $T : D(T) \to X$ be a linear operator, where $D(T)$ denotes the domain of T. With T, for a complex number λ, we associate an operator $T_\lambda = (T - \lambda I)$, where I is called identity operator on $D(T)$ and if T_λ has an inverse, we denote it by T_λ^{-1} i.e.

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and is called the *resolvent* operator of T. Many properties of T_λ and T_λ^{-1} depend on λ and the spectral theory is concerned with those properties. We are interested in the set of all λ in the complex plane such that T_λ^{-1} exists/ T_λ^{-1} is bounded/ domain of T_λ^{-1} is dense in X. For our investigation, we need some basic concepts in spectral theory which are given as some definitions and lemmas.

Definition 1.([32], pp. 371) *Let X and T be defined as above. A regular value of T is a complex number λ such that*

(R1) T_λ^{-1} exists;
(R2) T_λ^{-1} is bounded;
(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent* set $\rho(T,X)$ of T is the set of all *regular* values of T. Its complement $\sigma(T,X) = \mathbb{C} \setminus \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\rho(T,X)$ is partitioned into three disjoint sets as follows.

(I) ***Point spectrum*** $\sigma_p(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) does not hold. The elements of $\sigma_p(T,X)$ are called eigen values of T.

(II)***Continuous spectrum*** $\sigma_c(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds and satisfies (R3) but does not satisfy (R2).

(III)***Residual spectrum*** $\sigma_r(T,X)$**:** It is the set of all $\lambda \in \mathbb{C}$ such that (R1) holds but does not satisfy (R3). The condition (R2) may or may not hold .

Lemma 1.([33], pp. 59) *A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 2.([33], pp. 60) *The adjoint operator T^* is onto if and only if T has a bounded inverse.*

Let P,Q be two nonempty subsets of the space w of all real or complex sequences and $A=(a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n,k \in \mathbb{N}_0$. For every $x=(x_k) \in P$ and every positive integer n, we write

$$A_n(x)=\sum_k a_{nk}x_k.$$

The sequence $Ax=(A_n(x))$, if it exists, is called the transformation of x by the matrix A. Infinite matrix $A \in (P,Q)$ if and only if $Ax \in Q$ whenever $x \in P$.

Lemma 3.([34], pp. 253) *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 4.([34], pp. 245) *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 5.([34], pp. 254) *Let $1<p<\infty$ and $A \in (\ell_\infty,\ell_\infty)\cap(\ell_1,\ell_1)$, then $A \in (\ell_p,\ell_p)$.*

The basis of the space bv_p is also constructed and given by the following lemma:

Lemma 6.([9]) *Define the sequence $b^{(k)}=\{b_n^{(k)}\}_{n\in\mathbb{N}}$ of the elements of the space bv_p, for every fixed $k \in \mathbb{N}$, the set of positive integers, by*

$$b_n^{(k)}=\begin{cases}0, & (n<k)\\ 1, & (n\geq k)\end{cases}$$

Then, the sequence $b^{(k)}=\{b_n^{(k)}\}_{n\in\mathbb{N}}$ is a basis for the space bv_p and $x \in bv_p$ has a unique representation of the form

$$x=\sum_{k=1}^{\infty}\lambda_k b^{(k)}, \quad \text{where } \lambda_k=x_k-x_{k-1} \text{ for all } k\in\mathbb{N}.$$

3 The spectrum of the difference operator Δ_{ab} over the sequence space $\ell_p,(1<p<\infty)$

In this section, we compute the point spectrum, spectrum, the continuous spectrum and the residual spectrum of the operator Δ_{ab} on the sequence space ℓ_p.

Theorem 1.*The operator $\Delta_{ab}:\ell_p\to\ell_p$ is a linear operator satisfying the condition*

$$(|a_k|^p+|b_k|^p)^{1/p}\leq\|\Delta_{ab}\|_{(\ell_p:\ell_p)}\leq\sup_k(|a_k|+|b_k|).$$

*Proof.*Linearty of Δ_{ab} is trivial. Suppose we denote $e_k=(0,0,\ldots 0,1,0,\ldots)$ as a sequence whose k-th entry is 1 and otherwise 0. Clearly, $e_k\in\ell_p$ and

$$\begin{aligned}\|\Delta_{ab}(e_k)\|_{(\ell_p)}&=(|a_k|^p+|b_k|^p)^{1/p}\\&\leq\|\Delta_{ab}\|_{(\ell_p)}\|e_k\|_{\ell_p}\end{aligned}$$

Thus,

$$(|a_k|^p+|b_k|^p)^{1/p}\leq\|\Delta_{ab}\|_{(\ell_p)} \tag{2}$$

Let $x=(x_k)\in\ell_p$ and $1<p<\infty$ such that $\|x\|=1$. Now,

$$\begin{aligned}\|\Delta_{ab}(x)\|_{(\ell_p)}&=\left(\sum_{k=0}^{\infty}|a_kx_k+b_{k-1}x_{k-1}|^p\right)^{1/p}\\&\leq\left(\sum_{k=0}^{\infty}|a_kx_k|^p\right)^{1/p}+\left(\sum_{k=0}^{\infty}|b_{k-1}x_{k-1}|^p\right)^{1/p}\\&\leq\sup_k|a_k|\|x\|_{\ell_p}+\sup_k|b_k|\|x\|_{\ell_p}=\sup_k(|a_k|+|b_k|)\|x\|_{\ell_p}.\end{aligned}$$

Thus,

$$\|\Delta_{ab}\|_{(\ell_p)}\leq\sup_k(|a_k|+|b_k|). \tag{3}$$

Combining inequations (2) and (3) we complete the proof.

Theorem 2.*The spectrum of Δ_{ab} on the sequence space ℓ_p is given by*

$$\sigma(\Delta_{ab},\ell_p)=\Big\{\lambda\in\mathbb{C}:|\alpha-\lambda|\leq|\beta|\Big\}.$$

*Proof.*The proof of this theorem is divided into two parts.
Part 1: In the first part , we have to show that

$$\sigma(\Delta_{ab},\ell_p)\subseteq\Big\{\lambda\in\mathbb{C}:|\alpha-\lambda|\leq|\beta|\Big\}.$$

Equivalently, we need to show that if $\lambda\in\mathbb{C}$ with $|\alpha-\lambda|>|\beta|\Rightarrow\lambda\notin\sigma(\Delta_{ab},\ell_p)$. Let $\lambda\in\mathbb{C}$ with $|\alpha-\lambda|>|\beta|$. Now, solving the system of linear equations

$$\left.\begin{aligned}(a_0-\lambda)x_0&=y_0\\ b_0x_0+(a_1-\lambda)x_1&=y_1\\ b_1x_1+(a_2-\lambda)x_2&=y_2\\ &\cdots\cdots\\ b_{k-1}x_{k-1}+(a_k-\lambda)x_k&=y_k\\ &\cdots\cdots\end{aligned}\right\} \tag{4}$$

we obtain

$$\begin{aligned} x_0 &= \frac{y_0}{(a_0-\lambda)},\\ x_1 &= \frac{y_1}{(a_1-\lambda)} - \frac{b_0y_0}{(a_0-\lambda)(a_1-\lambda)},\\ \text{Similarly, } x_2 &= \frac{y_2}{(a_2-\lambda)} - \frac{b_1y_1}{(a_2-\lambda)(a_1-\lambda)}\\ &+ \frac{b_0b_1y_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)}. \end{aligned}$$

In fact, for $k \in \mathbb{N}_0$, we have

$$x_k = \frac{y_k}{(a_k-\lambda)} + \sum_{i=1}^{k}(-1)^i \left(\frac{\prod_{j=0}^{i-1} b_{k-(j+1)}}{\prod_{j=0}^{i}(a_{k-j}-\lambda)} \right) y_{k-i}.$$

Now, we obtain

$$\begin{aligned} \lim_{k\to\infty}|x_k| &\le \left| \frac{1}{(\alpha-\lambda)} + \sum_{i=1}^{\infty}\left(\frac{\beta^i}{(\alpha-\lambda)^{i+1}}\right)\right| \|y\|\\ &\le \left|\frac{1}{(\alpha-\lambda)}\right| + \sum_{i=1}^{\infty}\left(\frac{|\beta|^i}{|\alpha-\lambda|^{i+1}}\right)\|y\|\\ &= \frac{1}{|\alpha-\lambda|}\left(1+\left|\frac{\beta}{\alpha-\lambda}\right| + \left|\frac{\beta}{\alpha-\lambda}\right|^2 + \ldots\right)\|y\|\\ &= \frac{\|y\|}{|\alpha-\lambda|-|\beta|} \text{ (by the hypothesis).} \end{aligned}$$

Clearly, for each $k \in \mathbb{N}_0, x_k$ is finite and $\lim_{k\to\infty}|x_k| < \infty$, which implies that $\sup_k |x_k| < \infty$. Therefore, $(\Delta_{ab}-\lambda I)^{-1} \in B(\ell_1,\ell_1)$. Similarly we can prove that $(\Delta_{ab}-\lambda I)^{-1} \in B(\ell_\infty,\ell_\infty)$. By using Lemma 5 we have $(\Delta_{ab}-\lambda)^{-1} \in (\ell_p,\ell_p)$, thus

$$\sigma(\Delta_{ab},\ell_p) \subseteq \Big\{\lambda \in \mathbb{C} : |\alpha-\lambda| \le |\beta|\Big\}. \quad (5)$$

Part 2:
For the second part, we show that $\Big\{\lambda \in \mathbb{C} : |\alpha-\lambda| \le |\beta|\Big\} \subseteq \sigma(\Delta_{ab},\ell_p)$. Assume $\lambda \ne a_k$, for each $k \in \mathbb{N}_0$, then $(\Delta_{ab}-\lambda I)^{-1}$ exists. Choosing $y = e_k^1 = (\underbrace{1,1,\ldots 1}_{\text{k terms}},0,0\ldots.) \in \ell_p$ and $\lambda \in \mathbb{C}$, then we obtain

$$\lim_{k\to\infty}|x_k| \le \left|\frac{1}{(\alpha-\lambda)}\right| + \sum_{i=1}^{\infty}\left(\frac{|\beta|^i}{|\alpha-\lambda|^{i+1}}\right) = \infty,$$
$$\text{with } |\alpha-\lambda| < |\beta| \text{ and } |\alpha-\lambda| = |\beta|$$

Hence,
$\sup_k |x_k| = \infty, \quad \Rightarrow x \notin \ell_p$, which implies that $(\Delta_{ab}-\lambda I)^{-1} \notin B(\ell_p)$.
Furthermore, if $a_k = \lambda$, then $|x_k|$ is unbounded and as a result $(\Delta_{ab}-\lambda I)^{-1} \notin B(\ell_p)$.
Therefore,

$$\Big\{\lambda \in \mathbb{C} : |\alpha-\lambda| \le |\beta|\Big\} \subseteq \sigma(\Delta_{ab},\ell_p). \quad (6)$$

Combining inclusions (5) and (6), we complete the proof.

Theorem 3. *Point spectrum of the operator Δ_{ab} over ℓ_p is given by*

$$\sigma_p(\Delta_{ab},\ell_p) = \emptyset.$$

The notation p using in $\sigma_p(\Delta_{ab},\ell_p)$ has different meaning to that of in $\ell_p (1 < p < \infty)$.

Proof. Let λ be an eigen value of $\Delta_{ab}-\lambda I$, then there exists an eigen vector $\mathbf{0} \ne x \in \ell_p$ such that $\Delta_{ab}x = \lambda x$, which gives a system of linear equations:

$$\left.\begin{aligned} a_0x_0 &= \lambda x_0\\ b_0x_0 + a_1x_1 &= \lambda x_1\\ b_1x_1 + a_2x_2 &= \lambda x_2\\ &\ldots\ldots\\ b_{k-1}x_{k-1} + a_kx_k &= \lambda x_k\\ &\ldots\ldots \end{aligned}\right\} \quad (7)$$

On solving above system of equations, it is clear that if $x_0 = 0$ and $a_k - \lambda \ne 0$, then $x_k = 0$, for all $k \in \mathbb{N}_0$. Therefore, $x = \mathbf{0}$, which is a contradiction.

Again suppose $x_0 \ne 0$ and x_{ko} is the first zero entry of $x = (x_k)$. Now, from the above system of equations $b_{ko-1}x_{ko-1} + a_{ko}x_{ko} = \lambda x_{ko}$, which implies that $x_{ko-1} = 0$. Continuing this process it can be shown that $x_{ko-1} = x_{ko-2} = \ldots = x_1 = x_0 = 0 \Rightarrow x = \mathbf{0}$, which is a contradiction. Furthermore, for $x_k \ne 0$ for each $k \in \mathbb{N}_0$, then we obtain that $\lambda = a_0$ and

$$|x_k| = \frac{|b_{k-1}x_{k-1}|}{|\lambda - a_k|} \text{ for all } k = 1,2,3,\ldots$$
$$\lim_{k\to\infty}\left|\frac{x_k}{x_{k-1}}\right| = \frac{|\beta|}{|a_0-\alpha|} > 1, \text{ by the definition.}$$

Therefore, for $1 < p < \infty$, $x_k \notin \ell_p$. This completes the proof.

Theorem 4. *Point spectrum of the dual operator $(\Delta_{ab})^*$ of Δ_{ab} over $\ell_p^* \cong \ell_q$ is given by*

$$\sigma_p((\Delta_{ab})^*,\ell_\infty) = \Big\{\lambda \in \mathbb{C} : |\alpha-\lambda| < |\beta|\Big\}.$$

Proof. Suppose $(\Delta_{ab})^* f = \lambda f$ and $\mathbf{0} \ne f \in \ell_p^* \cong \ell_q$, where $1 < p < \infty$, $1/p + 1/q = 1$ and

$$(\Delta_{ab})^* = (\Delta_{ab})^T = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \ldots\\ 0 & a_1 & b_1 & 0 & \ldots\\ 0 & 0 & a_2 & b_2 & \ldots\\ 0 & 0 & 0 & a_3 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } f = \begin{pmatrix} f_0\\ f_1\\ f_2\\ \vdots \end{pmatrix}$$

Consider the system of linear equations

$$\left.\begin{array}{c} a_0 f_0 + b_0 f_1 = \lambda f_0 \\ a_1 f_1 + b_1 f_2 = \lambda f_1 \\ a_2 f_2 + b_2 f_3 = \lambda f_2 \\ \cdots\cdots\cdots \\ a_k f_k + b_k f_{k+1} = \lambda f_k \\ \cdots\cdots\cdots \end{array}\right\} \quad (8)$$

It is clear that for all $k \in \mathbb{N}_0$ the vector $f = (0, 0 \ldots f_k, f_{k+1}, 0, \ldots)$ is an eigen vector correspond to the the eigen value λ satisfying $|\alpha - \lambda| < |\beta|$. On solving the above system of equations, we have

$$f_{k+1} = \left(\frac{\lambda - a_k}{b_k}\right) f_k$$
$$= \left(\frac{\lambda - a_k}{b_k}\right)\left(\frac{\lambda - a_{k-1}}{b_{k-1}}\right)\cdots\left(\frac{\lambda - a_0}{b_0}\right) f_0$$

Now, $\lim_{k\to\infty} |f_{k+1}| \leq \sup_k \left(\left|\frac{\lambda - \alpha}{\beta}\right|^{k+1}\right) |f_0|$

Proceeding this way, we obtain that $|f_{k+1}| \leq |f_k| \ldots. \leq |f_0|$ if and only if $|\alpha - \lambda| < |\beta|$ and therefore, $\sum_{k=1}^{\infty} |f_k|^q < \infty$. Hence, $f \in \ell_q$, provided $|\alpha - \lambda| < |\beta|$ and $|f_0| < \infty$. This completes the proof.

Theorem 5. *Residual spectrum of the operator Δ_{ab} over ℓ_p is given by*

$$\sigma_r(\Delta_{ab}, \ell_p) = \Big\{\alpha \in \mathbb{C} : |\alpha - \lambda| < |\beta|\Big\}.$$

Proof. For $|\alpha - \lambda| < |\beta|$, the operator $\Delta_{ab} - \lambda I$ has an inverse. By Theorem 4 the operator $(\Delta_{ab})^* - \lambda I$ is not one to one for $\lambda \in \mathbb{C}$ with $|\alpha - \lambda| < |\beta|$. By using Lemma 2, we have $\overline{R(\Delta_{ab} - \lambda I)} \neq \ell_p$. Hence

$$\sigma_r(\Delta_{ab}, \ell_p) = \Big\{\alpha \in \mathbb{C} : |\alpha - \lambda| < |\beta|\Big\}.$$

Theorem 6. *Continuous spectrum of the operator Δ_{ab} over ℓ_p is given by*

$$\sigma_c(\Delta_{ab}, \ell_p) = \Big\{\lambda \in \mathbb{C} : |\alpha - \lambda = |\beta|\Big\}.$$

Proof. The proof of this theorem follows from Theorems 2,3,5 and with the fact that

$$\sigma(\Delta^r_{ab}, \ell_p) = \sigma_p(\Delta^r_{ab}, \ell_p) \cup \sigma_r(\Delta^r_{ab}, \ell_p) \cup \sigma_c(\Delta^r_{ab}, \ell_p).$$

4 The spectrum of the Difference operator Δ_{ab} over the sequence space $bv_p, (1 < p < \infty)$

In this section, we determine the spectrum of the generalized difference operator Δ_{ab} over the sequence space $bv_p, (1 < p < \infty)$. Since $\ell_p \subset bv_p$ strictly, the results examined for the sequence space ℓ_p are almost similar to that for the sequence space $bv_p, (1 < p < \infty)$.

Theorem 7. *The operator $\Delta_{ab} : bv_p \to bv_p$ is a bounded linear operator satisfying the condition*

$$\| \Delta_{ab} \|_{(bv_p : bv_p)} \leq \sup_k (|a_k| + |b_k|).$$

Proof. Linearity of the operator Δ_{ab} is trivial, hence omitted. Let $x = (x_k) \in bv_p$ for all $k \in \mathbb{N}_0$ and $1 < p < \infty$. Now, by using Minkowski's inequality, we have

$$\begin{aligned} \|\Delta_{ab}(x)\|_{(bv_p)} &= \left(\sum_{k=0}^{\infty} |a_k(x_k - x_{k-1}) + b_k(x_{k-1} - x_{k-2})|^p\right)^{1/p} \\ &\leq \left(\sum_{k=0}^{\infty} |a_k(x_k - x_{k-1})|^p\right)^{1/p} \\ &\quad + \left(\sum_{k=0}^{\infty} |b_k(x_{k-1} - x_{k-2})|^p\right)^{1/p} \\ &\leq \sup_k (|a_k| + |b_k|) \|x\|_{bv_p}. \end{aligned}$$

Thus,

$$\|\Delta_{ab}\|_{(bv_p)} \leq \sup_k (|a_k| + |b_k|).$$

This completes the proof.

Since the spectrum of the matrix Δ_{ab} as an operator over the sequence space bv_p are similar to that of the space ℓ_p. So, we avoid to repeat the similar statements as discussed in last sections. We give the results in the following theorems without proof.

Theorem 8. *(i)* $\sigma(\Delta_{ab}, bv_p) = \Big\{\lambda \in \mathbb{C} : |\alpha - \lambda| \leq |\beta|\Big\}$;
(ii) $\sigma_p(\Delta_{ab}, bv_p) = \emptyset$;
(iii) $\sigma_r(\Delta_{ab}, bv_p) = \Big\{\lambda \in \mathbb{C} : |\alpha - \lambda| < |\beta|\Big\}$;
(iv) $\sigma_c(\Delta_{ab}, bv_p) = \Big\{\lambda \in \mathbb{C} : |\alpha - \lambda| = |\beta|\Big\}$.

5 Conclusion

In the present article, we have determined the spectrum and fine spectrum of the difference operator Δ_{ab} over the sequence space ℓ_p and $bv_p(1 < p < \infty)$. The results and theorems presented by this article are more general and comprehensive than the works done by previous authors. Now, choosing sequences a and b suitably, our work generalizes various other known results studied by

(i) Akhmedov and Başar [8],
(ii) Akhmedov and Başar [9],
(iii) Kayaduman and Furkan [12],
(iv) Furkan et al. [22] and many others.

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