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Realizing a Four-Qubit Quantum Gate in a Cavity QED System

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Abstract: We propose a scheme to direct implementation the four-qubit phase gate by passing a five-level atom (which is initially in the lower state) across a multi-mode cavity QED. The four qubit are represented by the photons in the four modes of the cavity field. In particular, the four-qubit quantum phase gate can be implemented if certain conditions for transition strength, detuning and interaction time are better satisfied. Subsequently we will discus this phase gate scheme.

Keywords: Quantum gate, cavity *QED*

1 Introduction

The essential element of a computer is the logic gate, either in a classical computer or a quantum computer [1]. The important work of quantum computer architecture is to find an effective physical realization of quantum logic gates. In fact, almost any gate which can entangle two qubits can be used as a universal gate [2,3]. The quantum phase gate is one of them. Particularly, a quantum phase gate can be directly used in the implementation of Grover's search algorithm [5], quantum Fourier transformation [22], quantum error correction [23], and arbitrary superposed state preparation [24].

Some work has been done on the realization of quantum logic gates through the interaction of multi-level atom with the multi-mode cavity [4,6,7,8,9], but I will only mention two interesting references. In the work of Zubairy et al [8], a scheme to implement a two qubit quantum phase gate and one-qubit unitary operation implementation based on cavity QED was described; the logical states of a qubit are represented by the Fock states $|0\rangle$ and $|1\rangle$ of a high Q cavity mode; the two -qubit phase gate is accomplished by passing a ground state three-level atom through a three mode optical cavity. In the same way, Jun-Tao Chang et al[9] had also described a three-qubit quantum phase gate which is implemented by passing a four-level atom in a cascade configuration initially in its ground states through a three optical modes of the cavity. In this paper, we extended their method to realize a four-qubit quantum phase gate, since the decomposition of multiqubit gates into the elementary gates become mores complicated with the number of qubits increasing. Moreover It has been shown that a multiqubit quantum phase gate, can lead to faster computing. For example, according to Diao et al.[25] a four-qubit quantum phase gate needs a network of eight one-bit quantum gates and five two-qubit quantum phase gates. In our scheme that based on cavity QED, the four qubits are represented by four modes of the field inside the cavity. We will discuss the conditions on the coupling coefficients between the cavity modes and atomic transitions to complete this phase gate. The choice of interaction time and detuning will be also discussed and finally fidelity will be determined in order to examine these conditions.

2 Four-qubit quantum phase gate

The system that we consider here consists of a cascade five energy levels atom passing through an optical cavity. The relevant atomic level structure is shown in Figure 1(b) The atom which is initially in the lower state $|e\rangle$, interact with four modes of the cavity with frequency ω_1, ω_2, ω_3

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and ω_4 Figure 1(a). The cavity field frequencies ω_1 and ω_2 are assumed to be far detuned from the atomic transition respectively from the $|e\rangle$ state to the $|d\rangle$ state and from the $|d\rangle$ state to the $|c\rangle$ state; likewise the modes cavity ω_3 and ω_4 are far detuned from the atomic transition respectively from the $|c\rangle$ state to the $|b\rangle$ state and from the $|b\rangle$ state to the $|a\rangle$ state.

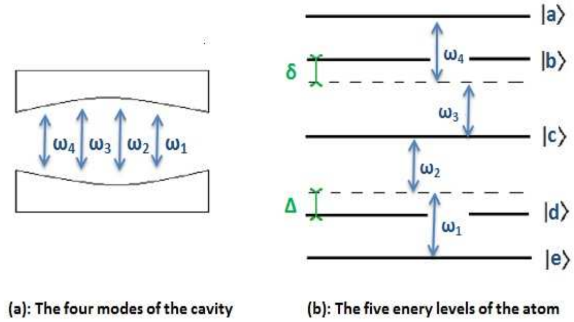

Fig. 1: (a): Frequencies of the four modes of the cavity. (b): Illustration of the different detunings between the high Q cavity mode frequencies and the transition between the five energy levels of the atom.

If we suppose initially one photon in each mode of the cavity and if we represent the four qubits by the quantum states of the field inside the cavity, the 2^4 states possible for the cavity modes are: $\{|\alpha_1\alpha_2\alpha_3\alpha_4\rangle\}$, where $|\alpha_1\alpha_2\alpha_3\alpha_4\rangle = |\alpha_1\rangle\otimes|\alpha_2\rangle\otimes|\alpha_3\rangle\otimes|\alpha_4\rangle$ and $|\alpha_i\rangle$ stand for the basis state $|0\rangle$ or $|1\rangle$ of the qubit $|i\rangle$ $(i = 1,2,3$ and $4)$.

The transformation of a four qubit quantum phase gate with phase η is then defined as:

$$Q_\eta\,|\alpha_1\alpha_2\alpha_3\alpha_4\rangle = \exp(i\eta\,\delta_{\alpha_1,1}\delta_{\alpha_2,1}\delta_{\alpha_3,1}\delta_{\alpha_4,1})\,|\alpha_1\alpha_2\alpha_3\alpha_4\rangle \quad (1)$$

Since the quantum phase gate introduces a phase η only when all the qubits in input state are 1, we can write Q_η as:

$$\begin{aligned} Q_\eta = {} & |0_10_20_30_4\rangle\langle 0_10_20_30_4| + |0_10_20_31_4\rangle\langle 0_10_20_31_4| \\ & + |0_10_21_30_4\rangle\langle 0_10_21_30_4| + |0_10_21_31_4\rangle\langle 0_10_21_31_4| \\ & + |0_11_20_30_4\rangle\langle 0_11_20_30_4| + |0_11_20_31_4\rangle\langle 0_11_20_31_4| \\ & + |0_11_21_30_4\rangle\langle 0_11_21_30_4| + |0_11_21_31_4\rangle\langle 0_11_21_31_4| \\ & + |1_10_20_30_4\rangle\langle 1_10_20_30_4| + |1_10_20_31_4\rangle\langle 1_10_20_30_4| \\ & + |1_10_21_30_4\rangle\langle 1_10_21_30_4| + |1_10_21_31_4\rangle\langle 1_10_21_31_4| \\ & + |1_11_20_30_4\rangle\langle 1_11_20_30_4| + |1_11_20_31_4\rangle\langle 1_11_20_31_4| \\ & + |1_11_21_30_4\rangle\langle 1_11_21_30_4| \\ & + \exp(i\eta)\,|1_11_21_31_4\rangle\langle 1_11_21_31_4| \end{aligned} \quad (2)$$

In the following, we take $\eta = \pi$.

It is seen that the following states remain unchanged during the passage of the atom within the cavity:

- The $|0_10_20_30_4\rangle$ states since there is no photon in the cavity

- Likewise all the states for which there is no photon in the cavity with frequency ω_1, in view of the atom is initially in the lower state $|e\rangle$ (It is $|0_10_20_30_4\rangle$, $|0_10_20_31_4\rangle$, $|0_10_21_30_4\rangle$, $|0_10_21_31_4\rangle$, $|0_11_20_30_4\rangle$, $|0_11_20_31_4\rangle$, $|0_11_21_30_4\rangle$ and $|0_11_21_31_4\rangle$).

-Else the states $|1_10_20_30_4\rangle$, $|1_10_20_31_4\rangle$, $|1_10_21_30_4\rangle$ and $|1_10_21_31_4\rangle$ since the cavity field frequencies are far detuned from the atomic transition frequencies.

Subsequently we will discus this phase gate scheme.

3 The theoretical model

The system's Hamiltonian model consists of a generalized scheme for the interaction between a five-level atom with a four-mode optical cavity and by taking into account the dipole and rotating-wave approximations. The Hamiltonian can be then written in the Jaynes-Cummings model as [9,10,11]:

$$H = H_0 + H_1 \quad (3)$$

with

$$\begin{aligned} H_0 = {} & \hbar\omega_{ab}|a\rangle\langle a| + \hbar\omega_{bc}|b\rangle\langle b| + \hbar\omega_{cd}|c\rangle\langle c| + \hbar\omega_{de}|d\rangle\langle d| \\ & + \hbar\omega_1\hat{a}_1^+\hat{a}_1 + \hbar\omega_2\hat{a}_2^+\hat{a}_2 + \hbar\omega_3\hat{a}_3^+\hat{a}_3 + \hbar\omega_4\hat{a}_4^+\hat{a}_4 \end{aligned} \quad (4)$$

$$\begin{aligned} H_1 = {} & \hbar g_1|d\rangle\langle e|\hat{a}_1 + \hbar g_2|c\rangle\langle d|\hat{a}_2 + \hbar g_3|b\rangle\langle c|\hat{a}_3 \\ & + \hbar g_4|a\rangle\langle b|\hat{a}_4 + H.C \end{aligned} \quad (5)$$

where $\hat{a}_i$ and $\hat{a}_i^+$ are the annihilation and the creation operators of the cavity mode i. g_i is the Rabi frequency of the interaction between the field mode ω_i (i from 1 to 4) and respectively the atomic transition $|d\rangle \leftrightarrows |e\rangle$, $|c\rangle \leftrightarrows |d\rangle$, $|b\rangle \leftrightarrows |c\rangle$ and $|a\rangle \leftrightarrows |b\rangle$.

In the interaction picture the Hamiltonian can be written as:

$$\begin{aligned} H_I = {} & \hbar g_1|d\rangle\langle e|\hat{a}_1 e^{it\Delta} + \hbar g_2|c\rangle\langle d|\hat{a}_2 e^{-it\Delta} \\ & + \hbar g_3|b\rangle\langle c|\hat{a}_3 e^{-it\delta} + \hbar g_4|a\rangle\langle b|\hat{a}_4 e^{it\delta} + H.C \end{aligned} \quad (6)$$

where $\omega_{de} - \omega_1 = -\Delta$, $\omega_{cd} - \omega_2 = \Delta$, $\omega_{bc} - \omega_3 = \delta$ and $\omega_{ab} - \omega_4 = -\delta$ (see Figure 1(b)).

Generally if we suppose initially n_i photons in each mode i and the atom is in the lower state $|e\rangle$, the initial state for the system $\{atom + cavity\}$ is: $|e,n_1,n_2,n_3,n_4\rangle = |e\rangle\otimes|n_1,n_2,n_3,n_4\rangle$. Then we can write the state for the system at time t as:

$$\begin{aligned} |\Psi(t)\rangle = {} & C_e(t)|e,n_1,n_2,n_3,n_4\rangle + C_d(t)|d,n_1-1,n_2,n_3,n_4\rangle \\ & + C_c(t)|c,n_1-1,n_2-1,n_3,n_4\rangle + C_b(t)|b,n_1-1,n_2-1,n_3-1,n_4\rangle \\ & + C_a(t)|a,n_1-1,n_2-1,n_3-1,n_4-1\rangle \end{aligned} \quad (7)$$

The Schrödinger equation $i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = H_I|\Psi(t)\rangle$, in the interaction picture give us:

$$\begin{cases} i\hbar\dot{C}_e(t) = \hbar g_1 C_d(t)e^{-i\Delta t}\sqrt{n_1} \\ i\hbar\dot{C}_d(t) = \hbar g_1 C_e(t)e^{i\Delta t}\sqrt{n_1} + \hbar g_2 C_c(t)e^{i\Delta t}\sqrt{n_2} \\ i\hbar\dot{C}_c(t) = \hbar g_2 C_d(t)e^{-i\Delta t}\sqrt{n_2} + \hbar g_3 C_b(t)e^{i\delta t}\sqrt{n_3} \\ i\hbar\dot{C}_b(t) = \hbar g_3 C_c(t)e^{-i\delta t}\sqrt{n_3} + \hbar g_4 C_a(t)e^{-i\delta t}\sqrt{n_4} \\ i\hbar\dot{C}_a(t) = \hbar g_4 C_b(t)e^{i\delta t}\sqrt{n_4} \end{cases} \quad (8)$$

To solve this system it's useful to makes a change to the amplitudes by setting: $Z_e = C_e$, $Z_d = C_d e^{-i\Delta t}$, $Z_c = C_c$, $Z_b = C_b e^{i\delta t}$, and $Z_a = C_a$. The system becomes:

$$\begin{cases} \dot{Z}_e = -ig_1\sqrt{n_1}Z_d \\ \dot{Z}_d = -ig_1\sqrt{n_1}Z_e - ig_2\sqrt{n_2}Z_c - i\Delta Z_d \\ \dot{Z}_c = -ig_2\sqrt{n_2}Z_d - ig_3\sqrt{n_3}Z_b \\ \dot{Z}_b = -ig_3\sqrt{n_3}Z_c - ig_4\sqrt{n_4}Z_a + i\delta Z_b \\ \dot{Z}_a = -ig_4\sqrt{n_4}Z_b \end{cases} \quad (9)$$

If we suppose there is no transition to the $|b\rangle$ and $|d\rangle$ states because of a very large detuning between the cavity modes frequencies and the transition frequencies to the $|b\rangle$ and $|d\rangle$ states(i.e $\dot{Z}_b = 0$ and $\dot{Z}_d = 0$) then the system becomes:

$$\begin{cases} \dot{Z}_b = 0 \\ \dot{Z}_d = 0 \\ \dot{Z}_e = i\frac{g_1^2 n_1}{\Delta}Z_e + i\frac{g_1 g_2\sqrt{n_1 n_2}}{\Delta}Z_c \\ \dot{Z}_c = i\frac{g_1 g_2\sqrt{n_1 n_2}}{\Delta}Z_e + i\frac{g_2^2 n_2}{\Delta}Z_c - i\frac{g_3^2 n_3}{\delta}Z_c - i\frac{g_3 g_4\sqrt{n_3 n_4}}{\delta}Z_a \\ \dot{Z}_a = -i\frac{g_3 g_4\sqrt{n_3 n_4}}{\delta}Z_c - i\frac{g_4^2 n_4}{\delta}Z_a \end{cases} \quad (10)$$

that we can rewrite as:

$$\begin{cases} \dot{Z}_b\,|b, n_1-1, n_2-1, n_3-1, n_4\rangle = 0\cdot|\Psi(t)\rangle \\ \dot{Z}_d\,|d, n_1-1, n_2, n_3, n_4\rangle = 0\cdot|\Psi(t)\rangle \\ \dot{Z}_e\,|e, n_1, n_2, n_3, n_4\rangle = \left(i\frac{g_1^2}{\Delta}\hat{a}_1^+\hat{a}_1|e\rangle\langle e| + i\frac{g_1g_2}{\Delta}\hat{a}_1^+\hat{a}_2^+|e\rangle\langle c|\right) \\ \quad\times|\Psi(t)\rangle \\ \dot{Z}_c\,|c, n_1-1, n_2-1, n_3, n_4\rangle = \left(i\frac{g_1g_2}{\Delta}\hat{a}_1\hat{a}_2|c\rangle\langle e|\right. \\ \quad + i\frac{g_2^2}{\Delta}\hat{a}_2\hat{a}_2^+|c\rangle\langle c| - i\frac{g_3^2}{\delta}\hat{a}_3^+\hat{a}_3|c\rangle\langle c| \\ \quad \left.- i\frac{g_3g_4}{\delta}\hat{a}_3^+\hat{a}_4^+|c\rangle\langle a|\right)|\Psi(t)\rangle \\ \dot{Z}_a\,|a, n_1-1, n_2-1, n_3-1, n_4-1\rangle = \left(-i\frac{g_3g_4}{\delta}\hat{a}_3\hat{a}_4|a\rangle\langle c|\right. \\ \quad \left.- i\frac{g_4^2}{\delta}\hat{a}_4\hat{a}_4^+|a\rangle\langle a|\right)|\Psi(t)\rangle \end{cases} \quad (11)$$

these evolution equations correspond to the following Hamiltonian in the interaction picture:

$$\begin{aligned} H_I^{'} = &-\frac{\hbar g_1^2}{\Delta}\hat{a}_1^+\hat{a}_1|e\rangle\langle e| - \frac{\hbar g_2^2}{\Delta}\hat{a}_2\hat{a}_2^+|c\rangle\langle c| + \frac{\hbar g_3^2}{\delta}\hat{a}_3^+\hat{a}_3|c\rangle\langle c| \\ &+ \frac{\hbar g_4^2}{\delta}\hat{a}_4\hat{a}_4^+|a\rangle\langle a| - \frac{\hbar g_1 g_2}{\Delta}\hat{a}_1^+\hat{a}_2^+|e\rangle\langle c| \\ &- \frac{\hbar g_1 g_2}{\Delta}\hat{a}_1\hat{a}_2|c\rangle\langle e| + \frac{\hbar g_3 g_4}{\delta}\hat{a}_3^+\hat{a}_4^+|c\rangle\langle a| \\ &+ \frac{\hbar g_3 g_4}{\delta}\hat{a}_3\hat{a}_4|a\rangle\langle c| \end{aligned} \quad (12)$$

3.1 Conditions for implementation of the phase gate

We consider the initial state of system $|\Psi_0\rangle = |e, 1, 1, 1, 0\rangle$ ($n_1 = 1$, $n_2 = 1$, $n_3 = 1$ and $n_4 = 0$).

Then the effectif Hamiltonian in the interaction picture becomes:

$$\begin{aligned} H_I^{'} = &-\frac{\hbar g_1^2}{\Delta}\hat{a}_1^+\hat{a}_1|e\rangle\langle e| - \frac{\hbar g_2^2}{\Delta}\hat{a}_2\hat{a}_2^+|c\rangle\langle c| + \frac{\hbar g_3^2}{\delta}\hat{a}_3^+\hat{a}_3|c\rangle\langle c| \\ &-\frac{\hbar g_1 g_2}{\Delta}\hat{a}_1^+\hat{a}_2^+|e\rangle\langle c| - \frac{\hbar g_1 g_2}{\Delta}\hat{a}_1\hat{a}_2|c\rangle\langle e| \end{aligned} \quad (13)$$

We suppose that $g_1 \gg g_2$ and $\frac{g_1^2}{\Delta} \gg \frac{g_3^2}{\delta}$ then we can write the Hamiltonian $H_I^{'}$ as:

$$\begin{aligned} H_I^{'} &\approx -\frac{\hbar g_1^2}{\Delta}\hat{a}_1^+\hat{a}_1|e\rangle\langle e| \\ &= -\frac{\hbar g_1^2}{\Delta}|e, 1, 1, 1, 0\rangle\langle e, 1, 1, 1, 0| \end{aligned} \quad (14)$$

If we choose the intercation time $\tau = \frac{(2k+1)\Delta}{g_1^2}\pi$ (k is an arbitrary integer) and by use of the Schrödinguer equation, the system $\{atom + cavity\}$ undergoes the following transformation:

$$|e, 1, 1, 1, 0\rangle \longrightarrow -|e, 1, 1, 1, 0\rangle \quad (15)$$

So we can realize the four-qubit phase gate. For to obtain the conditions more precisely, we will solve the Schrödinger equation in the interaction representation to find the probability amplitudes for all possible initial states.

For this, we pose for the normalized detuning $X_0 = \frac{\Delta}{g_1}$, $X_1 = \frac{\delta}{g_1}$, $X_2 = \frac{\Delta}{g_2}$, $X_3 = \frac{\delta}{g_3}$ and $X_4 = \frac{\delta}{g_4}$.

3.1.1 Initial state: $|\Psi_0\rangle = |e, 1, 0, 0, 0\rangle$

We use the interaction part in the Hamiltonian to determine the basic system states (atom + cavity) and which are $|e, 1, 0, 0, 0\rangle$ and $|d, 0, 0, 0, 0\rangle$. The possible state at time t is the superposition of these two states, then $|\Psi(t)\rangle = C_e(t)|e, 1, 0, 0, 0\rangle + C_d(t)|d, 0, 0, 0, 0\rangle$ (i.e $Z_a = Z_b = Z_c = 0$)

The system 9 becomes:

$$\begin{cases} \dot{Z}_e = -ig_1 Z_d \\ \dot{Z}_d = -ig_1 Z_e - i\Delta Z_d \end{cases} \quad (16)$$

that gives ($Z_e = C_e$):

$$\ddot{C}_e(t) + i\Delta\dot{C}_e(t) + g_1^2 C_e(t) = 0 \quad (17)$$

By considering for the initial state $C_e(t=0)=1$ and $C_d(t=0)=0$, solving this differential equation gives us at time $t=\tau$:

$$C_{e,1000}(\tau) = \frac{1}{2}e^{-i(2k+1)X_0^2\frac{\pi}{2}} \times \left[\left(1+\frac{X_0}{2\sqrt{1+\frac{X_0^2}{4}}}\right)e^{i\sqrt{1+\frac{X_0^2}{4}}(2k+1)X_0\pi} + \left(1-\frac{X_0}{2\sqrt{1+\frac{X_0^2}{4}}}\right)e^{-i\sqrt{1+\frac{X_0^2}{4}}(2k+1)X_0\pi}\right] \quad (18)$$

k is an arbitrary integer

We note that we will found the same expression for the following initial states: $|e,1,0,0,1\rangle$, $|e,1,0,1,0\rangle$ and $|e,1,0,1,1\rangle$.

In order that $C_{e,1000}(\tau)=1$, X_0 must satisfy the following conditions (with p and $p\prime$ are integers):

$$\begin{cases} (2k+1)X_0\left(\sqrt{1+\frac{X_0^2}{4}}+\frac{X_0}{2}\right)\pi = 2p\pi \\ (2k+1)X_0\left(\sqrt{1+\frac{X_0^2}{4}}-\frac{X_0}{2}\right)\pi = 2p\prime\pi \end{cases} \quad (19)$$

which give:

$$(2k+1)(p-p\prime)-2pp\prime = 0 \quad (20)$$

and

$$X_0 = \sqrt{\frac{2(p-p\prime)}{(2k+1)}} \quad (21)$$

For positive fixed values of k, p and p', the value of X_0 that allows us to have $C_{e,1000}(\tau)=1$ is then determined through the relation 21, with the condition bettween them satisfy the expression 20.

If we take for example $p=(2k+1)k$ and $p'=k$, equation 20 is verified and we have:

$$X_0 = \frac{2k}{\sqrt{2k+1}} \quad (22)$$

3.1.2 Initial state: $|\Psi_0\rangle = |e,1,1,0,0\rangle$

As before we consider a general state at time t as:

$$|\Psi(t)\rangle = C_e(t)\,|e,1,1,0,0\rangle + C_d(t)\,|d,1,0,0,0\rangle + C_c(t)\,|c,0,0,0,0\rangle$$

Then the system 10 becomes:

$$\begin{cases} \dot{Z}_e = i\frac{g_1^2}{\Delta}Z_e + i\frac{g_1g_2}{\Delta}Z_c \\ \dot{Z}_c = i\frac{g_1g_2}{\Delta}Z_e + i\frac{g_2^2}{\Delta}Z_c \end{cases} \quad (23)$$

Eliminating Z_c, we can get the following differential equation:

$$\ddot{Z}_e - i\left(\frac{g_1^2}{\Delta}+\frac{g_2^2}{\Delta}\right)\dot{Z}_e = 0 \quad (24)$$

The resolution of this differential equation give us at time t (we have used for the initial state $Z_e(t=0)=1$ and $\dot{Z}_e(t=0)=i\frac{g_1^2}{\Delta}$):

$$Z_e(t) = \frac{g_2^2}{g_1^2+g_2^2} + \frac{g_1^2}{g_1^2+g_2^2}e^{i\frac{\left(g_1^2+g_2^2\right)}{\Delta}t} \quad (25)$$

that yielding at time $t=\tau$ ($Z_e = C_e$):

$$C_{e,1100}(\tau) = \frac{X_0^2}{X_0^2+X_2^2} + \frac{X_2^2}{X_0^2+X_2^2}e^{i\frac{\left(X_0^2+X_2^2\right)}{X_2^2}(2k+1)\pi} \quad (26)$$

k is an arbitrary integer

We will found the same expression for the following initial state: $|e,1,1,0,1\rangle$

In order that $C_{e,1100}(\tau)=1$, X_0 and X_2 must satisfy the following condition (n is an arbitrary integer):

$$\frac{\left(X_0^2+X_2^2\right)}{X_2^2}(2k+1)\pi = 2n\pi \quad (27)$$

that implies:

$$X_2^2 = \frac{2k+1}{2(n-k)-1}X_0^2 \quad (28)$$

Using the condition on X_0 found for the initial state $|\Psi_0\rangle = |e,1,0,0,0\rangle$ ($p=(2k+1)k$ and $p'=k$), we find for X_2:

$$X_2 = \sqrt{\frac{2(p-p')}{2(n-k)-1}} = \frac{2k}{\sqrt{2(n-k)-1}} \quad (29)$$

3.1.3 Initial state: $|\Psi_0\rangle = |e,1,1,1,0\rangle$

$n_1=1$, $n_2=1$, $n_3=1$ and $n_4=0$ implies that:

$$|\Psi(t)\rangle = C_e(t)\,|e,1,1,1,0\rangle + C_d(t)\,|d,1,1,0,0\rangle + C_c(t)\,|c,1,0,0,0\rangle + C_b(t)\,|b,0,0,0,0\rangle$$

The system 10 becomes:

$$\begin{cases} \dot{Z}_e = i\frac{g_1^2}{\Delta}Z_e + i\frac{g_1 g_2}{\Delta}Z_c \\ \dot{Z}_c = i\frac{g_1 g_2}{\Delta}Z_e + i\left(\frac{g_2^2}{\Delta} - \frac{g_3^2}{\delta}\right)Z_c \end{cases} \tag{30}$$

By eliminating Z_c, we obtain the following differential equation:

$$\ddot{Z}_e - i(\frac{g_1^2}{\Delta} + \frac{g_2^2}{\Delta} - \frac{g_3^2}{\delta})\dot{Z}_e + \frac{g_1^2 g_3^2}{\Delta\delta}Z_e = 0 \tag{31}$$

The resolution of this differential equation give us at time t (we have used for the initial state $Z_e(t=0) = 1$ and $\dot{Z}_e(t=0) = i\frac{g_1^2}{\Delta}$):

$$\begin{aligned} Z_e(t) = {} & e^{\frac{i}{2}\left(\frac{g_1^2}{\Delta}+\frac{g_2^2}{\Delta}-\frac{g_3^2}{\delta}\right)t}\left[\cos\left(\sqrt{\frac{g_1^2 g_3^2}{\Delta\delta} + \frac{1}{4}\left(\frac{g_1^2}{\Delta} + \frac{g_2^2}{\Delta} - \frac{g_3^2}{\delta}\right)^2}\, t\right)\right. \\ & + \frac{i}{2}\frac{\frac{g_1^2}{\Delta} - \frac{g_2^2}{\Delta} + \frac{g_3^2}{\delta}}{\sqrt{\frac{g_1^2 g_3^2}{\Delta\delta} + \frac{1}{4}\left(\frac{g_1^2}{\Delta} + \frac{g_2^2}{\Delta} - \frac{g_3^2}{\delta}\right)^2}} \\ & \left.\times \sin\left(\sqrt{\frac{g_1^2 g_3^2}{\Delta\delta} + \frac{1}{4}\left(\frac{g_1^2}{\Delta} + \frac{g_2^2}{\Delta} - \frac{g_3^2}{\delta}\right)^2}\, t\right)\right] \end{aligned} \tag{32}$$

that implies at time $t = \tau$ ($Z_e = C_e$):

$$\begin{aligned} C_{e,1110}(\tau) = {} & e^{i\left(1+\frac{X_0^2}{X_2^2}-\frac{X_0 X_1}{X_3^2}\right)(2k+1)\frac{\pi}{2}} \\ & \times\left[\cos\left(\sqrt{\frac{4X_0X_1}{X_3^2} + \left(1 + \frac{X_0^2}{X_2^2} - \frac{X_0X_1}{X_3^2}\right)^2}(2k+1)\frac{\pi}{2}\right)\right. \\ & + i\frac{1 - \frac{X_0^2}{X_2^2} + \frac{X_0X_1}{X_3^2}}{\sqrt{\frac{4X_0X_1}{X_3^2} + \left(1 + \frac{X_0^2}{X_2^2} - \frac{X_0X_1}{X_3^2}\right)^2}} \\ & \left.\times \sin\left(\sqrt{\frac{4X_0X_1}{X_3^2} + \left(1 + \frac{X_0^2}{X_2^2} - \frac{X_0X_1}{X_3^2}\right)^2}(2k+1)\frac{\pi}{2}\right)\right] \end{aligned} \tag{33}$$

By using the condition $g_1 \gg g_2$ as we have already mentioned to achieve a phase gate (i.e $X_0 \ll X_2$), we can simplify the expression of $C_{e,1110}(\tau)$ $\left(\frac{X_0^2}{X_2^2} \ll 1\right)$.

So we have:

$$\frac{1 - \frac{X_0^2}{X_2^2} + \frac{X_0X_1}{X_3^2}}{\sqrt{\frac{4X_0X_1}{X_3^2} + \left(1 + \frac{X_0^2}{X_2^2} - \frac{X_0X_1}{X_3^2}\right)^2}} \simeq \frac{1 + \frac{X_0X_1}{X_3^2}}{\sqrt{\frac{4X_0X_1}{X_3^2} + \left(1 - \frac{X_0X_1}{X_3^2}\right)^2}} = 1 \tag{34}$$

which gives the approximate value of $C_{e,1110}(\tau)$:

$$\begin{aligned} C_{e,1110}(\tau) & \simeq e^{i\left(1+\frac{X_0^2}{X_2^2}-\frac{X_0X_1}{X_3^2}+\sqrt{\frac{4X_0X_1}{X_3^2}+\left(1+\frac{X_0^2}{X_2^2}-\frac{X_0X_1}{X_3^2}\right)^2}\right)(2k+1)\frac{\pi}{2}} \\ & \simeq e^{i(2k+1)\pi} \\ & = -1 \end{aligned} \tag{35}$$

We will verify the accuracy of these approximations by plotting the variation of the amplitude $C_{e,1110}(\tau)$ according to the parameters X_1 and X_3. X_0 and X_2 are determined from equations 22 and 29. To do this we choose the following values for the integer k ($k = 51$) and the integer n ($n = 52$) and that will be adopted in the following. So we get $X_0 \simeq 10.05$ and $X_2 = 102$. We plot the curve of the real and imaginary part of the $C_{e,1110}(\tau)$ state amplitude at time $t = \tau$ as a function of X_1 and X_3 (see figure 2). we note that for values of X_1 greater than 100 ($X_1 > 100$) and $X_3 \sim 10$, the real part of $C_{e,1110}(\tau)$ approaches -1 and the immaginary part approaches 0.

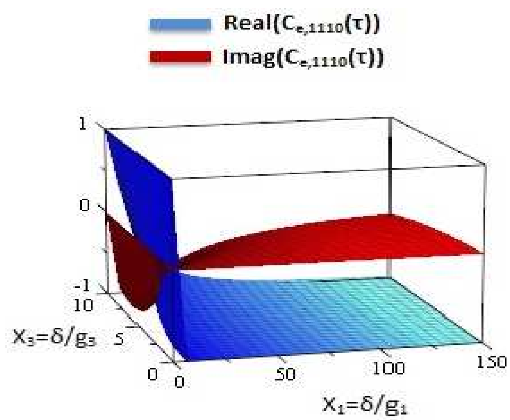

Fig. 2: The plot of the real and imaginary part of the $C_{e,1110}(\tau)$ state amplitude at time $t = \tau$ as a function of $X_1 = \frac{\delta}{g_1}$ and $X_3 = \frac{\delta}{g_3}$. We take $k = 51$, $n = 52$, $X_0 \simeq 10.05$ and $X_2 = 102$.

3.1.4 Initial state: $|\Psi_0\rangle = |e,1,1,1,1\rangle$

We proceed in the same way as for the initial state $|\Psi_0\rangle = |e,1,1,1,0\rangle$ and we consider the general state of the system ($n_1 = 1$, $n_2 = 1$, $n_3 = 1$ and $n_4 = 1$):

$$\begin{aligned} |\Psi(t)\rangle = {} & C_e(t)|e,1,1,1,1\rangle + C_d(t)|d,1,1,1,0\rangle \\ & + C_c(t)|c,1,1,0,0\rangle + C_b(t)|b,1,0,0,0\rangle \\ & + C_a(t)|a,0,0,0,0\rangle \end{aligned} \tag{36}$$

The system 10 becomes:

$$\begin{cases} \dot{Z}_e = i\frac{g_1^2}{\Delta}Z_e + i\frac{g_1g_2}{\Delta}Z_c \\ \dot{Z}_c = i\frac{g_1g_2}{\Delta}Z_e + i\left(\frac{g_2^2}{\Delta} - \frac{g_3^2}{\delta}\right)Z_c - i\frac{g_3g_4}{\delta}Z_a \\ \dot{Z}_a = -i\frac{g_3g_4}{\delta}Z_c - i\frac{g_4^2}{\delta}Z_a \end{cases} \quad (37)$$

By eliminating Z_c and Z_a, we obtain the following differential equation:

$$\dddot{Z}_e - i\left(\frac{g_1^2}{\Delta} + \frac{g_2^2}{\Delta} - \frac{g_3^2}{\delta} - \frac{g_4^2}{\delta}\right)\ddot{Z}_e + \left(\frac{g_1^2g_3^2}{\Delta\delta} + \frac{g_1^2g_4^2}{\Delta\delta} + \frac{g_2^2g_4^2}{\Delta\delta}\right)\dot{Z}_e = 0 \quad (38)$$

By solving this equation and introducing the normalized parameters X_0, X_1, X_2, X_3 and X_4, we find the following solution at time $t = \tau$ ($Z_e = C_e$):

$$C_{e,1111}(\tau) = 1 + \frac{2i}{\sqrt{\left(1 + \frac{X_0^2}{X_2^2} + \frac{X_0X_1}{X_3^2} + \frac{X_0X_1}{X_4^2}\right)^2 - \frac{4X_0^3X_1}{X_2^2X_3^2}}} \times e^{i\left(1+\frac{X_0^2}{X_2^2} - \frac{X_0X_1}{X_3^2} - \frac{X_0X_1}{X_4^2}\right)(2k+1)\frac{\pi}{2}} \times \sin\left(\sqrt{\left(1 + \frac{X_0^2}{X_2^2} + \frac{X_0X_1}{X_3^2} + \frac{X_0X_1}{X_4^2}\right)^2 - \frac{4X_0^3X_1}{X_2^2X_3^2}}(2k+1)\frac{\pi}{2}\right) \quad (39)$$

From the implementation condition that was already mentioned $\frac{g_1^2}{\Delta} \gg \frac{g_3^2}{\delta}$, we can deduce the following condition $\frac{X_0X_1}{X_3^2} \gg 1$. The condition $\frac{X_0^2}{X_2^2} \ll 1$ implies that $\frac{4X_0^3X_1}{X_2^2X_3^2} \ll \frac{X_0X_1}{X_3^2}$, So the term $\frac{X_0X_1}{X_3^2} + \frac{X_0X_1}{X_4^2}$ under the root in the above equation will be the determining factor. Then:

$$\sqrt{\left(1 + \frac{X_0^2}{X_2^2} + \frac{X_0X_1}{X_3^2} + \frac{X_0X_1}{X_4^2}\right)^2 - \frac{4X_0^3X_1}{X_2^2X_3^2}} \gg 1$$
$$\Longrightarrow \frac{1}{\sqrt{\left(1 + \frac{X_0^2}{X_2^2} + \frac{X_0X_1}{X_3^2} + \frac{X_0X_1}{X_4^2}\right)^2 - \frac{4X_0^3X_1}{X_2^2X_3^2}}} \simeq 0$$
$$\Longrightarrow C_{e,1111}(\tau) \simeq 1 \quad (40)$$

We will verify the accuracy of these approximations by plotting the variation of the amplitude $C_{e,1111}(\tau)$ according to the parameters X_1 and X_4 (see Figure 3). We remark that for similar normalized detuning to those considered in the case of the other initial states (i.e $X_0 \simeq 10.05$, $X_2 = 102$, $X_1 > 100$ and $X_3 \sim 10$), the real part of $C_{e,1111}(\tau)$ approaches 1 and the immaginary part approaches 0.

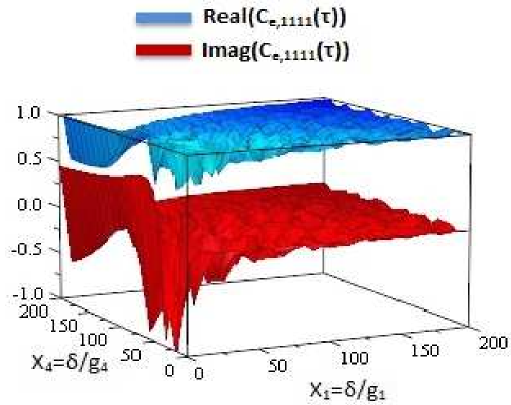

Fig. 3: The plot of the real and imaginary part of the $C_{e,1111}(\tau)$ state amplitude at time $t = \tau$ as a function of $X_1 = \frac{\delta}{g_1}$ and $X_4 = \frac{\delta}{g_4}$. We take $k = 51$, $n = 52$, $X_0 \simeq 10.05$, $X_2 = 102.$ and $X_3 = 10$.

4 Discussion

First, we collect all the conditions to achieve the four-qubit phase gate such that:

∘ The interaction time should be as: $\tau = \frac{(2k+1)\Delta}{g_1^2}\pi = \frac{(2k+1)X_0}{g_1}\pi$ (k is an arbitrary integer).

∘ $\Delta \gg g_1, g_2$ and $\delta \gg g_3, g_4$: it implies that $X_0, X_2, X_3, X_4 \gg 1$

∘ $g_1 \gg g_2$: then $X_0 \ll X_2$

∘ $\frac{g_1^2}{\Delta} \gg \frac{g_3^2}{\delta}$: that gives $\frac{X_0X_1}{X_3^2} \gg 1$

∘ $X_0 = \sqrt{\frac{2(p-p\prime)}{(2k+1)}}$ and $X_2 = \sqrt{\frac{2(p-p')}{2(n-k)-1}}$ with n is an integer number, p and p' are integer numbers satisfying $(2k+1)(p-p\prime) - 2pp\prime = 0$.

We also note that X_0, X_1, X_2, X_3 and X_4 are normalized parameters and the values that verify the conditions for implémentation of the phase gate remain general and don't correspond to any specific values of the constants of coupling (g_1, g_2, g_3 and g_4) and detunings (Δ and δ).

To investigate the experimental feasibility of this proposal, let us consider the microwave cavity-QED experiment in [18], highly excited Rydberg atoms (typically ^{85}Rb) have been used to interact with a superconducting cavity with $Q = 4 \times 10^{10}$. The photon lifetime inside the cavity is in order $\tau_{ph} \sim 130$ *ms*, and the coupling strength is around $g = 2\pi \times 50$ *kHz*. We consider the numerical values considered in our proposal ($k = 51$, $X_0 = 10.5$). Then the direct calculation shows that the time required to implement the phase gate is $\tau \simeq 10.8$ *ms* which is much shorter than the photon lifetime τ_{ph}.

Secondly, we note that this gate is based on the marked state $|1,1,1,0\rangle$, and we can built the others fifteen gates for the marking job by application of single-qubit rotations. For example the transformation operator Q_π that corresponds to the marked state $|1,1,1,1\rangle$ (already defined in the section 2) can be built as:

$$Q_\pi = (I_1 \otimes I_2 \otimes I_3 \otimes \sigma_{x,4}) \times U \times (I_1 \otimes I_2 \otimes I_3 \otimes \sigma_{x,4})$$

We note that U is the transformation operator that corresponds to the marked state $|1,1,1,0\rangle$, σ_x is the NOT transformation having the following matrix representation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and I_j is the identity matrix corresponding to the qubits j.

5 Fidelity

This analysis allows to fully built the quantum phase gate operation, by calculating the fidelity which is considered as a very useful tool to characterizes the performance of this operation as a deterministic gate. For two quantum systems given by the density matrices ρ_1 and ρ_2 the fidelity $F(\rho_1,\rho_2)$ can be defined as [12,13]:

$$F(\rho_1,\rho_2) = \left(Tr\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} \right)^2$$

In our case both of the wavefunction $|\Psi(t)\rangle$ describing the system in Equation 7 and the target $|\Psi(0)\rangle = |e,1,1,1,0\rangle$ are pure states; consequently, one can set $\rho_1 = |\Psi(t)\rangle\langle\Psi(t)|$ and $\rho_2 = |e,1,1,1,0\rangle\langle e,1,1,1,0|$. The Fidelity $F(\rho_1,\rho_2)$ then is nothing but the probability of the system to be in the state $|e,1,1,1,0\rangle$. The fidelity becomes:

$$F(|\Psi\rangle\langle\Psi|, |e,1,1,1,0\rangle\langle e,1,1,1,0|) = |\langle e,1,1,1,0|\,\Psi\rangle|^2$$

In Figure 4 ,we show the plot of fidelity as a function of $X_1 = \frac{\delta}{g_1}$ and $X_3 = \frac{\delta}{g_3}$. We remark that quite promising result are obtained over a broad range of system parameters and the correspondingly fidelity are always higher $(F > 96\%)$

6 Conclusion

In conclusion, we have proposed a scheme to realize a cavity QED based five-qubit quantum phase gate which may simplify the implementation of certain quantum computing problems. It consists of a five-level atom in a cascade configuration initially in its lower state passing through a four-mode cavity. The photons in the four modes represent the four qubits. In general, the system must have a strong atom-field interaction and have a lifetime longer than the required interaction time. The multi-level cascade atomic structure is realistic and has been investigated a lot in the past [14,15,16].

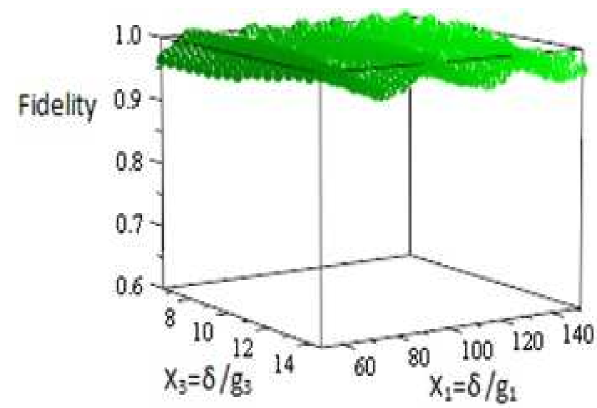

Fig. 4: The plot of fidelity as a function of $\frac{\delta}{g_1}$ and $\frac{\delta}{g_3}$.

Nevertheless, our scheme requires that the cavity runs in the regime of strong coupling, which generally requires the optical modes to be confined in a small mode volume for extended periods of time. Therefore, the cavity must have an extremely high Q factor. Various modern cavities with a high Q factor have been fabricated to realize strong coupling (see for example [17,18,19,20,21]) .

Especially, Gotzinger et al. successfully achieved the strong coupling between multiple whispering-gallery modes and two individual nanoemitters which have different center emission or absorption frequency in one silicon microsphere resonator cavity [21]. With this such development in the resonator systems, it is promising that, the strong interaction between a multimode field and a multilevel atom simultaneously inside a cavity can be experimentally performed.

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