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Asymptotic Quantile Tukey-Lambda-Normality induced by the Hadamard Walk in the Central Limit Theorem of the Open Quantum Random Walk with Time Dependence

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Abstract: We generalize the technique of Aljarrah et.al [Mohammad A Aljarrah, Carl Lee and Felix Famoye, On generating T-X family of distributions using quantile functions, Journal of Statistical Distributions and Applications 2014, 1:2] and show a special distribution arising from this technique, a so-called Quantile Tukey-Lambda-Normal distribution, gives the exact asymptotic behavior of the open quantum random walk in the central limit theorem with time dependence induced by the Hadamard walk as observed in[Clement Boateng Ampadu(2017) Asymptotic Behavior of the Hadamard Walk in the Central Limit Theorem of the Open Quantum Random Walk with Time Dependence SF J Quan Phy 1:1], under certain conditions.

Keywords: quantile, tukey-lambda distribution, normal distribution, open quantum random walk, time dependence, central limit theorem, hadamard walk

1 Introduction

1.1 *The Beta Generated Distribution*

Eugene et.al [Eugene, N, Lee, C, Famoye, F: The beta-normal distribution and its applications. Communications in Statistics-Theory and Methods 31(4), 497?512 (2002)] proposed the beta-generated family of distributions, where the beta distribution with PDF say b is used as the generator. The CDF of the beta generated distribution is then defined as

$$G(x) = \int_0^{F(x)} b(t)dt$$

where F is the CDF of any random variable. If X is continuous, the corresponding PDF of the beta generated distribution is

$$g(x) = \frac{f(x)}{B(\alpha,\beta)} F^{\alpha-1}(x)(1-F(x))^{\beta-1}$$

$\alpha > 0, \beta > 0$, where $B(\alpha,\beta)$ is the beta function. The PDF of the beta-generated distribution can be considered as a generalization of the distribution of order statistic [Jones, MC: Families of distributions arising from distributions of order statistics. Test 13, 1?43 (2004); Eugene, N, Lee, C, Famoye, F: The beta-normal distribution and its applications. Communications in Statistics-Theory and Methods 31(4), 497?512 (2002)]. By applying different $F(x)$ many authors have studied variants of the beta-generated distribution and its applications, and for examples, see [Akinsete, A, Famoye, F, Lee, C: The beta-Pareto distribution. Statistics 42, 547?563 (2008); Cordeiro, GM, Lemonte, AJ: The β-Birnbaum-Saunders distribution: an improved distribution for fatigue life modeling. Computational Statistics and Data Analysis 55(3), 1445?1461 (2011); Alshawarbeh, A, Lee, C, Famoye, F: The beta-Cauchy distribution. Journal of Probability and Statistical Science 10, 41?58 (2012)].

1.2 *The $T-X(W)$ Family of Distributions*

Alzaatreh et.al [Alzaatreh, A, Lee, C, Famoye, F: A new method for generating families of continuous distributions. Metron 71(1), 63?79 (2013b)] proposed a general method by replacing the beta PDF of Eugene et.al

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[Eugene, N, Lee, C, Famoye, F: The beta-normal distribution and its applications. Communications in Statistics-Theory and Methods 31(4), 497?512 (2002)] with a general PDF say r of a continuous random variable say T and replacing $F(x)$, the CDF of X, with a weighted version, $W(F(x))$, where $W(F(x))$ admits the following properties

$(a)W(F(x)) \in [a,b]$
$(b)W$ is a monotone increasing and differentiable function
$(c)\lim_{x\to -\infty} W(F(x)) = a$ and $\lim_{x\to+\infty} W(F(x)) = b$

where $[a,b]$ is the support of the random variable T for $-\infty \leq a < b \leq \infty$. The CDF of the $T-X(W)$ family is then defined as

$$G(x) = \int_a^{W(F(x))} r(t)dt$$

If R is the CDF of T, then the CDF of the $T-X(W)$ family can be written as

$$G(x) = R(W(F(x)))$$

and the corresponding PDF (if it exists) can be written as

$$g(x) = r(W(F(x)))\frac{d}{dx}W(F(x))$$

By applying different $F(x)$ and W, variants of the $T-X(W)$ family have been investigated, and for examples, see [Alzaatreh, A, Famoye, F, Lee, C: Gamma-Pareto distribution and its applications. Journal of Modern Applied Statistical Methods 11(1), 78?94 (2012a); Alzaatreh, A, Lee, C, Famoye, F: On the discrete analogues of continuous distributions. Statistical Methodology 9, 589?603 (2012b); Alzaatreh, A, Famoye, F, Lee, C: Weibull-Pareto distribution and its applications. Communications in Statistics-Theory and Methods 42, 1673?1691 (2013a)].

1.3 The $T-X(Y)$ Family of Distributions

Aljarrah et.al [Mohammad A Aljarrah, Carl Lee and Felix Famoye, On generating T-X family of distributions using quantile functions, Journal of Statistical Distributions and Applications 2014, 1:2] proposed a generalization of the method of Alzaatreh et.al [Alzaatreh, A, Lee, C, Famoye, F: A new method for generating families of continuous distributions. Metron 71(1), 63?79 (2013b)] by introducing a new weight function that is based on the quantile function associated with a random variable Y. Let Q_Y be the quantile function associated with the random variable Y whose cumulative distribution function (CDF) is continuous and strictly increasing, then the CDF of the $T-X(Y)$ family is then defined as

$$G(x) = \int_a^{Q_Y(F(x))} r(t)dt$$

where $r(t)$ is the probability density function (PDF) of random variable $T \in [a,b]$, for $-\infty \leq a < b \leq \infty$, and $F(x)$ is the CDF of any random variable X. If R is the CDF of T, then the CDF of the $T-X(Y)$ family can be written as

$$G(x) = R(Q_Y(F(x)))$$

and the corresponding PDF is given by

$$g(x) = \frac{f(x)}{p(Q_Y(F(x)))} r(Q_Y(F(x)))$$

where p is the PDF of Y. Variants of the $T-X(Y)$ family have been explored, for example, see [Mohammad A. Aljarrah, Felix Famoye, and Carl Lee, A New Weibull-Pareto Distribution, Communications in Statistics - Theory and Methods Vol. 44 , Iss. 19,2015].

2 Generalization of $T-X(Y)$ Family

In this section, we introduce a weight function which is more general than Q_Y, and obtain the technique of Aljarrah et.al [Mohammad A Aljarrah, Carl Lee and Felix Famoye, On generating T-X family of distributions using quantile functions, Journal of Statistical Distributions and Applications 2014, 1:2] as Corollary.

Definition 21*Let V be any function such that the following holds:*

$(a)F(x) \in [V(a),V(b)]$
$(b)F(x)$ *is differentiable and strictly increasing*
$(c)\lim_{x\to -\infty} F(x) = V(a)$ *and* $\lim_{x\to\infty} F(x) = V(b)$

then the CDF for this generalization of the $T-X(Y)$ family which we call the ***T-X family induced by V*** *is given by*

$$G(x) = \int_a^{V(F(x))} r(t)dt$$

where $r(t)$ is the PDF of random variable $T \in [a,b]$, for $-\infty \leq a < b \leq \infty$, and $F(x)$ is the CDF of any random variable X.

Theorem 22*The CDF of the* ***T-X family induced by V*** *is given by $G(x) = R[V(F(x))]$*

*Proof.*Follows from the definition of $G(x)$ and noting that $R(t)$ is the CDF of the random variable T, thus, $R' = r$

Theorem 23*The PDF of the* ***T-X family induced by V*** *is given by*

$$g(x) = r[V(F(x))]V'[F(x)]f(x)$$

*Proof.*Note that $g(x) = \frac{dG}{dx}$, $R' = r$, and $F' = f$, and G is given by Theorem 2.2

Corollary 24*The CDF of the $T-X(Y)$ family is given by $G(x) = R[Q_Y(F(x))]$*

*Proof.*Let $V = Q_Y$ in Theorem 2.2, where Q_Y is the quantile function of Y whose CDF is continuous and strictly increasing

Corollary 25*The PDF of the $T-X(Y)$ family is given by* $g(x) = \frac{f(x)}{p(Q_Y(F(x)))} r[Q_Y(F(x))]$

*Proof.*Let $V = Q_Y$, where P is the CDF of Y and p is the PDF of Y. Since

$$P(Q_Y(x)) = x$$

then

$$P'(Q_Y(x))Q'_Y(x) = 1$$

Since $P' = p$ and $Q'_Y = V'$, we deduce that

$$Q'_Y(x) = V'(x) = \frac{1}{p(Q_Y(x))}$$

So by Theorem 2.3, the result follows.

3 The $q_T - X$ Family induced by V

Consider the CDF of the $T-X$ family induced by V from Definition 2.1 which is given by

$$G(x) = \int_a^{V(F(x))} r(t)dt$$

Since T has an absolutely continuous distribution with PDF $r(t)$ and CDF $R(t)$, then the quantile function $Q(t)$ is written as $Q(t) = R^{-1}(t)$, $0 < t < 1$, and the quantile density function is written as $q(t) = \frac{dQ(t)}{dt} = \frac{1}{r(Q(t))}$, $0 < t < 1$. Now replacing the integrand of $T-X$ family induced by V with the quantile density function associated with T we get the following

Definition 31*Let V be any function such that the following holds:*

$(a)F(x) \in [V(a), V(b)]$
$(b)F(x)$ *is differentiable and strictly increasing*
$(c)\lim_{x\to -\infty} F(x) = V(a)$ *and* $\lim_{x\to\infty} F(x) = V(b)$

then the CDF for this generalization of the $T-X$ family induced by V which we call the $q_T - X$ ***family induced by V*** *is given by*

$$K(x) = \int_a^{V(F(x))} \frac{1}{r(Q(t))} dt$$

where $\frac{1}{r(Q(t))}$ is the quantile density function of random variable $T \in [a,b]$, for $-\infty \le a < b \le \infty$, and $F(x)$ is the CDF of any random variable X.

Theorem 32*The CDF of the $q_T - X$* ***family induced by V*** *is given by*

$$K(x) = Q[V(F(x))]$$

*Proof.*Follows from the previous definition and noting that $Q' = \frac{1}{r\circ Q}$

Theorem 33*The PDF of the $q_T - X$* ***family induced by V*** *is given by*

$$k(x) = \frac{f(x)}{r[Q(V(F(x)))]} V'[F(x)]$$

Proof.$k = K'$, $Q' = \frac{1}{r\circ Q}$, $F' = f$, and K is given by Theorem 3.2

4 The Tukey-Lambda Distribution

According to [Mahmoud Aldeni , Carl Lee and Felix Famoye, Families of distributions arising from the quantile of generalized lambda distribution, Journal of Statistical Distributions and Applications (2017) 4:25], the four-parameter generalized lambda distribution is defined in terms of its quantile function, this distribution was proposed by [Ramberg, J.S., Schmeiser, B.W.: An approximate method for generating asymmetric random variables. Communications of the ACM. 17(2), 78?82 (1974)]. In particular with parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $0 < u < 1$, the quantile function is given by

$$Q_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}$$

When $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \lambda_4$, we obtain the Tukey lambda distribution [Tukey, J.W.: The practical relationship between the common transformations of percentages of counts and of amounts, Technical Report 36. Princeton University, Statistical Techniques Research Group (1960)], and write

$$Q_\lambda(u) = \frac{u^\lambda - (1-u)^\lambda}{\lambda}$$

where $0 < u < 1$ and $\lambda \neq 0$ and the corresponding quantile density function is given by

$$q_\lambda(u) = (1-u)^{-1+\lambda} + u^{-1+\lambda}$$

where $0 < u < 1$ and $\lambda \neq 0$

5 The Quantile Tukey-Lambda-X Family induced by V

Assuming $V(F(x)) = F(x)$, then from Definition 3.1, the CDF of the Quantile Tukey-Lambda-X Family has the following integral representation

$$K(x;\lambda) = \int_0^{F(x)} (1-t)^{-1+\lambda} + t^{-1+\lambda} dt$$

Theorem 51*The CDF of the Quantile Tukey-Lambda-X Family induced by*

$$V(F(x)) = F(x)$$

has the following explicit representation

$$K(x;\lambda) = \frac{F(x)^{\lambda} - (1-F(x))^{\lambda}}{\lambda}$$

where $F(x)$ is the CDF of any random variable X and $\lambda \in (-\infty,0)\cup(0,\infty)$

*Proof.*Since $V(F(x)) = F(x)$ and $Q_{\lambda}(u) = \frac{u^{\lambda}-(1-u)^{\lambda}}{\lambda}$, the result follows from Theorem 3.2

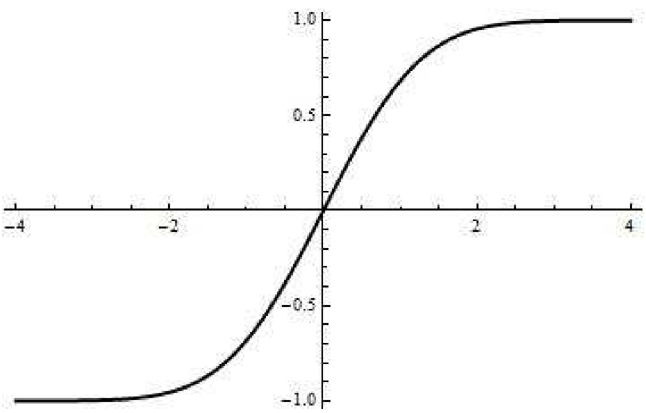

Fig. 1: The graph of $K(x;1)$ when $F(x)$ is the CDF of the Standard Normal Distribution

Theorem 52*The PDF of the Quantile Tukey-Lambda-X Family induced by*

$$V(F(x)) = F(x)$$

has the following explicit representation

$$k(x;\lambda) = f(x)\left\{(1-F(x))^{-1+\lambda} + F(x)^{-1+\lambda}\right\}$$

where $f(x)$ and $F(x)$ are the PDF and CDF, respectively, of any random variable X, and $\lambda \in (-\infty,0)\cup(0,\infty)$

*Proof.*Follows from differentiating the CDF given by the previous Theorem

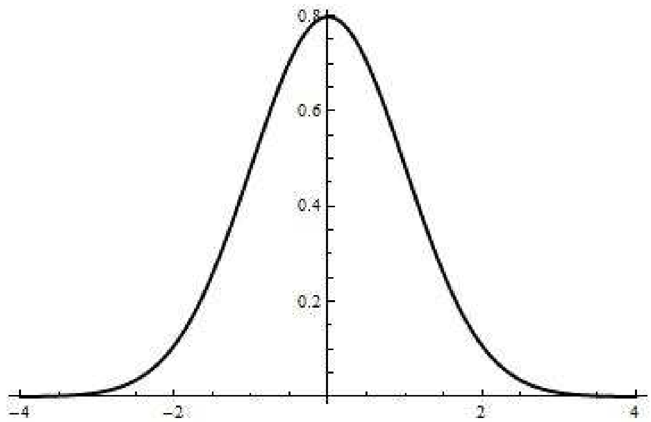

Fig. 2: The graph of $k(x;1)$ when $f(x)$ and $F(x)$ are the PDF and CDF, respectively, of the Standard Normal Distribution

6 The Discrete Analogue

Using discretization criterion to obtain the discrete analogue of continuous distributions is popular in statistical distribution theory, and for a survey of methods the reader should consult [Subrata Chakraborty, Generating discrete analogues of continuous probability distributions-A survey of methods and constructions, Journal of Statistical Distributions and Applications (2015) 2:6]. In [Roy, D: The discrete normal distribution. Commun. Stat. Theor. Methods. 32(10), 1871?1883 (2003)] the discrete analogue of the normal distribution was introduced as follows:

$$P(Y=k) = \Phi\left(\frac{k+1-\mu}{\sigma}\right) - \Phi\left(\frac{k-\mu}{\sigma}\right)$$

for $k = \cdots,-2,-1,0,1,2,\cdots;$ where $\sigma > 0;$ $-\infty < \mu < +\infty;$ $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

If $F(\cdot)$ is the cumulative distribution function of the standard normal distribution, then it follows from Theorem 5.1 that the discrete analogue of the Quantile Tukey-Lambda-Normal distribution is given by

$$P(Y=k) = \left[\frac{F\left(\frac{k+1-\mu}{\sigma}\right)^{\lambda} - \left(1-F\left(\frac{k+1-\mu}{\sigma}\right)\right)^{\lambda}}{\lambda}\right] - \left[\frac{F\left(\frac{k-\mu}{\sigma}\right)^{\lambda} - \left(1-F\left(\frac{k-\mu}{\sigma}\right)\right)^{\lambda}}{\lambda}\right]$$

for $k = \cdots,-2,-1,0,1,2,\cdots;$ where $\sigma > 0;$ $-\infty < \mu < +\infty;$ $F(\cdot)$ is the cumulative distribution function of the standard normal distribution.

7 The Hadamard Walk Revisited

Recall from [Clement Boateng Ampadu(2017) Asymptotic Behavior of the Hadamard Walk in the

Central Limit Theorem of the Open Quantum Random Walk with Time Dependence SF J Quan Phy 1:1] we introduced the continuous-time open quantum walk in the central limit theorem and used a similar discretization process as described in the previous section to enable us answer the open question in the conclusions section of Chaobin Liu et.al [arXiv:1604.05652v1 [quant-ph] 19 Apr 2016] in which the swap operator is related to the Hadamard gate. In particular with

$$P(x,t) := \Phi\left(\frac{x+1}{\sqrt{3cosh(2t)}}\right) - \Phi\left(\frac{x}{\sqrt{3cosh(2t)}}\right)$$

where $x = \cdots, -2, -1, 0, 1, 2, \cdots$, $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, we gave an alternate answer to the question in Liu et.al [arXiv:1604.05652v1 [quant-ph] 19 Apr 2016] by examining $\lim_{t\to\infty} \frac{P(x,t)}{\sqrt{t}}$, the graph below (in green) showed in the CLT of the OQRW with time-dependence, the asymptotic behavior is close to normal.

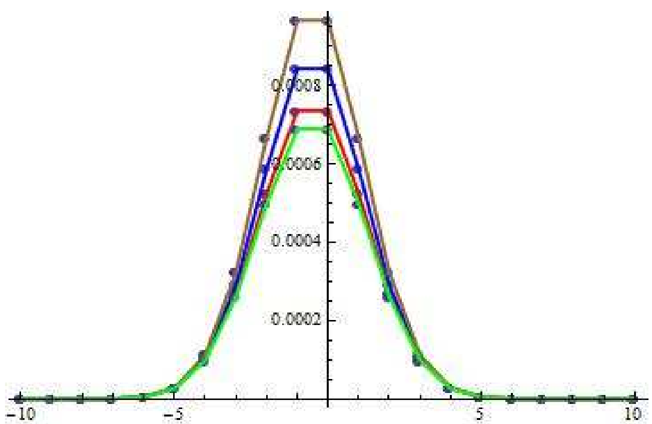

Fig. 3: Asymptotic Normality (green) as observed in [Clement Boateng Ampadu(2017) Asymptotic Behavior of the Hadamard Walk in the Central Limit Theorem of the Open Quantum Random Walk with Time Dependence SF J Quan Phy 1:1] versus Asymptotic Quantile Tukey-Lambda-Normality (brown, blue, red)

Now put

$$J(x,t,\lambda) := \left[\frac{F\left(\frac{x+1}{\sqrt{3cosh(2t)}}\right)^{\lambda} - \left(1 - F\left(\frac{x+1}{\sqrt{3cosh(2t)}}\right)\right)^{\lambda}}{\lambda}\right] - \left[\frac{F\left(\frac{x}{\sqrt{3cosh(2t)}}\right)^{\lambda} - \left(1 - F\left(\frac{x}{\sqrt{3cosh(2t)}}\right)\right)^{\lambda}}{\lambda}\right] \quad (1)$$

where $x = \cdots, -2, -1, 0, 1, 2, \cdots$, $F(\cdot)$ is the cumulative distribution function of the standard normal distribution. When we examine

$$\lim_{t\to\infty}\left\{\lim_{\lambda\to 2^-} \frac{J(x,t,\lambda)}{\sqrt{t}}\right\}$$

it is observed that the asymptotic behavior becomes EXACTLY as observed in [Clement Boateng Ampadu(2017) Asymptotic Behavior of the Hadamard Walk in the Central Limit Theorem of the Open Quantum Random Walk with Time Dependence SF J Quan Phy 1:1]. Note that in the above figure we have the following

$$\lim_{t\to\infty} \frac{J(x,t,1.5)}{\sqrt{t}} = \textit{Graph in Brown}$$

$$\lim_{t\to\infty} \frac{J(x,t,1.7)}{\sqrt{t}} = \textit{Graph in Blue}$$

$$\lim_{t\to\infty} \frac{J(x,t,1.9)}{\sqrt{t}} = \textit{Graph in Red}$$

$$\lim_{t\to\infty} \frac{P(x,t)}{\sqrt{t}} = \lim_{t\to\infty} \frac{J(x,t,2)}{\sqrt{t}} = \textit{Graph in Green}$$

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