Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040403

On the Local M-Derivative

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Received: 18 Jan. 2018, Revised: 8 Feb. 2018, Accepted: 23 Mar. 2018
Published online: 1 Oct. 2018

Abstract: We denote a new differential operator by $\mathscr{D}_M^{\alpha,\beta}(\cdot)$, where the parameter α, associated with the order, is such that $0 < \alpha < 1$, $\beta > 0$ and M is used to denote that the function to be derived involves a Mittag-Leffler function with one parameter. This new derivative satisfies some properties of integer-order calculus, e.g. linearity, product rule, quotient rule, function composition and the chain rule. Besides as in the case of the Caputo derivative, the derivative of a constant is zero. Because Mittag-Leffler function is a natural generalization of the exponential function, we can extend some of the classical results, namely: Rolle's theorem, the mean-value theorem and its extension. We present the corresponding M-integral from which, as a natural consequence, new results emerge which can be interpreted as applications. Specifically, we generalize the inversion property of the fundamental theorem of calculus and prove a theorem associated with the classical integration by parts. Finally, we present an application involving linear differential equations by means of local M-derivative with some graphs.

Keywords: Local M-derivative, local M-differential equation, M-integral, Mittag-Leffler function.

1 Introduction

The integral and differential calculus of integer-order developed by Leibniz and Newton was a great discovery in mathematics, having numerous applications in several areas of physics, biology, engineering and others. But something intriguing and interesting to the mathematicians of the day was still to come. In 1695 [1,2,3], ℓ'Hospital, in a letter to Leibniz, asked him about the possibility of extending the meaning of an integer-order derivative $d^n y/dx^n$ to the case in which the order is a fraction. This question initiated the history of a new calculus which was called arbitrary order calculus and which nowadays is usually called fractional calculus.

Although fractional calculus emerged at the same time as the integer-order calculus proposed by Newton and Leibniz, it did not attract the attention of the scientific community and for many years remained hidden. It was only after an international congress in 1974 that fractional calculus began to be known and consolidated in numerous applications in several fields such as applied mathematic, physics, biology and engineering.

Several types of fractional derivatives have been introduced to date, among which the Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, Riesz and other types [4,5,6]. Most of these derivatives are defined on the basis of the corresponding fractional integral in the Riemann-Liouville sense.

Recently, Khalil et al. [7] proposed the conformable derivative with order α, $0 < \alpha < 1$, in order to generalize properties of classical calculus. Some applications of the conformable derivative and the alternative fractional derivative are gaining space in the field of the fractional calculus and numerous works using such derivatives are published of which we mention: the heat equation, the Taylor formula and some inequalities of convex functions [8,9]. More recently, in 2014, Katugampola [10] also proposed a new fractional derivative with classical properties, similar to the conformable derivative.

The main motivation to propose a new derivative called M-derivative, and consequently to study property and theorems as well as to realize applications comes from the recently-introduced alternative derivative [10] and some new results involving the conformable derivative [7,11,12], all of which constitute to particular cases of our results. In this sense, as an application of local M-derivative, we present the general solution of a linear differential equation with graphs. Those graphs present the particular case of the conformable derivative, and two of them the M-derivative.

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This paper is organized as follows: in section 2 we present the concepts of fractional derivatives in the Riemann-Liouville and Caputo sense, the definition of fractional derivative by Khalil et al. [7] and the alternative definition proposed by Katugampola [10], together with their properties. In section 3, our principal result, we introduce the definition of a local M-derivative involving a Mittag-Leffler function and demonstrate several theorems. In section 4 we introduce the corresponding M-integral, for which we also present several results; in particular, a generalization of the fundamental theorem of calculus. In section 5, we present the relation between the local M-derivatives, introduced here, and the alternative proposed in [10]. In section 6, we present an application involving linear differential equation by means of local M-derivative with some graphs. Concluding remarks close the paper.

2 Preliminaries

The most explored and studied fractional derivatives of fractional calculus are the so-called Riemann-Liouville and Caputo derivatives. Both types are fundamental in the study of fractional differential equations; their definitions are presented below.

Definition 1. *Let $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$ and $m-1 < \alpha \leq m$. The fractional derivative of order α of a causal function f in the Riemann-Liouville sense, $\mathscr{D}^{\alpha}_{RL} f(t)$, is defined by* [13,14,15]

$$\mathscr{D}^{\alpha}_{RL} f(t) := D^m \mathbf{J}^{m-\alpha} f(t), \tag{1}$$

or

$$\mathscr{D}^{\alpha}_{RL} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \left[\displaystyle\int_0^t f(\tau)(t-\tau)^{m-\alpha-1} d\tau \right], & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \tag{2}$$

where $D^m = d^m/dt^m$ is the usual derivative of integer-order m and $\mathbf{J}^{m-\alpha}$ is the Riemann-Liouville fractional integral. If $\alpha = 0$, we define $\mathscr{D}^0_{RL} = I$, where I is the identity operator.

Definition 2. *Let $\alpha \in \mathbb{C}$ such that $Re(\alpha) > 0$ and m the smallest integer greater than or equal to $Re(\alpha)$, with $m-1 < \alpha \leq m$. The fractional derivative of order α of a causal function f in the Caputo sense, $\mathscr{D}^{\alpha}_{C} f(t)$, is defined by* [13,14,15]

$$\mathscr{D}^{\alpha}_{C} f(t) := \mathbf{J}^{m-\alpha} D^m f(t), \; m \in \mathbb{N} \tag{3}$$

or

$$\mathscr{D}^{\alpha}_{C} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t f^{(m)}(\tau)(t-\tau)^{m-\alpha-1} d\tau, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \tag{4}$$

where D^m is the usual derivative of integer-order m, $\mathbf{J}^{m-\alpha}$ is the Riemann-Liouville fractional integral and $f^{(m)}(\tau) = \dfrac{d^m f(\tau)}{d\tau^m}$.

We present now the definitions of two new types of "*fractional*" derivatives. As we shall show later, these definitions coincide, for a particular value of their parameters, with the derivative of order one of integer-order calculus.

Definition 3. *Consider $f : [0,\infty) \to \mathbb{R}$ and $t > 0$. The conformable derivative of order α of f is given by* [7]

$$T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{5}$$

$\forall t > 0$ and $\alpha \in (0,1)$.

A function f is called α-differentiable if it has a fractional derivative. If f is α-differentiable in some interval $(0,a)$, $a > 0$ and if $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then we define

$$T_\alpha(0) = \lim_{t \to 0^+} T_\alpha f(t).$$

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Definition 4.*Consider $f:[0,\infty)\to\mathbb{R}$ and $t>0$. The alternative derivative of order α of f is given by* [10]

$$\mathscr{D}_\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon}, \tag{6}$$

$\forall t>0$ and $\alpha\in(0,1)$.

Also, if f is α-differentiable in some interval $(0,a)$, $a>0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then we define

$$\mathscr{D}_\alpha(0)=\lim_{t\to 0^+}\mathscr{D}_\alpha f(t).$$

In this paper, if both conformable and alternative derivatives of α order of a function exist, we will simply say that the function f is α-differentiable.

The main result of this paper is to discuss a new type of derivative, the local M-derivative, that generalizes the alternative derivative. The new definition seems to be a natural extension of the usual, integer-order derivative, and satisfies the eight properties mentioned above. Also, as in the case of conformable and alternative derivatives, our definition coincides with the known fractional derivatives of polynomials. Finally, we are able to define a corresponding integral for which we can prove the fundamental theorem of calculus, the inversion theorem and a theorem of integration by parts.

3 Local M-derivative

In this section we present the main definition of this article and obtain several results that generalize equivalent results valid for the alternative derivative and which bear a great similarity to the results found in classical calculus.

On the basis of this definition we could demonstrate that our local M-derivative is linear and satisfies the product rule, the composition rule for two α-differentiable functions, the quotient rule and the chain rule. We show that the derivative of a constant is zero and present "fractional" versions of Rolle's theorem, the mean-value theorem and the extended mean-value theorem. Further, the continuity of the "fractional" derivative is demonstrated, as in integer-order calculus.

Thus, let us begin with a definition, which is a generalization of the classical definition of a derivative as a special limit.

Definition 5.*Let $f:[0,\infty)\to\mathbb{R}$ and $t>0$. For $0<\alpha<1$ we define the local M-derivative of order α of function f, denoted $\mathscr{D}_M^{\alpha,\beta}f(t)$, by*

$$\mathscr{D}_M^{\alpha,\beta}f(t):=\lim_{\varepsilon\to 0}\frac{f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-f(t)}{\varepsilon}, \tag{7}$$

$\forall t>0$, *where $\mathbb{E}_\beta(\cdot)$, $\beta>0$ is the classical Mittag-Leffler function* [16,17]. Consider f is α-differentiable in $(0,a)$, $a>0$, and $\lim_{t\to 0^+}\mathscr{D}_M^{\alpha,\beta}f(t)$, we have

$$\mathscr{D}_M^{\alpha,\beta}f(0)=\lim_{t\to 0^+}\mathscr{D}_M^{\alpha,\beta}f(t).$$

Theorem 1*If $f:[0,\infty)\to\mathbb{R}$ is α-differentiable at $t_0>0$, $0<\alpha\le 1$, $\beta>0$, then f is continuous at t_0.*

Proof 1*Indeed, let us consider*

$$f\left(t_0\mathbb{E}_\beta\left(\varepsilon t_0^{-\alpha}\right)\right)-f(t_0)=\left(\frac{f\left(t_0\mathbb{E}_\beta\left(\varepsilon t_0^{-\alpha}\right)\right)-f(t_0)}{\varepsilon}\right)\varepsilon. \tag{8}$$

Taking the limit $\varepsilon\to 0$ on both sides of Eq.(8), *we have*

$$\begin{aligned}\lim_{\varepsilon\to 0}f\left(t_0\mathbb{E}_\beta\left(\varepsilon t_0^{-\alpha}\right)\right)-f(t_0)&=\lim_{\varepsilon\to 0}\left(\frac{f\left(t_0\mathbb{E}_\beta\left(\varepsilon t_0^{-\alpha}\right)\right)-f(t_0)}{\varepsilon}\right)\lim_{\varepsilon\to 0}\varepsilon\\&=\mathscr{D}_M^{\alpha,\beta}f(t)\lim_{\varepsilon\to 0}\varepsilon\\&=0.\end{aligned}$$

Then, f is continuous at t_0.

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Using the definition of the one-parameter Mittag-Leffler function, we have

$$f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)=f\left(t\sum_{k=0}^{\infty}\frac{\left(\varepsilon t^{-\alpha}\right)^k}{\Gamma\left(\beta k+1\right)}\right). \tag{9}$$

Taking the limit $\varepsilon\to 0$ on both sides of Eq.(9); since f is a continuous function, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right) &= \lim_{\varepsilon\to 0} f\left(t\sum_{k=0}^{\infty}\frac{\left(\varepsilon t^{-\alpha}\right)^k}{\Gamma\left(\beta k+1\right)}\right)\\ &= f\left(t\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\frac{\left(\varepsilon t^{-\alpha}\right)^k}{\Gamma\left(\beta k+1\right)}\right)\\ &= f\left(t\right), \end{aligned}\tag{10}$$

because when $\varepsilon\to 0$ in the Mittag-Leffler function, the only term that contributes to the sum is $k=0$, so that

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\frac{\left(\varepsilon t^{-\alpha}\right)^k}{\Gamma\left(\beta k+1\right)}=1.$$

We present here a theorem that encompasses the main classical properties of integer-order derivatives, in particular of order one. As for the chain rule, it is verified by means of an example, as we will see in the sequence.

Theorem 2 *Let $0<\alpha\leq 1$, $\beta>0$, $a,b\in\mathbb{R}$ and f,g α-differentiable for $t>0$. Then:*

1. (Linearity) $\mathscr{D}_M^{\alpha,\beta}\left(af+bg\right)(t)=a\mathscr{D}_M^{\alpha,\beta}f\left(t\right)+b\mathscr{D}_M^{\alpha,\beta}g\left(t\right)$.

Proof 2 *Using* Definition 5, *we have*

$$\begin{aligned}\mathscr{D}_M^{\alpha,\beta}\left(af+bg\right)(t) &= \lim_{\varepsilon\to 0}\frac{\left(af+bg\right)\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-\left(af+bg\right)(t)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{af\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)+bg\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-af\left(t\right)-bg\left(t\right)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{af\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-af\left(t\right)}{\varepsilon}+\lim_{\varepsilon\to 0}\frac{bg\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-bg\left(t\right)}{\varepsilon}\\ &= a\mathscr{D}_M^{\alpha,\beta}f\left(t\right)+b\mathscr{D}_M^{\alpha,\beta}g\left(t\right)\end{aligned}$$

2. (Product Rule) $\mathscr{D}_M^{\alpha,\beta}\left(f\cdot g\right)(t)=f\left(t\right)\mathscr{D}_M^{\alpha,\beta}g\left(t\right)+g\left(t\right)\mathscr{D}_M^{\alpha,\beta}f\left(t\right)$.

Proof 3 *Using* Definition 5, *we have*

$$\begin{aligned}\mathscr{D}_M^{\alpha,\beta}\left(f\cdot g\right)(t) &= \lim_{\varepsilon\to 0}\frac{f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)g\left(t\right)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\left(\frac{\begin{array}{c}f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)+f\left(t\right)g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-\\ -f\left(t\right)g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)g\left(t\right)\end{array}}{\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0}\left(\frac{f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-f\left(t\right)}{\varepsilon}\right)\lim_{\varepsilon\to 0}g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)+\\ &\quad+\lim_{\varepsilon\to 0}\left(\frac{g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-g\left(t\right)}{\varepsilon}\right)f\left(t\right)\\ &= \mathscr{D}_M^{\alpha,\beta}\left(f\right)(t)\lim_{\varepsilon\to 0}g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)+\mathscr{D}_M^{\alpha,\beta}\left(g\right)(t)f\left(t\right)\\ &= \mathscr{D}_M^{\alpha,\beta}\left(f\right)(t)g\left(t\right)+\mathscr{D}_M^{\alpha,\beta}\left(g\right)(t)f\left(t\right),\end{aligned}$$

because $\lim_{\varepsilon\to 0}g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)=g\left(t\right)$.

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3.(Quotient rule) $\mathscr{D}_M^{\alpha,\beta}\left(\frac{f}{g}\right)(t)=\frac{g(t)\mathscr{D}_M^{\alpha,\beta}f(t)-f(t)\mathscr{D}_M^{\alpha,\beta}g(t)}{[g(t)]^2}$.

Proof 4 *Using* Definition 5, *we have*

$$\begin{aligned}\mathscr{D}_M^{\alpha,\beta}\left(\frac{f}{g}\right)(t)&=\lim_{\varepsilon\to 0}\frac{\frac{f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)}{g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)}-\frac{f(t)}{g(t)}}{\varepsilon}\\&=\lim_{\varepsilon\to 0}\frac{\frac{g(t)f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-f(t)g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)+f(t)g(t)-f(t)g(t)}{\varepsilon}}{g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)g(t)}\\&=\frac{\lim_{\varepsilon\to 0}\frac{g(t)\left(f\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-f(t)\right)}{\varepsilon}-\lim_{\varepsilon\to 0}\frac{f(t)\left(g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)-g(t)\right)}{\varepsilon}}{\lim_{\varepsilon\to 0}g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)g(t)}\\&=\frac{g(t)\mathscr{D}_M^{\alpha,\beta}f(t)-f(t)\mathscr{D}_M^{\alpha,\beta}g(t)}{[g(t)]^2},\end{aligned}$$

because $\lim_{\varepsilon\to 0}g\left(t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)=g(t)$.

4.$\mathscr{D}_M^{\alpha,\beta}(c)=0$, *with* $f(t)=c$ *is a constant.*

Proof 5 *The result follows directly from* Definition 5.

5.*If, furthermore,* f *is differentiable, then* $\mathscr{D}_M^{\alpha,\beta}(f)(t)=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{df(t)}{dt}$.

Proof 6 *We can write*

$$\begin{aligned}t\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)&=\sum_{k=0}^{\infty}\frac{\left(\varepsilon t^{-\alpha}\right)^k}{\Gamma(\beta k+1)}=t+\frac{\varepsilon t^{1-\alpha}}{\Gamma(\beta+1)}+\frac{t\left(\varepsilon t^{-\alpha}\right)^2}{\Gamma(2\beta+1)}+\frac{t\left(\varepsilon t^{-\alpha}\right)^3}{\Gamma(3\beta+1)}+\\&\quad+\cdots+\frac{t\left(\varepsilon t^{-\alpha}\right)^n}{\Gamma(n\beta+1)}+\cdots\\&=t+\frac{\varepsilon t^{1-\alpha}}{\Gamma(\beta+1)}+O\left(\varepsilon^2\right).\end{aligned}\tag{11}$$

Using Definition 5 *and introducing the change*

$$h=\varepsilon t^{1-\alpha}\left(\frac{1}{\Gamma(\beta+1)}+O(\varepsilon)\right)\Rightarrow\varepsilon=\frac{h}{t^{1-\alpha}\left(\frac{1}{\Gamma(\beta+1)}+O(\varepsilon)\right)},$$

in Eq.(11), *we conclude that*

$$\begin{aligned}\mathscr{D}_M^{\alpha,\beta}f(t)&=\lim_{\varepsilon\to 0}\frac{f\left(t+\frac{\varepsilon t^{1-\alpha}}{\Gamma(\beta+1)}+O\left(\varepsilon^2\right)\right)-f(t)}{\varepsilon}\\&=\lim_{\varepsilon\to 0}\frac{\frac{f(t+h)-f(t)}{ht^{\alpha-1}}}{\frac{1}{\Gamma(\beta+1)}\left(1+\Gamma(\beta+1)O(\varepsilon)\right)}\\&=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\lim_{\varepsilon\to 0}\frac{\frac{f(t+h)-f(t)}{h}}{1+\Gamma(\beta+1)O(\varepsilon)}\\&=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{df(t)}{dt},\end{aligned}$$

with $\beta>0$ *and* $t>0$.

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6.(Chain rule) $\mathscr{D}_M^{\alpha,\beta}(f\circ g)(t) = f'(g(t))\mathscr{D}_M^{\alpha,\beta}g(t)$, *for* f *differentiable at* $g(t)$.

Proof 7*If* $g(t) = a$ *is a constant, then*

$$\mathscr{D}_M^{\alpha,\beta}(f\circ g)(t) = \mathscr{D}_M^{\alpha,\beta} f(g(t)) = \mathscr{D}_M^{\alpha,\beta}(f)(a) = 0.$$

On the other hand, assume that g *is not a constant in the neighborhood of* a*, that is, suppose an* $\varepsilon_0 > 0$ *such that* $g(x_1) \neq g(x_2)$, $\forall x_1, x_2 \in (a_0-\varepsilon_0, a_0+\varepsilon_0)$*. Now, since* g *is continuous at* a*, for* ε *small enough we have*

$$\begin{aligned}\mathscr{D}_M^{\alpha,\beta}(f\circ g)(a) &= \lim_{\varepsilon\to 0}\frac{f\left(g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)\right) - f(g(a))}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f\left(g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)\right) - f(g(a))}{g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right) - g(a)}\,\frac{g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right) - g(a)}{\varepsilon}.\end{aligned} \tag{12}$$

Introducing the change

$$\varepsilon_1 = g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right) - g(a) \Rightarrow g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right) = g(a) + \varepsilon_1$$

in Eq.(12)*, we conclude that*

$$\begin{aligned}\mathscr{D}_M^{\alpha,\beta}(f\circ g)(a) &= \lim_{\varepsilon_1\to 0}\frac{f\left(g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right)\right) - f(g(a))}{\varepsilon_1}\lim_{\varepsilon\to 0}\frac{g\left(a\mathbb{E}_\beta\left(\varepsilon t^{-\alpha}\right)\right) - g(a)}{\varepsilon} \\ &= f'(g(a))\,\mathscr{D}_M^{\alpha,\beta}g(a),\end{aligned}$$

with $a > 0$.

The two theorems below, present some results used in the calculus of integer order.

Theorem 3*Let* $a \in \mathbb{R}$, $\beta > 0$ *and* $0 < \alpha \leq 1$. *Then we have the following results:*

1. $\mathscr{D}_M^{\alpha,\beta}(1) = 0$.
2. $\mathscr{D}_M^{\alpha,\beta}(e^{at}) = \dfrac{t^{1-\alpha}}{\Gamma(\beta+1)}ae^{at}$.
3. $\mathscr{D}_M^{\alpha,\beta}(\sin(at)) = \dfrac{t^{1-\alpha}}{\Gamma(\beta+1)}a\cos(at)$.
4. $\mathscr{D}_M^{\alpha,\beta}(\cos(at)) = -\dfrac{t^{1-\alpha}}{\Gamma(\beta+1)}a\sin(at)$.
5. $\mathscr{D}_M^{\alpha,\beta}\left(\frac{t^\alpha}{\alpha}\right) = \dfrac{1}{\Gamma(\beta+1)}$.
6. $\mathscr{D}_M^{\alpha,\beta}(t^a) = \dfrac{a}{\Gamma(\beta+1)}t^{a-\alpha}$.

Theorem 4*Let* $0 < \alpha \leq 1$, $\beta > 0$ *and* $t > 0$. *Then, we have*

1. $\mathscr{D}_M^{\alpha,\beta}\left(\sin\left(\frac{1}{\alpha}t^\alpha\right)\right) = \dfrac{\cos\left(\frac{1}{\alpha}t^\alpha\right)}{\Gamma(\beta+1)}$.
2. $\mathscr{D}_M^{\alpha,\beta}\left(\cos\left(\frac{1}{\alpha}t^\alpha\right)\right) = -\dfrac{\sin\left(\frac{1}{\alpha}t^\alpha\right)}{\Gamma(\beta+1)}$.
3. $\mathscr{D}_M^{\alpha,\beta}\left(e^{\frac{t^\alpha}{\alpha}}\right) = \dfrac{e^{\frac{t^\alpha}{\alpha}}}{\Gamma(\beta+1)}$.

The identities in Theorem 3 and Theorem 4 are direct consequences of item 5 of Theorem 2.

We now prove the extensions, for α-differentiable functions in the sense of the local M-derivative defined in Eq.(7), associated with the Rolle's theorem and the mean-value and extended mean-value theorems.

Theorem 5(Rolle's theorem for α-differentiable functions) *Let* $a > 0$ *and* $f : [a,b] \to \mathbb{R}$ *be a function such that:*

1. f *is continuous on* $[a,b]$*;*
2. f *is* α*-differentiable on* (a,b) *for some* $\alpha \in (0,1)$*;*

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3. $f(a) = f(b)$.

Then, there exists $c \in (a,b)$, such that $\mathscr{D}_M^{\alpha,\beta} f(c) = 0$, $\beta > 0$.

Proof 8 *Since f is continuous on $[a,b]$ and $f(a) = f(b)$, there exists a point $c \in (a,b)$ at which function f has a local extreme. Then,*

$$\mathscr{D}_M^{\alpha,\beta} f(c) = \lim_{\varepsilon \to 0^-} \frac{f\left(c\mathbb{E}_\beta\left(\varepsilon c^{-\alpha}\right)\right) - f(c)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{f\left(c\mathbb{E}_\beta\left(\varepsilon c^{-\alpha}\right)\right) - f(c)}{\varepsilon}.$$

As $\lim_{\varepsilon \to 0^\pm} \mathbb{E}_\beta\left(\varepsilon c^{-\alpha}\right) = 1$, the two limits in the right hand side of this equation have opposite signs. Hence, $\mathscr{D}_M^{\alpha,\beta} f(c) = 0$.

Theorem 6 (Mean-value theorem for α-differentiable functions) *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a function such that:*

1. *f is continuous on $[a,b]$;*
2. *f is α-differentiable on (a,b) for some $\alpha \in (0,1)$.*

Then, there exists $c \in (a,b)$ given by

$$\mathscr{D}_M^{\alpha,\beta} f(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}},$$

with $\beta > 0$.

Proof 9 *Consider $g(x)$ as*

$$g(x) = f(x) - f(a) - \Gamma(\beta+1)\left(\frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\right)\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right). \tag{13}$$

As g satisfies the conditions of Rolle's theorem, there exists $c \in (a,b)$ such that $\mathscr{D}_M^{\alpha,\beta} f(c) = 0$. Applying the local M-derivative $\mathscr{D}_M^{\alpha,\beta}$ to both sides of Eq.(13) *and using the fact that $\mathscr{D}_M^{\alpha,\beta}\left(\frac{t^\alpha}{\alpha}\right) = \frac{1}{\Gamma(\beta+1)}$ and $\mathscr{D}_M^{\alpha,\beta}(c) = 0$, c is a constant, we conclude that*

$$\mathscr{D}_M^{\alpha,\beta} f(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}.$$

Theorem 7 (Extended mean-value theorem) *Let $a > 0$ and $f, g : [a,b] \to \mathbb{R}$ functions such that:*

1. *f, g are continuous on $[a,b]$;*
2. *f, g are α-differentiable for some $\alpha \in (0,1)$.*

Then, there exists $c \in (a,b)$ such that

$$\frac{\mathscr{D}_M^{\alpha,\beta} f(c)}{\mathscr{D}_M^{\alpha,\beta} g(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \tag{14}$$

with $\beta > 0$.

Proof 10 *Consider the function*

$$F(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)(g(x) - g(a)). \tag{15}$$

As F is a continuous function on $[a,b]$, α-differentiable on (a,b) and $F(a) = 0 = F(b)$, by means of Rolle's theorem there exists $c \in (a,b)$ such that $\mathscr{D}_M^{\alpha,\beta} F(c) = 0$ for some $\alpha \in (0,1)$. Then, applying the local M-derivative $\mathscr{D}_M^{\alpha,\beta}$ to both sides of Eq.(15) *and using the fact that $\mathscr{D}_M^{\alpha,\beta}(c) = 0$ where c is a constant, we conclude that*

$$\frac{\mathscr{D}_M^{\alpha,\beta} f(c)}{\mathscr{D}_M^{\alpha,\beta} g(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

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Definition 6. *We consider $\beta > 0$, $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$ and f n times differentiable* (in the classical sense) for $t > 0$. Then the local M-derivative of order n of f is defined by

$$\mathscr{D}_M^{\alpha,\beta;n} f(t) := \lim_{\varepsilon \to 0} \frac{f^{(n)}\left(t\mathbb{E}_\beta\left(\varepsilon t^{n-\alpha}\right)\right) - f^{(n)}(t)}{\varepsilon}, \tag{16}$$

if and only if the limit exists.

From Definition 6 and the chain rule, using induction on n, we can prove that $\mathscr{D}_M^{\alpha,\beta,n} f(t) = \dfrac{t^{n+1-\alpha}}{\Gamma(\beta+1)} f^{(n+1)}(t)$, $\alpha \in (n, n+1]$ and so f is $(n+1)$-differentiable for $t > 0$.

4 M-Integral

Here, we introduce the concept of M-integral of a function f. From this definition we can prove some results similar to classical results such as the inverse property, the fundamental theorem of calculus and the theorem of integration by parts. Other results about the M-integral are also presented. In preparing this section we made extensive use of references [7, 10, 11, 12].

Definition 7. (M-integral) *Let $a \geq 0$ and $t \geq a$. Consider f defined in $(a,t]$ and $0 < \alpha < 1$. Then, the M-integral of order α of a function f is given by*

$${}_M\mathscr{I}_a^{\alpha,\beta} f(t) = \Gamma(\beta+1) \int_a^t \frac{f(x)}{x^{1-\alpha}} dx, \tag{17}$$

with $\beta > 0$.

Theorem 8 (Inverse) *Consider $a \geq 0$ and $0 < \alpha < 1$. Also, consider f a continuous function such that there exists ${}_M\mathscr{I}_a^{\alpha,\beta} f$. Then*

$$\mathscr{D}_M^{\alpha,\beta}\left({}_M\mathscr{I}_a^{\alpha,\beta} f(t)\right) = f(t), \tag{18}$$

with $t \geq a$ and $\beta > 0$.

Proof 11 *Indeed, using the chain rule proved in* Theorem 2 *we have*

$$\begin{aligned}
\mathscr{D}_M^{\alpha,\beta}\left({}_M\mathscr{I}_a^{\alpha,\beta} f(t)\right) &= \frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{d}{dt}\left({}_M\mathscr{I}_a^{\alpha,\beta} f(t)\right) \\
&= \frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{d}{dt}\left(\Gamma(\beta+1)\int_a^t \frac{f(x)}{x^{1-\alpha}} dx\right) \\
&= \frac{t^{1-\alpha}}{\Gamma(\beta+1)}\left(\frac{\Gamma(\beta+1)}{t^{1-\alpha}} f(t)\right) \\
&= f(t).
\end{aligned} \tag{19}$$

We now prove the fundamental theorem of calculus in the sense of the M-derivative mentioned at the beginning of the paper.

Theorem 9 (Fundamental theorem of calculus) *Let $f : (a,b) \to \mathbb{R}$ be an α-differentiable function and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$${}_M\mathscr{I}_a^{\alpha,\beta}\left(\mathscr{D}_M^{\alpha,\beta} f(t)\right) = f(t) - f(a),$$

with $\beta > 0$.

Proof 12 *In fact, since function f is differentiable, using the chain rule (non-integer order) and the fundamental theorem of calculus, we have*

$$\begin{aligned}
{}_M\mathscr{I}_a^{\alpha,\beta}\left(\mathscr{D}_M^{\alpha,\beta} f(t)\right) &= \Gamma(\beta+1)\int_a^t \frac{\mathscr{D}_M^{\alpha,\beta} f(t)}{x^{1-\alpha}} dx \\
&= \Gamma(\beta+1)\int_a^t \frac{x^{1-\alpha}}{\Gamma(\beta+1)} \frac{1}{x^{1-\alpha}} \frac{df(t)}{dt} dx \\
&= \int_a^t \frac{df(t)}{dt} dx \\
&= f(t) - f(a)
\end{aligned} \tag{20}$$

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If the condition $f(a)=0$ holds, then by Theorem 9, Eq.(20), we have ${}_M\mathscr{I}_a^{\alpha,\beta}\left[\mathscr{D}_M^{\alpha\beta}f(t)\right]=f(t)$.

Theorem 9 can be generalized to a larger order as follows.

Theorem 10*Let $\alpha\in(n,n+1]$ and $f:[a,\infty)\to\mathbb{R}$ be $(n+1)$-times differentiable for $t>a$. Then, $\forall\, t>a$, we have*

$$ {}_M\mathscr{I}_a^{\alpha,\beta}\left(\mathscr{D}_M^{\alpha,\beta}f(t)\right)=f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(a)(t-a)^k}{k!}, $$

with $\beta>0$.

Proof 13*Using the definition of M-integral and the chain rule proved in* Theorem 2*, we have*

$$\begin{aligned} {}_M\mathscr{I}_a^{\alpha,\beta}\left(\mathscr{D}_M^{\alpha,\beta}f(t)\right) &= {}_M\mathscr{I}_a^{n+1}\left[(t-a)^{\beta-1}\mathscr{D}_M^{\alpha,\beta}f^{(n)}(t)\right] \\ &= {}_M\mathscr{I}_a^{n+1}\left[(t-a)^{\beta-1}(t-a)^{1-\beta}f^{(n+1)}(t)\right] \\ &= {}_M\mathscr{I}_a^{n+1}\left(f^{(n+1)}(t)\right). \end{aligned} \tag{21}$$

Then, performing piecewise integration for the integer-order derivative in Eq.(21)*, we have*

$$ {}_M\mathscr{I}_a^{\alpha,\beta}\left(\mathscr{D}_M^{\alpha,\beta}f(t)\right)=f(t)-\sum_{k=0}^{n}\frac{f^{(k)}(a)(t-a)^k}{k!}. $$

As integer-order calculus has a result known as integration by parts, we shall now present, through a theorem, a similar result which we might call fractional integration by parts.

Using the notation of Eq.(17) for the M-integral:

$$ {}_M\mathscr{I}_a^{\alpha,\beta}f(t)=\Gamma(\beta+1)\int_a^t\frac{f(x)}{x^{1-\alpha}}dx=\int_a^t f(x)\,d_\alpha x, $$

where $d_\alpha x=\dfrac{\Gamma(\beta+1)}{x^{1-\alpha}}dx$.

Theorem 11*Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that f,g are differentiable and $0<\alpha<1$. Then*

$$ \int_a^b f(x)\,\mathscr{D}_M^{\alpha,\beta}g(x)\,d_\alpha x=f(x)g(x)\,|_a^b-\int_a^b g(x)\,\mathscr{D}_M^{\alpha,\beta}f(x)\,d_\alpha x, \tag{22}$$

with $\beta>0$.

Proof 14*Indeed, using M-integral and applying the chain rule (non-integer order) and the fundamental theorem of calculus, we have*

$$\begin{aligned} \int_a^b f(x)\,\mathscr{D}_M^{\alpha,\beta}g(x)\,d_\alpha x &= \Gamma(\beta+1)\int_a^b\frac{f(x)}{x^{1-\alpha}}\mathscr{D}_M^{\alpha,\beta}g(x)\,dx \\ &= \Gamma(\beta+1)\int_a^b\frac{f(x)}{x^{1-\alpha}}\frac{x^{1-\alpha}}{\Gamma(\beta+1)}\frac{dg(x)}{dt}dx \\ &= \int_a^b f(x)g'(x)\,dx \\ &= f(x)g(x)\,|_a^b-\Gamma(\beta+1)\int_a^b\frac{g(x)}{x^{1-\alpha}}\frac{x^{1-\alpha}}{\Gamma(\beta+1)}\frac{df(x)}{dt}dx \\ &= f(x)g(x)\,|_a^b-\int_a^b g(x)\,\mathscr{D}_M^{\alpha,\beta}f(x)\,d_\alpha x, \end{aligned}$$

where $d_\alpha x=\frac{\Gamma(\beta+1)}{x^{1-\alpha}}dx$.

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Theorem 12 *Let $0 < a < b$ and let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, for $0 < \alpha < 1$ we have*

$$\left| {}_M\mathscr{I}_a^{\alpha,\beta} f(t) \right| \leq ({}_M\mathscr{I}_a^{\alpha,\beta} |f(t)|), \tag{23}$$

with $\beta > 0$.

Proof 15 *From the definition of M-integral of order α, we have*

$$\begin{aligned} \left| {}_M\mathscr{I}_a^{\alpha,\beta} f(t) \right| &= \left| \Gamma(\beta+1) \int_a^t \frac{f(x)}{x^{1-\alpha}} dx \right| \\ &\leq |\Gamma(\beta+1)| \int_a^t \left| \frac{f(x)}{x^{1-\alpha}} \right| dx \\ &= {}_M\mathscr{I}_a^{\alpha,\beta} |f(t)|. \end{aligned}$$

Corollary 1 *Considering $f : [a,b] \to \mathbb{R}$ a continuous function such that*

$$N = \sup_{t\in[a,b]} |f(t)|. \tag{24}$$

Then, $\forall t \in [a,b]$ and $0 < \alpha < 1$, we have

$$\left| {}_M\mathscr{I}_a^{\alpha,\beta} f(t) \right| \leq \Gamma(\beta+1) N \left(\frac{t^\alpha}{\alpha} - \frac{a^\alpha}{\alpha} \right), \tag{25}$$

with $\beta > 0$.

Proof 16 *By* Theorem 12*, we have*

$$\begin{aligned} \left| {}_M\mathscr{I}_a^{\alpha,\beta} f(t) \right| &\leq {}_M\mathscr{I}_a^{\alpha,\beta} |f(t)| \\ &= \Gamma(\beta+1) \int_a^t |f(x)| x^{\alpha-1} dx \\ &\leq \Gamma(\beta+1) N \int_a^t x^{\alpha-1} dx \\ &= \Gamma(\beta+1) N \left(\frac{t^\alpha}{\alpha} - \frac{a^\alpha}{\alpha} \right). \end{aligned} \tag{26}$$

5 Relation With Alternative Derivative

We discuss a possible relation between the alternative derivative and the local M-derivative proposed here.

Katugampola [10] proposed a new fractional derivative which he called alternative derivative, given by

$$\mathscr{D}^\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \tag{27}$$

with $\alpha \in (0,1)$ and $t > 0$.

It is easily seen that our definition of local M-derivative Eq.(7)) is more general than the alternative derivative Eq.(27).

The definition in Eq.(7) contains the classical Mittag-Leffler function $\mathbb{E}_\beta(\cdot)$, which is a generalization of the exponential function. Indeed, choosing $\beta = 1$ in the definition of the classical Mittag-Leffler function [16,17], we have

$$\mathbb{E}_\beta(x) = \mathbb{E}_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = e^x. \tag{28}$$

In particular, introducing $x = \varepsilon t^{-\alpha}$ in Eq.(7) and taking the limit $\varepsilon \to 0$ we recover the alternative derivative $\mathscr{D}^\alpha$:

$$\mathscr{D}_M^{\alpha,\beta} f(t) = \lim_{\varepsilon\to 0} \frac{f(t\mathbb{E}_1(\varepsilon t^{-\alpha})) - f(t)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon} = \mathscr{D}^\alpha f(t). \tag{29}$$

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6 Application

Fractional linear differential equations are important in the study of fractional calculus and applications. In this section, we present the general solution of a linear differential equation by means of the local M-derivative. In this sense, as a particular case, we study an example and perform a graph analysis of the solution.

The general first-order differential equation based on the local M-derivative is represented by

$$\mathscr{D}_M^{\alpha,\beta}u(t)+P(t)\,u(t)=Q(t)\,. \tag{30}$$

where $P(t)$, $Q(t)$ are α-differentiable functions and $u(t)$ is unknown.

Using the item 5 of Theorem (2) in the Eq.(30), we have

$$\frac{d}{dt}u(t)+\frac{\Gamma(\beta+1)}{t^{1-\alpha}}P(t)\,u(t)=\frac{\Gamma(\beta+1)}{t^{1-\alpha}}Q(t)\,. \tag{31}$$

The Eq.(31) is a first order equation, whose general solution is given by

$$u(t)=e^{-\Gamma(\beta+1)\int\frac{P(t)}{t^{1-\alpha}}dt}\left(\Gamma(\beta+1)\int\frac{Q(t)}{t^{1-\alpha}}e^{\Gamma(\beta+1)\int\frac{P(t)}{t^{1-\alpha}}dt}dt+C\right),$$

where C is an arbitrary constant.

By definition of M-integral, we conclude that the solution is given by

$$u(t)=e^{-{}_MI_a^{\alpha,\beta}(P(t))}\left({}_MI_a^{\alpha,\beta}\left(Q(t)\,e^{{}_MI_a^{\alpha,\beta}(P(t))}\right)+C\right).$$

Now let us choose some values and functions and make an example using the linear differential equation previously studied by means of the local M-derivative. Then, taking $P(t)=-\lambda$, $Q(t)=0$, $u(0)=u_0$, $a=0$, $0<\alpha\leq 1$ e $\beta>0$, we have the following linear differential equation

$$\mathscr{D}_M^{\alpha,\beta}u(t)=\lambda\,u(t)\,, \tag{32}$$

whose solution is given by

$$u(t)=u_0\,e^{\frac{-\lambda}{\alpha}\Gamma(\beta+1)t^{\alpha}}=u_0\mathbb{E}_1\left(\frac{-\lambda}{\alpha}\Gamma(\beta+1)t^{\alpha}\right),$$

where $\mathbb{E}_1(\cdot)$ is Mittag-Leffler function.

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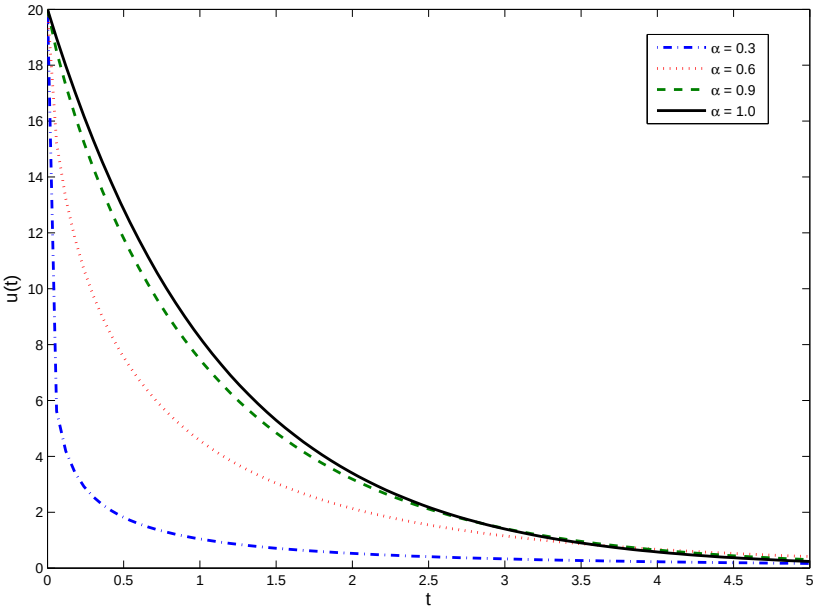

Fig. 1: Analytical solution of the Eq.(32). We consider the values $\beta = 0.5$, λ=1 and u_0=20.

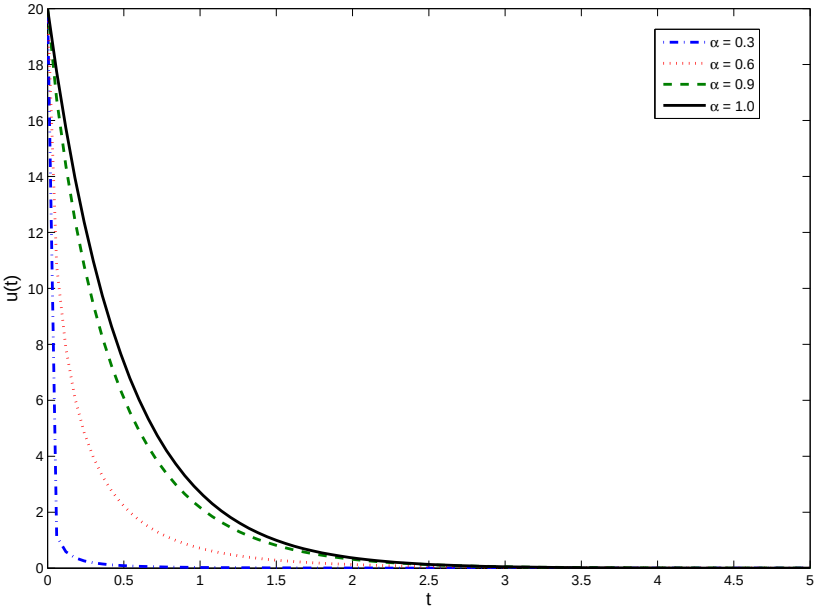

Fig. 2: Analytical solution of the Eq.(32). We take the values $\beta = 1.0$, λ=2 and u_0=20

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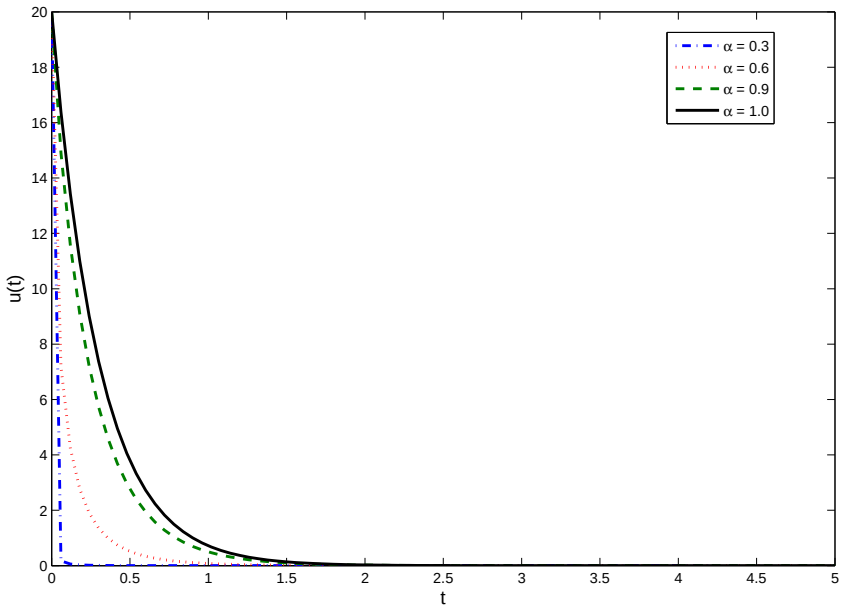

Fig. 3: Analytical solution of the Eq.(32). We choose the values $\beta = 1.5$, λ=2.5 and u_0=20.

7 Conclusion

We have introduced the so-called local M-derivative, and its corresponding M-integral. We could prove important results concerning integer-order derivatives of this kind, in particular, derivatives of order one. For α-differentiable functions in the context of local M-derivatives we could show that the derivative as proposed here behaves well with respect to the product rule, the quotient rule, composition of functions and the chain rule. The local M-derivative of a constant is zero, differently from the case of the fractional derivative in the Riemann-Liouville sense. Moreover, we present α-differentiable functions versions associated with the Rolle's theorem, the mean-value theorem and the extended mean value theorem.

An M-integral is introduced and some results bearing relations to results in the calculus of integer order are obtained, among which the M-fractional versions of the inverse theorem, the fundamental theorem of calculus and a theorem involving integration by parts.

We obtain a relation between our local M-derivative with the local alternative derivative, as presented in section 5 of the paper, as well as possible applications in several areas, particularly as we show, in the solution of a linear differential equation. We conclude from this result that the definition presented here can be considered a generalization of the so-called alternative derivative [10].

Acknowledgment

We are grateful to Dr. J. Emílio Maiorino for several and fruitful discussions and we are also grateful to the anonymous referee with their suggestions improved the manuscript.

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