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Hilfer-Hadamard Fractional Differential Equations and Inclusions Under Weak Topologies

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Abstract: In this article, by applying some Mönch's fixed-point theorems associated with the technique of measure of weak noncompactness, we prove some results concerning the existence of weak solutions for some classes of Hilfer-Hadamard fractional differential equations and inclusions.

Keywords: Differential equation, inclusion, mixed Pettis Riemann-Liouville integral of fractional order, Hilfer-Hadamard fractional derivative, weak solution, multifunction, fixed-point.

1 Introduction

Fractional differential equations and inclusions have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [1,2,3,4]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer to the monographs of Abbas *et al.* [5,6], Ahmad *et al.* [7], Samko *et al.* [8], Kilbas *et al.* [9] and Zhou [10].

The measure of weak noncompactness was introduced by De Blasi [11]. The strong measure of noncompactness was developed first by Banaŝ and Goebel [12] and subsequently developed and used in many papers; see for example, Akhmerov *et al.* [13], Alvàrez [14], Benchohra *et al.* [15], Guo *et al.* [16], and the references therein. In [15,17] the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see [6,18,19], and the references therein.

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; [20,21,2,22,23,24], and other problems with Hilfer-Hadamard fractional derivative; see [25,26]. In this article, we discuss the existence of weak solutions for the following problem of Hilfer-Hadamard fractional differential equation of the form

$$\begin{cases} ({}^H D_1^{\alpha,\beta} u)(t) = f(t,u(t));\ t \in I := [1,T], \\ \\ ({}^H I_1^{1-\gamma} u)(t)|_{t=1} = \phi, \end{cases} \tag{1}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha+\beta-\alpha\beta$, $T>1$, $\phi \in E$, $f : I \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, ${}^H I_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^H D_1^{\alpha,\beta}$ is the Hilfer-Hadamard derivative operator of order α and type β.

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Next, we consider the following problem of Hilfer-Hadamard fractional differential inclusion of the form

$$\begin{cases} ({}^H D_1^{\alpha,\beta} u)(t) \in F(t,u(t));\ t \in I, \\ \\ ({}^H I_1^{1-\gamma} u)(t)|_{t=1} = \phi, \end{cases} \tag{2}$$

where $F : I \times E \to \mathscr{P}(E)$ is a given multi-valued map, and $\mathscr{P}(E)$ is the family of all nonempty subsets of a separable Banach space E.

Our goal in this work is to give some existence results for functional Hilfer-Hadamard fractional differential equations and inclusions.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} \|v(t)\|_E.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E. We denote by $AC^1(I)$ the space defined by

$$AC^1(I) := \{w : I \to E : \frac{d}{dt} w(t) \in AC(I)\}.$$

For a function $u \in C$, set

$$\delta[u(t)] = t\frac{d}{dt}u(t).$$

Let $q > 0,\ n = [q]+1$, where $[q]$ is the integer part of q. Define the space

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(t)] \in AC(I)\}.$$

Let $\gamma \in (0,1]$, by $C_{\gamma,\ln}(I),\ C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_{\gamma,\ln}(I) = \{w(t) : (\ln t)^{1-\gamma} w(t) \in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in I} \|(\ln t)^{1-\gamma} w(t)\|_E,$$

$$C_\gamma(I) = \{w : (1,T] \to E : t^{1-\gamma} w(t) \in C\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in I} \|t^{1-\gamma} w(t)\|_E,$$

and

$$C_\gamma^1(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

with the norm

$$\|w\|_{C_\gamma^1} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

In the following we denote $\|w\|_{C_{\gamma,\ln}}$ by $\|w\|_C$. Let $(E,w) = (E,\sigma(E,E^*))$ be the Banach space E with its weak topology.

Definition 1. *A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set K whose linear span is dense in X.*

Examples:
1. Every separable Banach space is WCG.
2. Every reflexive Banach space is WCG.
3. Every $L_1(\mu)-$space, with μ being a $\sigma-$finite, non-negative measure, is WCG.

Definition 2. *A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E,w) then $h(u_n) \to h(u)$ in (E,w)).*

Definition 3. *[27] The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, $u_J = \int_J u(s)ds$).*

Let $P(I,E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I,E)$ be the Banach space of Bocher integrable functions $u : I \to E$. Define the class $P_1(I,E)$ by

$$P_1(I,E) = \{u \in P(I,E) : \varphi(u) \in L^1(I,\mathbb{R});\ for\ every\ \varphi \in E^*\}.$$

The space $P_1(I,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \le 1} \int_1^T |(\varphi u)(x)| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [[27], Theorem 3.4 and Corollary 3.41]).

Proposition 1. *[28,27] If $u \in P_1(I,E)$ and h is a measurable and essentially bounded $E-$valued function, then $uh \in P_1(I,E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral.
Now, we give some results and properties of fractional calculus.

Definition 4. *[5,9,8] (Riemann-Liouville fractional integral). The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by*

$$(I_1^r w)(t) = \frac{1}{\Gamma(r)} \int_1^t (t-s)^{r-1} w(s) ds;\ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_1^{r_1} I_1^{r_2} w)(t) = (I_1^{r_1+r_2} w)(t);\ for\ a.e.\ t \in I.$$

Definition 5. *[5,9,8] (Riemann-Liouville fractional derivative). The Riemann-Liouville fractional derivative of order $r > 0$ of a function $w \in L^1(I)$ is defined by*

$$\begin{aligned}(D_1^r w)(t) &= \left(\frac{d^n}{dt^n} I_1^{n-r} w\right)(t) \\ &= \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_1^t (t-s)^{n-r-1} w(s) ds;\ for\ a.e.\ t \in I,\end{aligned}$$

where $n = [r] + 1$ and $[r]$ is the integer part of r.

In particular, if $r \in (0,1]$, then

$$\begin{aligned}(D_1^r w)(t) &= \left(\frac{d}{dt} I_1^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_1^t (t-s)^{-r} w(s) ds;\ for\ a.e.\ t \in I.\end{aligned}$$

Let $r\in(0,1]$, $\gamma\in[0,1)$ and $w\in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows:

$$(D_1^r I_1^r w)(t)=w(t);\ for\ all\ t\in(1,T].$$

Moreover, if $I_1^{1-r}w\in C^1_{1-\gamma}(I)$, then the following composition is proved in [8]

$$(I_1^r D_1^r w)(t)=w(t)-\frac{(I_1^{1-r}w)(1^+)}{\Gamma(r)}t^{r-1};\ for\ all\ t\in(1,T].$$

Definition 6.*[5,9,8] (Caputo fractional derivative). The Caputo fractional derivative of order $r>0$ of a function $w\in AC^n(I)$ is defined by*

$$\begin{aligned}({}^cD_1^r w)(t) &= \left(I_1^{n-r}\frac{d^n}{dt^n}w\right)(t)\\ &= \frac{1}{\Gamma(n-r)}\int_1^t (t-s)^{n-r-1}\frac{d^n}{ds^n}w(s)ds;\ for\ a.e.\ t\in I.\end{aligned}$$

In particular, if $r\in(0,1]$, then

$$\begin{aligned}({}^cD_1^r w)(t) &= \left(I_1^{1-r}\frac{d}{dt}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_1^t (t-s)^{-r}\frac{d}{ds}w(s)ds;\ for\ a.e.\ t\in I.\end{aligned}$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [9] for a more detailed analysis.

Definition 7.*[9] (Hadamard fractional integral). The Hadamard fractional integral of order $q>0$ for a function $g\in L^1(I,E)$, is defined as*

$$({}^HI_1^q g)(x)=\frac{1}{\Gamma(q)}\int_1^x\left(\ln\frac{x}{s}\right)^{q-1}\frac{g(s)}{s}ds,$$

provided the integral exists.

*Example 1.*Let $0<q<1$. Then

$${}^HI_1^q \ln t=\frac{1}{\Gamma(2+q)}(\ln t)^{1+q},\ for\ a.e.\ t\in[0,e].$$

*Remark.*Let $g\in P_1(I,E)$. For every $\varphi\in E^*$, we have

$$\varphi({}^HI_1^q g)(t)=({}^HI_1^q\varphi g)(t),\ for\ a.e.\ t\in I.$$

Set

$$\delta=x\frac{d}{dx},\ q>0,\ n=[q]+1,$$

and

$$AC_\delta^n:=\{u:[1,T]\to E:\delta^{n-1}[u(x)]\in AC(I)\}.$$

Analogous to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

Definition 8.*[9] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q>0$ applied to the function $w\in AC_\delta^n$ is defined as*

$$({}^HD_1^q w)(x)=\delta^n({}^HI_1^{n-q}w)(x).$$

In particular, if $q\in(0,1]$, then

$$({}^HD_1^q w)(x)=\delta({}^HI_1^{1-q}w)(x).$$

*Example 2.*Let $0 < q < 1$. Then

$$ {}^{H}D_1^q \ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{1-q}, \; for\ a.e.\ t \in [0,e]. $$

It has been proved (see e.g. Kilbas [[29], Theorem 4.8]) that in the space $L^1(I,E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$ ({}^{H}D_1^q)({}^{H}I_1^q w)(x) = w(x). $$

From Theorem 2.3 of [9], we have

$$ ({}^{H}I_1^q)({}^{H}D_1^q w)(x) = w(x) - \frac{({}^{H}I_1^{1-q}w)(1)}{\Gamma(q)}(\ln x)^{q-1}. $$

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way:

Definition 9.*(Caputo-Hadamard fractional derivative). The Caputo-Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as*

$$ ({}^{Hc}D_1^q w)(x) = ({}^{H}I_1^{n-q}\delta^n w)(x). $$

In particular, if $q \in (0,1]$, then

$$ ({}^{Hc}D_1^q w)(x) = ({}^{H}I_1^{1-q}\delta w)(x). $$

In [2], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [22,23]).

Definition 10.*(Hilfer fractional derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, $I_1^{(1-\alpha)(1-\beta)}w \in AC^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined as*

$$ (D_1^{\alpha,\beta}w)(t) = \left(I_1^{\beta(1-\alpha)}\frac{d}{dt}I_1^{(1-\alpha)(1-\beta)}w\right)(t); \; for\ a.e.\ t \in I. \tag{3} $$

Properties. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.
1. The operator $(D_1^{\alpha,\beta}w)(t)$ can be written as

$$ (D_1^{\alpha,\beta}w)(t) = \left(I_1^{\beta(1-\alpha)}\frac{d}{dt}I_1^{1-\gamma}w\right)(t) = \left(I_1^{\beta(1-\alpha)}D_1^{\gamma}w\right)(t); \; for\ a.e.\ t \in I. $$

Moreover, the parameter γ satisfies

$$ \gamma \in (0,1], \; \gamma \geq \alpha, \; \gamma > \beta, \; 1-\gamma < 1-\beta(1-\alpha). $$

2. The generalization (3) for $\beta = 0$ coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$ D_1^{\alpha,0} = D_1^{\alpha}, \; and\ D_1^{\alpha,1} = {}^{c}D_1^{\alpha}. $$

3. If $D_1^{\beta(1-\alpha)}w$ exists and in $L^1(I)$, then

$$ (D_1^{\alpha,\beta}I_1^{\alpha}w)(t) = (I_1^{\beta(1-\alpha)}D_1^{\beta(1-\alpha)}w)(t); \; for\ a.e.\ t \in I. $$

Furthermore, if $w \in C_\gamma(I)$ and $I_1^{1-\beta(1-\alpha)}w \in C_\gamma^1(I)$, then

$$ (D_1^{\alpha,\beta}I_1^{\alpha}w)(t) = w(t); \; for\ a.e.\ t \in I. $$

4. If $D_1^{\gamma}w$ exists and in $L^1(I)$, then

$$(I_1^{\alpha}D_1^{\alpha,\beta}w)(t) = (I_1^{\gamma}D_1^{\gamma}w)(t) = w(t) - \frac{I_1^{1-\gamma}(1^+)}{\Gamma(\gamma)}t^{\gamma-1};\ for\ a.e.\ t \in I.$$

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [25]) is defined in the following way:

Definition 11.*(Hilfer-Hadamard fractional derivative). Let* $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta,\ w \in L^1(I),$ *and* ${}^H I_1^{(1-\alpha)(1-\beta)} w \in AC^1(I)$. *The Hilfer-Hadamard fractional derivative of order* α *and type* β *applied to the function* w *is defined as*

$$\begin{aligned}({}^H D_1^{\alpha,\beta}w)(t) &= \left({}^H I_1^{\beta(1-\alpha)}({}^H D_1^{\gamma}w)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)}\delta({}^H I_1^{1-\gamma}w)\right)(t);\ for\ a.e.\ t \in I.\end{aligned} \quad (4)$$

This new fractional derivative (11) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

$${}^H D_1^{\alpha,0} = {}^H D_1^{\alpha},\ and\ {}^H D_1^{\alpha,1} = {}^{Hc}D_1^{\alpha}.$$

From Theorem 21 in [26], we conclude with the following lemma

Lemma 1.*Let* $f : I \times E \to E$ *be such that* $f(\cdot,u(\cdot)) \in C_{\gamma,\ln}(I)$ *for any* $u \in C_{\gamma,\ln}(I)$. *Then problem (1) is equivalent to the problem of the solutions of the Volterra integral equation*

$$u(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^{\alpha} f(\cdot,u(\cdot)))(t).$$

Definition 12.*[11] Let* E *be a Banach space,* Ω_E *the bounded subsets of* E *and* B_1 *the unit ball of* E. *The De Blasi measure of weak noncompactness is the map* $\beta : \Omega_E \to [0,\infty)$ *defined by*

$$\beta(X) = \inf\{\varepsilon > 0 : there\ exists\ a\ weakly\ compact\ \Omega \subset E \text{ such that } X \subset \varepsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a)$A \subset B \Rightarrow \beta(A) \leq \beta(B)$,
(b)$\beta(A) = 0 \Leftrightarrow A$ is weakly relatively compact,
(c)$\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,
(d)$\beta(\overline{A}^{\omega}) = \beta(A)$, ($\overline{A}^{\omega}$ denotes the weak closure of A),
(e)$\beta(A + B) \leq \beta(A) + \beta(B)$,
(f)$\beta(\lambda A) = |\lambda|\beta(A)$,
(g)$\beta(conv(A)) = \beta(A)$,
(h)$\beta(\cup_{|\lambda| \leq h}\lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 2.*Let* E *be a normed space, and* $x_0 \in E$ *with* $x_0 \neq 0$. *Then, there exists* $\varphi \in E^*$ *with* $\|\varphi\| = 1$ *and* $\varphi(x_0) = \|x_0\|$.

For a given set V of functions $v : I \to E$ let us denote by

$$V(t) = \{v(t) : v \in V\};\ t \in I,\ and\ V(I) = \{v(t) : v \in V,\ t \in I\}.$$

Lemma 2.[16] *Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I, and*

$$\beta_C(H) = \max_{t\in I} \beta(H(t)),$$

and

$$\beta\left(\int_I u(s)ds\right) \leq \int_I \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H,\ s \in I\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

Let $\mathscr{P}(E)$ the family of all nonempty subsets of E. In what follows $\mathscr{P}_{cl}(E) = \{Y \in \mathscr{P}(E) : Y \text{ is closed}\}$, $\mathscr{P}_b(E) = \{Y \in \mathscr{P}(E) : Y \text{ is bounded}\}$, $\mathscr{P}_{cp}(E) = \{Y \in \mathscr{P}(E) : Y \text{ is compact}\}$, and $\mathscr{P}_{cp,cv}(E) = \{Y \in \mathscr{P}(E) : Y \text{ is compact and convex}\}$.

Definition 13.*A multivalued map $G : E \to \mathscr{P}(E)$ is* convex (closed) valued *if $G(x)$ is convex (closed) for all $x \in E$. We say that G is* bounded on bounded sets *if $G(B)$ is bounded in E for each bounded set B of E (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$). The mapping G is called* upper semi-continuous *(u.s.c.) on E if for each $x_0 \in E$, the set $G(x_0)$ is a nonempty closed subset of E, and for each open set N of E containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. The mapping G has a* fixed-point *if there is $x \in E$ such that $x \in G(x)$.*

Definition 14.*A multivalued map $G : I \to \mathscr{P}_{cl}(E)$ is said to be measurable if for each $\omega \in E$ the function*

$$t \to d(\omega, G(t)) = \inf\{\|\omega - \upsilon\| : \upsilon \in G(t)\}$$

is measurable.

Definition 15.*The selection set of a multivalued map $G : I \to \mathscr{P}(E)$ is defined by*

$$S_G = \{u \in L^1(I) : u(t) \in G(t)\ ,\ a.e.\ t \in I\}.$$

For each $u \in C_{\gamma,\ln}$, the set $S_{F\circ u}$ known as the set of selectors *from $F \circ u$ is defined by*

$$S_{F\circ u} = \{v \in L^1(I) : v(t) \in F(t, u(t));\ a.e.\ t \in I\}.$$

For more details on multivalued maps we refer to the books of Aubin and Cellina [30] and Deimling [31].

Definition 16.*A function $F : Q \to P_{cl,cv}(Q)$ has a weakly sequentially closed graph, if for any sequence $(x_n, y_n) \in Q \times Q, y_n \in F(x_n)$ for $n \in \{1, 2, \ldots\}$, with $x_n \to x$ in (E, ω), and $y_n \to y$ in (E, ω), then $y \in F(x)$.*

3 Hilfer-Hadamard Fractional Differential Equations

Let us start in this section by defining what we mean by a weak solution of the problem (1).

Definition 17.*By a weak solution of the problem (1) we mean a measurable function $u \in C_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+) = \phi$, and the equation $({}^H D_1^{\alpha,\beta} u)(t) = f(t, u(t))$ on I.*

For our purpose we need the following fixed-point theorem:

Theorem 1.[32] *Let Q be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(I, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{5}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

The following hypotheses is used in the sequel.

$(H_1)$for a.e. $t \in I$, the function $v \to f(t, v)$ is weakly sequentially continuous,

$(H_2)$for each $v \in E$, the function $t \to f(t,v)$ is Pettis integrable a.e. on I,
(H_3)there exists $p \in C(I,[0,\infty))$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t,u))| \leq \frac{p(t)}{1+\|\varphi\|+\|u\|_E}, \ for\ a.e.\ t \in I,\ and\ each\ u \in E,$$

$(H_4)$for each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(f(t,B) \leq (\ln t)^{1-\gamma} p(t)\beta(B).$$

Set

$$p^* = \sup_{t\in I} p(t),$$

Theorem 2. *Assume that the hypotheses $(H_1)-(H_4)$ hold. If*

$$L := \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{6}$$

then the problem (1) has at least one weak solution defined on I.

Proof. Consider the operator $N : C_{\gamma,\ln} \to C_{\gamma,\ln}$ defined by:

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s\Gamma(\alpha)} ds.$$

First, note that the hypotheses imply that for each $u \in C_{\gamma,\ln}$, the function $t \mapsto \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s}$, $for\ a.e.\ t \in I$, is Pettis integrable. Thus, the operator N is well defined. Let $R > 0$ be such that

$$R > \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)},$$

and consider the set

$$\begin{aligned} Q = \Big\{ & u \in C_\gamma : \|u\|_C \leq R \text{ and } \|(\ln t_2)^{1-\gamma}u(t_2) - (\ln t_1)^{1-\gamma}u(t_1)\|_E \\ & \leq \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\left(\ln\frac{t_2}{t_1}\right)^\alpha \\ & + \frac{p^*}{\Gamma(\alpha)}\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right| ds\Big\}. \end{aligned}$$

Clearly, the subset Q is closed, convex end equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 1. The proof is given in several steps.

Step 1. *N maps Q into itself.*
Let $u \in Q$, $t \in I$ and assume that $(Nu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|(\ln t)^{1-\gamma}(Nu)(t)\|_E = |\varphi((\ln t)^{1-\gamma}(Nu)(t))|$. Thus

$$\|(\ln t)^{1-\gamma}(Nu)(t)\|_E = \varphi\left(\frac{\phi}{\Gamma(\gamma)} + \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1} f(s,u(s))\frac{ds}{s}\right).$$

Then

$$\begin{aligned} \|(\ln t)^{1-\gamma}(Nu)(t)\|_E &\leq \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}|\varphi(f(s,u(s)))|\frac{ds}{s} \\ &\leq \frac{p^*(\ln T)^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\ &\leq \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \\ &\leq R. \end{aligned}$$

Next, let $t_1,t_2 \in I$ such that $t_1 < t_2$ and let $u \in Q$, with

$$(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1)\|_E = |\varphi((\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1))|,$$

and $\|\varphi\| = 1$. Then

$$\begin{aligned}
&\|(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1)\|_E = |\varphi((\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1))|\\
&\leq \varphi\left((\ln t_2)^{1-\gamma}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s\Gamma(\alpha)}ds-(\ln t_1)^{1-\gamma}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s\Gamma(\alpha)}ds\right)\\
&\leq (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{|\varphi(f(s,u(s)))|}{s\Gamma(\alpha)}ds\\
&\quad+\int_1^{t_1}|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}-(\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}|\frac{|\varphi(f(s,u(s)))|}{s\Gamma(\alpha)}ds\\
&\leq (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{p(s)}{s\Gamma(\alpha)}ds\\
&\quad+\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}-(\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{p(s)}{s\Gamma(\alpha)}ds.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1)\|_E &\leq \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\left(\ln\frac{t_2}{t_1}\right)^{\alpha}\\
&+\frac{p^*}{\Gamma(\alpha)}\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}-(\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|ds.
\end{aligned}$$

Hence $N(Q) \subset Q$.

Step 2. *N is weakly-sequentially continuous.*
Let (u_n) be a sequence in Q and let $(u_n(t)) \to u(t)$ in (E,ω) for each $t \in I$. Fix $t \in I$, since f satisfies the assumption (H_1), we have $f(t,u_n(t))$ converges weakly uniformly to $f(t,u(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies $(Nu_n)(t)$ converges weakly uniformly to $(Nu)(t)$ in (E,ω), for each $t \in I$. Thus, $N(u_n) \to N(u)$. Hence, $N : Q \to Q$ is weakly-sequentially continuous.

Step 3. *The implication (5) holds.*
Let V be a subset of Q such that $\overline{V} = \overline{conv}(N(V) \cup \{0\})$. Obviously

$$V(t) \subset \overline{conv}(NV)(t)) \cup \{0\}),\ t \in I.$$

Further, as V is bounded and equicontinuous, by Lemma 3 in [33] the function $t \to v(t) = \beta(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 2 and the properties of the measure β, for any $t \in I$, we have

$$\begin{aligned}
(\ln t)^{1-\gamma}v(t) &\leq \beta((\ln t)^{1-\gamma}(NV)(t) \cup \{0\})\\
&\leq \beta((\ln t)^{1-\gamma}(NV)(t))\\
&\leq \frac{(\ln T)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}p(s)\beta(V(s))ds\\
&\leq \frac{(\ln T)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}(\ln s)^{1-\gamma}p(s)v(s)ds\\
&\leq \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\|v\|_C.
\end{aligned}$$

Thus

$$\|v\|_C \leq L\|v\|_C.$$

From (6), we get $\|v\|_C = 0$, that is $v(t) = \beta(V(t)) = 0$, for each $t \in I$. and then by Theorem 2 in [34], V is weakly relatively compact in $C_{\gamma,\ln}$. Applying now Theorem 1, we conclude that N has a fixed-point which is a weak solution of the problem (1).

4 Hilfer-Hadamard Fractional Differential Inclusions

Let us start in this section by defining what we mean by a weak solution of the problem (2).

Definition 18. *By a weak solution of the problem (2) we mean a measurable function $u \in C_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+) = \phi$, and the equation $({}^H D_1^{\alpha,\beta} u)(t) = h(t)$ on I, where $h \in S_{F\circ u}$.*

From Lemma 1, we conclude with the following lemma.

Lemma 3. *Let $F : I \times E \to E$ be such that $S_{F\circ u} \subset C_{\gamma,\ln}(I)$ for any $u \in C_{\gamma,\ln}(I)$. Then problem (2) is equivalent to the problem of the solutions of the integral equation*

$$u(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^{\alpha} v)(t),$$

where $v \in S_{F\circ u}$.

For our purpose we shall need the following fixed-point theorem:

Theorem 3. *[32] Let E be a Banach space with Q a nonempty, bounded, closed, convex and equicontinuous subset of a metrizable locally convex vector space C such that $0 \in Q$. Suppose $T : Q \to \mathscr{P}_{cl,cv}(Q)$ has weakly sequentially closed graph. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{7}$$

holds for every subset $V \subset Q$, then the operator T has a fixed-point.

The following hypotheses are used in the sequel.

(H_1') $F : I \times E \to \mathscr{P}_{cp,cl,cv}(E)$ has weakly sequentially closed graph,

(H_2') for each continuous $u : I \to E$, there exists a measurable function $v \in S_{F\circ u}$ a.e. on I and v is Pettis integrable on I,

(H_3') there exists $q \in C(I, [0,\infty))$ such that for all $\varphi \in E^*$, we have

$$\|F(t,u)\|_{\mathscr{P}} = \sup_{v \in S_{F\circ u}} |\varphi(v)| \le \frac{q(t)}{1 + \|\varphi\| + \|u\|_E}, \ for\ a.e.\ t \in I,\ and\ each\ u \in E,$$

(H_4') for each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(F(t,B) \le (\ln t)^{1-\gamma} q(t)\beta(B).$$

Set

$$q^* = \sup_{t \in I} q(t),$$

Theorem 4. *Assume that the hypotheses $(H_1') - (H_4')$ hold. If*

$$L' := \frac{q^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{8}$$

then the problem (2) has at least one weak solution defined on I.

Proof. Consider the multi-valued map $\overline{N} : C_{\gamma,\ln} \to \mathscr{P}_{cl}(C_{\gamma,\ln})$ defined by:

$$(\overline{N}u)(t) = \left\{ h \in C_{\gamma,\ln} : h(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s\Gamma(\alpha)} ds;\ v \in S_{F\circ u} \right\}.$$

Note that the hypotheses imply that for each $u \in C_{\gamma,\ln}$, there exists a Pettis integrable function $v \in S_{F\circ u}$, and for each $s \in [1,t]$, the function

$$t \mapsto \left(\ln \frac{t}{s}\right)^{\alpha-1} v(s);\ for\ a.e.\ t \in I,$$

is Pettis integrable. Thus, the multi-function $\overline{N}$ is well defined. Let $R' > 0$ be such that

$$R' > \frac{q^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)},$$

and consider the set

$$\begin{aligned} Q' = \Big\{ u \in C_{\gamma,\ln} : \|u\|_C \le R' \text{ and } &\|(\ln t_2)^{1-\gamma}u(t_2) - (\ln t_1)^{1-\gamma}u(t_1)\|_E \\ &\le \frac{q^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\left(\ln\frac{t_2}{t_1}\right)^{\alpha} \\ &+ \frac{q^*}{\Gamma(\alpha)}\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right| ds \Big\}. \end{aligned}$$

Clearly, the subset Q' is closed, convex end equicontinuous. We shall show that the operator $\overline{N}$ satisfies all the assumptions of Theorem 3. The proof is given in several steps.

Step 1. *$\overline{N}(u)$ is convex for each $u \in Q'$.*
For that, let $h_1, h_2 \in \overline{N}(u)$. Then there exist $v_1, v_2 \in S_{F\circ u}$ such that, for each $t \in I$, and for any $i = 1,2$, we have

$$h_i(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v_i(s)}{s\Gamma(\alpha)}ds.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in I$, we have

$$[\lambda h_1 + (1-\lambda)h_2](t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\lambda v_1(s) + (1-\lambda)v_2(s)}{s\Gamma(\alpha)}ds.$$

Since $S_{F\circ u}$ is convex (because F has convex values), it follows that

$$\lambda h_1 + (1-\lambda)h_2 \in \overline{N}(u).$$

Step 2. *$\overline{N}$ maps Q' into itself.*
Take $h \in \overline{N}(Q')$. Then there exists $u \in Q'$ with $h \in \overline{N}(u)$, and there exists a Pettis integrable $v : I \to E$ with $v(t) \in F(t, u(t))$; for a.e. $t \in I$. Assume that $h(t) \neq 0$, then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ such that

$$\|(\ln t)^{1-\gamma}h(t)\|_E = |\varphi((\ln t)^{1-\gamma}h(t))|.$$

Then

$$\|(\ln t)^{1-\gamma}h(t)\|_E = \varphi\left(\frac{\phi}{\Gamma(\gamma)} + \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}v(s)\frac{ds}{s}\right).$$

Thus

$$\begin{aligned} \|(\ln t)^{1-\gamma}h(t)\|_E &\le \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}|\varphi(v(s))|\frac{ds}{s} \\ &\le \frac{q^*(\ln T)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \\ &\le \frac{q^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \\ &\le R'. \end{aligned}$$

Next, let $t_1, t_2 \in I$ such that $t_1 < t_2$ and let $h \in \overline{N}(u)$, with

$$(\ln t_2)^{1-\gamma}h(t_2) - (\ln t_1)^{1-\gamma}h(t_1) \neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\|(\ln t_2)^{1-\gamma}h(t_2) - (\ln t_1)^{1-\gamma}h(t_1)\|_E = |\varphi((\ln t_2)^{1-\gamma}h(t_2) - (\ln t_1)^{1-\gamma}h(t_1))|,$$

and $\|\varphi\| = 1$. Then, we have

$$\begin{aligned}
&\|(\ln t_2)^{1-\gamma}h(t_2) - (\ln t_1)^{1-\gamma}h(t_1)\|_E = |\varphi((\ln t_2)^{1-\gamma}h(t_2) - (\ln t_1)^{1-\gamma}h(t_1))| \\
&\le \varphi\left((\ln t_2)^{1-\gamma}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{v(s)}{s\Gamma(\alpha)}ds - (\ln t_1)^{1-\gamma}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\frac{v(s)}{s\Gamma(\alpha)}ds\right) \\
&\le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{|\varphi(v(s))|}{s\Gamma(\alpha)}ds \\
&\quad + \int_1^{t_1}\left|(\ln t_2)^{1-\gamma}(t_2-s)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{|\varphi(v(s))|}{s\Gamma(\alpha)}ds \\
&\le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{q(s)}{s\Gamma(\alpha)}ds \\
&\quad + \int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{q(s)}{s\Gamma(\alpha)}ds.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\|(\ln t_2)^{1-\gamma}h(t_2) - (\ln t_1)^{1-\gamma}h(t_1)\|_E &\le \frac{q^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\left(\ln\frac{t_2}{t_1}\right)^{\alpha} \\
&+ \frac{q^*}{\Gamma(\alpha)}\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|ds.
\end{aligned}$$

This implies that $h \in Q'$. Hence $\overline{N}(Q') \subset Q'$.

Step 3. *$\overline{N}$ has weakly-sequentially closed graph.*
Let (u_n, w_n) be a sequence in $Q' \times Q'$, with $u_n(t) \to u(t)$ in (E,ω) for each $t \in I$, $w_n(t) \to w(t)$ in (E,ω) for each $t \in I$, and $w_n \in \overline{N}(u_n)$ for $n \in \{1,2,\ldots\}$.
We show that $w \in \overline{N}(u)$. Since $w_n \in \overline{N}(u_n)$, there exists $v_n \in S_{F\circ u_n}$ such that

$$w_n(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v_n(s)}{s\Gamma(\alpha)}ds.$$

We show that there exists $v \in S_{F\circ u}$ such that, for each $t \in I$,

$$w(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s\Gamma(\alpha)}ds.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence v_{n_m} such that v_{n_m} is Pettis integrable,

$$v_{n_m}(t) \in F(t, u_n(t)) \text{ a.e. } t \in I,$$

$$v_{n_m}(\cdot) \to v(\cdot) \text{ in } (E,\omega) \text{ as } m \to \infty.$$

As $F(t,\cdot)$ has weakly-sequentially closed graph, $v(t) \in F(t,u(t))$. Then by the Lebesgue dominated convergence theorem for the Pettis integral, we obtain

$$\varphi(w_n(t)) \to \varphi\left(\frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s\Gamma(\alpha)}ds\right),$$

i.e. $w_n(t) \to (\overline{N}u)(t)$ in (E,ω). Since this holds, for each $t \in I$, then we get $w \in \overline{N}(u)$.

Step 4. *The implication (7) holds.*
Let V be a subset of Q', such that $\overline{V} = \overline{conv}(\overline{N}(V) \cup \{0\})$. Obviously $V(t) \subset \overline{conv}(\overline{N}(V)(t)) \cup \{0\})$ for each $t \in I$. Further, as V is bounded and equicontinuous, the function $t \to v(t) = \beta(V(t))$ is continuous on I. By (H_4') and the properties of

the measure β, for any $t \in I$ we have

$$\begin{aligned}
(\ln t)^{1-\gamma}v(t) &\leq \beta((\ln t)^{1-\gamma}(NV)(t) \cup \{0\}) \\
&\leq \beta((\ln t)^{1-\gamma}(NV)(t)) \\
&\leq \beta\{(\ln t)^{1-\gamma}(Nu)(t) : u \in V\} \\
&\leq \beta\left\{(\ln T)^{1-\gamma}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s\Gamma(\alpha)}ds : v(t) \in S_{F\circ u},\ u \in V\right\} \\
&\leq \beta\left\{(\ln T)^{1-\gamma}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{F(s,V(s))}{s\Gamma(\alpha)}ds\right\} \\
&\leq (\ln T)^{1-\gamma}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\beta(V(s))}{s\Gamma(\alpha)}ds \\
&\leq (\ln T)^{1-\gamma}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{(\ln s)^{1-\gamma}q(s)v(s)}{s\Gamma(\alpha)}ds \\
&\leq \frac{q^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\|v\|_C.
\end{aligned}$$

In particular,

$$\|v\|_C \leq L'\|v\|_C.$$

By (8) it follows that $\|v\|_C = 0$, that is, $v(t) = \beta(V(t)) = 0$ for each $t \in I$, and then V is weakly relatively compact in C. Applying now Theorem 3, we conclude that $\overline{N}$ has a fixed-point which is a weak solution of the problem (2).

5 Examples

Let

$$E = l^1 = \left\{u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty}|u_n| < \infty\right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty}|u_n|.$$

Example 1. Consider the problem of Hilfer-Hadamard fractional differential equation of the form

$$\begin{cases} ({}^H D_1^{\frac{1}{2},\frac{1}{2}}u_n)(t) = f_n(t,u(t));\ t \in [1,e], \\ ({}^H I_1^{\frac{1}{4}}u)(t)|_{t=1} = (2^{-1}, 2^{-2}, \ldots, 2^{-n}, \ldots), \end{cases} \tag{9}$$

where

$$f_n(t,u(t)) = \frac{ct^2}{1+\|u(t)\|_E}\frac{u_n(t)}{e^{t+4}};\ \ t \in [1,e],$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots),\ and\ c := \frac{e^3}{8}\Gamma\left(\frac{1}{2}\right).$$

Set

$$f = (f_1, f_2, \ldots, f_n, \ldots).$$

Clearly, the function f is continuous.
For each $u \in E$ and $t \in [1,e]$, we have

$$\|f(t,u(t))\|_E \leq ct^2\frac{1}{e^{t+4}}.$$

Hence, the hypothesis (H_3) is satisfied with $p^* = ce^{-3}$. We shall show that condition (6) holds with $T = e$. Indeed,

$$\frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} = \frac{2ce^{-3}}{\Gamma(\frac{1}{2})} = \frac{1}{4} < 1.$$

Simple computations show that all conditions of Theorem 2 are satisfied. It follows that the problem (9) has at least one weak solution defined on $[1,e]$.

Example 2. Consider the problem of Hilfer-Hadamard fractional differential inclusion of the form

$$\begin{cases} ({}^H D_1^{\frac{1}{2},\frac{1}{2}} u_n)(t) \in F_n(t,u(t));\ t\in[1,e], \\ ({}^H I_1^{\frac{1}{4}} u)(t)|_{t=1} = (1,0,\ldots,0,\ldots), \end{cases} \tag{10}$$

where

$$F_n(t,u(t)) = \frac{ct^2 e^{-4-t}}{1+\|u(t)\|_E}[u_n(t)-1,u_n(t)];\ t\in[1,e],$$

with

$$u=(u_1,u_2,\ldots,u_n,\ldots),\ and\ c := \frac{e^3}{8}\Gamma\left(\frac{1}{2}\right).$$

Set

$$F=(F_1,F_2,\ldots,F_n,\ldots).$$

We assume that F is closed and convex valued. Clearly, the function F is continuous.
For each $u\in E$ and $t\in[1,e]$, we have

$$\|F(t,u(t))\|_{\mathscr{P}} \le ct^2\frac{1}{e^{t+4}}.$$

Hence, the hypothesis (H_3') is satisfied with $q^* = ce^{-3}$. We shall show that condition (8) holds with $T=e$. Indeed,

$$\frac{q^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} = \frac{2ce^{-3}}{\Gamma(\frac{1}{2})} = \frac{1}{4} < 1.$$

Simple computations show that all conditions of Theorem 4 are satisfied. It follows that the problem (10) has at least one weak solution defined on $[1,e]$.

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