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Canavati Fractional Approximation by Max-Product Operators

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Abstract: Here we study the approximation of functions by sublinear positive operators with applications to a large variety of Max-Product operators under Canavati fractional differentiability. Our approach is based on our general fractional results about positive sublinear operators. We derive Jackson type inequalities under simple initial conditions. So our way is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of Canavati fractional derivative of the function under approximation.

Keywords: Positive sublinear operators, Max-product operators, modulus of continuity, Canavati fractional derivative, generalized fractional derivative.

1 Introduction, Motivation and Preliminaries

The inspiring motivation here is the monograph by B. Bede, L. Coroianu and S. Gal [1], 2016.

Let $N \in \mathbb{N}$, the well-known Bernstein polynomials [2] are positive linear operators, defined by the formula

$$B_N(f)(x) = \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f\left(\frac{k}{N}\right), \quad x \in [0,1], \ f \in C([0,1]). \tag{1}$$

T. Popoviciu in [3], 1935, proved for $f \in C([0,1])$ that

$$|B_N(f)(x) - f(x)| \le \frac{5}{4}\omega_1\left(f, \frac{1}{\sqrt{N}}\right), \quad \forall x \in [0,1], \tag{2}$$

where

$$\omega_1(f,\delta) = \sup_{\substack{x,y\in[a,b]:\\ |x-y|\le\delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{3}$$

is the first modulus of continuity, here $[a,b] = [0,1]$.

G. G. Lorentz in [2], 1986, p. 21, proved for $f \in C^1([0,1])$ that

$$|B_N(f)(x) - f(x)| \le \frac{3}{4\sqrt{N}}\omega_1\left(f', \frac{1}{\sqrt{N}}\right), \quad \forall x \in [0,1], \tag{4}$$

In [1], p. 10, the authors introduced the basic Max-product Bernstein operators,

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad N \in \mathbb{N}, \tag{5}$$

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where $\bigvee$ stands for maximum, and $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$ and $f : [0,1] \to \mathbb{R}_+ = [0,\infty)$.

These are nonlinear and piecewise rational operators.

The authors in [1] studied similar such nonlinear operators such as: the Max-product Favard-Szász-Mirakjan operators and their truncated version, the Max-product Baskakov operators and their truncated version, also many other similar specific operators. The study in [1] is based on presented there general theory of sublinear operators. These Max-product operators tend to converge faster to the on hand function.

So we mention from [1], p. 30, that for $f : [0,1] \to \mathbb{R}_+$ continuous, we have the estimate

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq 12\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \text{ for all } N \in \mathbb{N},\ x \in [0,1], \tag{6}$$

Also from [1], p. 36, we mention that for $f : [0,1] \to \mathbb{R}_+$ being concave function we get that

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq 2\omega_1\left(f, \frac{1}{N}\right), \text{ for all } x \in [0,1], \tag{7}$$

a much faster convergence.

In this paper we expand the study of [1] by considering Canavati fractional smoothness of functions. So our inequalities are with respect to $\omega_1(D^\alpha f, \delta)$, $\delta > 0$, where $D^\alpha f$ with $\alpha > 0$ is the Canavati fractional derivative. The structure of the manuscript is as follows. In Section 2 the main results are presented. The applications can be seen in Section 3.A conclusion part in mentioned in Section 4.

2 Main Results

We make

*Remark.*I) Here see [4], pp. 7-10.

Let $x, x_0 \in [a,b]$ such that $x \geq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, such that $p = [\nu]$, $[\cdot]$ the integral part, $\alpha = \nu - p$ $(0 < \alpha < 1)$.

Let $f \in C^p([a,b])$ and define

$$\left(J_\nu^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} f(t)\,dt,\ x_0 \leq x \leq b. \tag{8}$$

the left generalized Riemann-Liouville fractional integral.

Here Γ stands for the gamma function.

Clearly here it holds $\left(J_\nu^{x_0} f\right)(x_0) = 0$. We define $\left(J_\nu^{x_0} f\right)(x) = 0$ for $x < x_0$. By [4], p. 388, $\left(J_\nu^{x_0} f\right)(x)$ is a continuous function in x, for a fixed x_0.

We define the subspace $C_{x_0+}^\nu([a,b])$ of $C^p([a,b])$:

$$C_{x_0+}^\nu([a,b]) := \left\{ f \in C^p([a,b]) : J_{1-\alpha}^{x_0} f^{(p)} \in C^1([x_0,b]) \right\}. \tag{9}$$

So let $f \in C_{x_0+}^\nu([a,b])$, we define the left generalized ν-fractional derivative of f over $[x_0,b]$ as

$$D_{x_{0+}}^\nu f = \left(J_{1-\alpha}^{x_0} f^{(p)}\right)', \tag{10}$$

that is

$$\left(D_{x_{0+}}^\nu f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x_0}^x (x-t)^{-\alpha} f^{(p)}(t)\,dt, \tag{11}$$

which exists for $f \in C_{x_0+}^\nu([a,b])$, for $a \leq x_0 \leq x \leq b$.

Canavati in [5] first introduced this kind of left fractional derivative over $[0,1]$.

We mention the following left generalized fractional Taylor formula ($f \in C_{x_0+}^\nu([a,b])$, $\nu > 1$).

It holds

$$f(x) - f(x_0) = \sum_{k=1}^{p-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} \left(D_{x_0+}^\nu f\right)(t)\,dt, \tag{12}$$

for $x, x_0 \in [a,b]$ with $x \geq x_0$.

II) Here see [6], p. 333, and again [6], pp. 345-348.
Let $x, x_0 \in [a,b]$ such that $x \leq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, such that $p = [\nu]$, $\alpha = \nu - p$ $(0 < \alpha < 1)$.
Let $f \in C^p([a,b])$ and define

$$\left(J_{x_0-}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^{x_0} (z-x)^{\nu-1} f(z)\, dz, \;\; a \leq x \leq x_0. \tag{13}$$

the right generalized Riemann-Liouville fractional integral.
Define the subspace of functions

$$C_{x_0-}^{\nu}([a,b]) := \left\{ f \in C^p([a,b]) : J_{x_0-}^{1-\alpha} f^{(p)} \in C^1([a,x_0]) \right\}. \tag{14}$$

Define the right generalized ν-fractional derivative of f over $[a,x_0]$ as

$$D_{x_0-}^{\nu} f = (-1)^{p-1} \left(J_{x_0-}^{1-\alpha} f^{(p)} \right)'. \tag{15}$$

Notice that

$$J_{x_0-}^{1-\alpha} f^{(p)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^{x_0} (z-x)^{-\alpha} f^{(p)}(z)\, dz, \tag{16}$$

exists for $f \in C_{x_0-}^{\nu}([a,b])$, and

$$\left(D_{x_0-}^{\nu} f\right)(x) = \frac{(-1)^{p-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^{x_0} (z-x)^{-\alpha} f^{(p)}(z)\, dz. \tag{17}$$

I.e.

$$\left(D_{x_0-}^{\nu} f\right)(x) = \frac{(-1)^{p-1}}{\Gamma(p-\nu+1)} \frac{d}{dx} \int_x^{x_0} (z-x)^{p-\nu} f^{(p)}(z)\, dz, \tag{18}$$

which exists for $f \in C_{x_0-}^{\nu}([a,b])$, for $a \leq x \leq x_0 \leq b$.
We mention the following right generalized fractional Taylor formula ($f \in C_{x_0-}^{\nu}([a,b])$, $\nu > 1$).
It holds

$$f(x) - f(x_0) = \sum_{k=1}^{p-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\nu)} \int_x^{x_0} (z-x)^{\nu-1} \left(D_{x_0-}^{\nu} f\right)(z)\, dz, \tag{19}$$

for $x, x_0 \in [a,b]$ with $x \leq x_0$.

We need

Definition 1. *Let $D_{x_0}^{\nu} f$ denote any of $D_{x_0-}^{\nu} f$, $D_{x_0+}^{\nu} f$, and $\delta > 0$. We set*

$$\omega_1\left(D_{x_0}^{\nu} f, \delta\right) := \max\left\{ \omega_1\left(D_{x_0-}^{\nu} f, \delta\right)_{[a,x_0]}, \omega_1\left(D_{x_0+}^{\nu} f, \delta\right)_{[x_0,b]} \right\}, \tag{20}$$

where $x_0 \in [a,b]$. Here the moduli of continuity are considered over $[a,x_0]$ and $[x_0,b]$, respectively.

We need

Theorem 1. *Let $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x_0 \in [a,b]$ and $f \in C_{x_0+}^{\nu}([a,b]) \cap C_{x_0-}^{\nu}([a,b])$. Assume that $f^{(k)}(x_0) = 0$, $k = 1, \ldots, p-1$, and $\left(D_{x_0+}^{\nu} f\right)(x_0) = \left(D_{x_0-}^{\nu} f\right)(x_0) = 0$. Then*

$$|f(x) - f(x_0)| \leq \frac{\omega_1\left(D_{x_0}^{\nu} f, \delta\right)}{\Gamma(\nu+1)} \left[|x-x_0|^{\nu} + \frac{|x-x_0|^{\nu+1}}{(\nu+1)\delta} \right], \;\; \delta > 0, \tag{21}$$

for all $a \leq x \leq b$.

Proof. We use (12) and (19), and the assumption $f^{(k)}(x_0)=0$, $k=1,...,p-1$ and $\left(D_{x_0+}^{\nu}f\right)(x_0)=\left(D_{x_0-}^{\nu}f\right)(x_0)=0$. We have that

$$f(x)-f(x_0)=\frac{1}{\Gamma(\nu)}\int_{x_0}^{x}(x-z)^{\nu-1}\left(\left(D_{x_0+}^{\nu}f\right)(z)-\left(D_{x_0+}^{\nu}f\right)(x_0)\right)dz, \tag{22}$$

for all $x_0\leq x\leq b$,

and

$$f(x)-f(x_0)=\frac{1}{\Gamma(\nu)}\int_{x}^{x_0}(z-x)^{\nu-1}\left(\left(D_{x_0-}^{\nu}f\right)(z)-\left(D_{x_0-}^{\nu}f\right)(x_0)\right)dz, \tag{23}$$

for all $a\leq x\leq x_0$.

We observe that ($x_0\leq x\leq b$)

$$\begin{aligned}
|f(x)-f(x_0)| &\leq \frac{1}{\Gamma(\nu)}\int_{x_0}^{x}(x-z)^{\nu-1}\left|\left(D_{x_0+}^{\nu}f\right)(z)-\left(D_{x_0+}^{\nu}f\right)(x_0)\right|dz \underset{(\delta_1>0)}{\leq} \\
&\frac{1}{\Gamma(\nu)}\int_{x_0}^{x}(x-z)^{\nu-1}\omega_1\left(D_{x_0+}^{\nu}f,\frac{\delta_1|z-x_0|}{\delta_1}\right)_{[x_0,b]}dz\leq \\
&\frac{\omega_1\left(D_{x_0+}^{\nu}f,\delta_1\right)_{[x_0,b]}}{\Gamma(\nu)}\int_{x_0}^{x}(x-z)^{\nu-1}\left(1+\frac{(z-x_0)}{\delta_1}\right)dz= \\
&\frac{\omega_1\left(D_{x_0+}^{\nu}f,\delta_1\right)_{[x_0,b]}}{\Gamma(\nu)}\left[\frac{(x-x_0)^{\nu}}{\nu}+\frac{1}{\delta_1}\int_{x_0}^{x}(x-z)^{\nu-1}(z-x_0)^{2-1}dz\right]= \\
&\frac{\omega_1\left(D_{x_0+}^{\nu}f,\delta_1\right)_{[x_0,b]}}{\Gamma(\nu)}\left[\frac{(x-x_0)^{\nu}}{\nu}+\frac{1}{\delta_1}\frac{\Gamma(\nu)\Gamma(2)}{\Gamma(\nu+2)}(x-x_0)^{\nu+1}\right]= \\
&\frac{\omega_1\left(D_{x_0+}^{\nu}f,\delta_1\right)_{[x_0,b]}}{\Gamma(\nu)}\left[\frac{(x-x_0)^{\nu}}{\nu}+\frac{1}{\delta_1}\frac{1}{(\nu+1)\nu}(x-x_0)^{\nu+1}\right]= \\
&\frac{\omega_1\left(D_{x_0+}^{\nu}f,\delta_1\right)_{[x_0,b]}}{\Gamma(\nu+1)}\left[(x-x_0)^{\nu}+\frac{(x-x_0)^{\nu+1}}{(\nu+1)\delta_1}\right].
\end{aligned} \tag{24}$$

We have proved

$$|f(x)-f(x_0)|\leq\frac{\omega_1\left(D_{x_0+}^{\nu}f,\delta_1\right)_{[x_0,b]}}{\Gamma(\nu+1)}\left[(x-x_0)^{\nu}+\frac{(x-x_0)^{\nu+1}}{(\nu+1)\delta_1}\right], \tag{25}$$

$\delta_1>0$, and $x_0\leq x\leq b$.

Similarly acting, we get ($a\leq x\leq x_0$)

$$\begin{aligned}
|f(x)-f(x_0)| &\leq \frac{1}{\Gamma(\nu)}\int_{x}^{x_0}(z-x)^{\nu-1}\left|D_{x_0-}^{\nu}f(z)-D_{x_0-}^{\nu}f(x_0)\right|dz\leq \\
&\frac{1}{\Gamma(\nu)}\int_{x}^{x_0}(z-x)^{\nu-1}\omega_1\left(D_{x_0-}^{\nu}f,|z-x_0|\right)_{[a,x_0]}dz=
\end{aligned}$$

($\delta_2>0$)

$$\begin{aligned}
&\frac{1}{\Gamma(\nu)}\int_{x}^{x_0}(z-x)^{\nu-1}\omega_1\left(D_{x_0-}^{\nu}f,\frac{\delta_2|x_0-z|}{\delta_2}\right)_{[a,x_0]}dz\leq \\
&\frac{\omega_1\left(D_{x_0-}^{\nu}f,\delta_2\right)_{[a,x_0]}}{\Gamma(\nu)}\left[\int_{x}^{x_0}(z-x)^{\nu-1}\left(1+\frac{x_0-z}{\delta_2}\right)dz\right]= \\
&\frac{\omega_1\left(D_{x_0-}^{\nu}f,\delta_2\right)_{[a,x_0]}}{\Gamma(\nu)}\left[\frac{(x_0-x)^{\nu}}{\nu}+\frac{1}{\delta_2}\int_{x}^{x_0}(x_0-z)^{2-1}(z-x)^{\nu-1}dz\right]=
\end{aligned} \tag{26}$$

$$\frac{\omega_1\left(D^{\nu}_{x_0-}f,\delta_2\right)_{[a,x_0]}}{\Gamma(\nu)}\left[\frac{(x_0-x)^{\nu}}{\nu}+\frac{1}{\delta_2}\frac{\Gamma(\nu)\Gamma(2)}{\Gamma(\nu+2)}(x_0-x)^{\nu+1}\right]=$$
$$\frac{\omega_1\left(D^{\nu}_{x_0-}f,\delta_2\right)_{[a,x_0]}}{\Gamma(\nu)}\left[\frac{(x_0-x)^{\nu}}{\nu}+\frac{1}{\delta_2}\frac{(x_0-x)^{\nu+1}}{(\nu+1)\nu}\right]= \tag{27}$$
$$\frac{\omega_1\left(D^{\nu}_{x_0-}f,\delta_2\right)_{[a,x_0]}}{\Gamma(\nu+1)}\left[(x_0-x)^{\nu}+\frac{(x_0-x)^{\nu+1}}{(\nu+1)\delta_2}\right].$$

We have proved

$$|f(x)-f(x_0)|\leq\frac{\omega_1\left(D^{\nu}_{x_0-}f,\delta_2\right)_{[a,x_0]}}{\Gamma(\nu+1)}\left[(x_0-x)^{\nu}+\frac{(x_0-x)^{\nu+1}}{(\nu+1)\delta_2}\right], \tag{28}$$

$\delta_2>0$, and $(a\leq x\leq x_0)$. Choosing $\delta=\delta_1=\delta_2>0$, by (25) and (28), we get (21).

We need

Definition 2. *Here $C_+([a,b]):=\{f:[a,b]\to\mathbb{R}_+,\text{ continuous functions}\}$. Let $L_N:C_+([a,b])\to C_+([a,b])$, operators, $\forall N\in\mathbb{N}$, such that*

(i)

$$L_N(\alpha f)=\alpha L_N(f),\ \forall\alpha\geq 0,\forall f\in C_+([a,b]), \tag{29}$$

(ii) if $f,g\in C_+([a,b]):f\leq g$, then

$$L_N(f)\leq L_N(g),\ \ \forall N\in\mathbb{N}, \tag{30}$$

(iii)

$$L_N(f+g)\leq L_N(f)+L_N(g),\ \ \forall\ f,g\in C_+([a,b]). \tag{31}$$

We call $\{L_N\}_{N\in\mathbb{N}}$ positive sublinear operators.

We make

Remark. By [1], p. 17, we get: let $f,g\in C_+([a,b])$, then

$$|L_N(f)(x)-L_N(g)(x)|\leq L_N(|f-g|)(x),\ \ \forall\ x\in[a,b]. \tag{32}$$

Furthermore, we also have that

$$|L_N(f)(x)-f(x)|\leq L_N(|f(\cdot)-f(x)|)(x)+|f(x)|\,|L_N(e_0)(x)-1|, \tag{33}$$

$\forall\ x\in[a,b]$; $e_0(t)=1$.

From now on we assume that $L_N(1)=1$. Hence it holds

$$|L_N(f)(x)-f(x)|\leq L_N(|f(\cdot)-f(x)|)(x),\ \ \forall\ x\in[a,b]. \tag{34}$$

Using Theorem 1 and (21) with (34) we get:

$$|L_N(f)(x_0)-f(x_0)|\leq\frac{\omega_1\left(D^{\nu}_{x_0}f,\delta\right)}{\Gamma(\nu+1)}\cdot \tag{35}$$
$$\left[L_N\left(|\cdot-x_0|^{\nu}\right)(x_0)+\frac{L_N\left(|\cdot-x_0|^{\nu+1}\right)(x_0)}{(\nu+1)\delta}\right],\quad\delta>0.$$

We have proved

Theorem 2. *Let $\nu>1$, $\nu\notin\mathbb{N}$, $p=[\nu]$, $x_0\in[a,b]$ and $f:[a,b]\to\mathbb{R}_+$, $f\in C^{\nu}_{x_0+}([a,b])\cap C^{\nu}_{x_0-}([a,b])$. Assume that $f^{(k)}(x_0)=0$, $k=1,...,p-1$, and $\left(D^{\nu}_{x_0+}f\right)(x_0)=\left(D^{\nu}_{x_0-}f\right)(x_0)=0$. Let $L_N:C_+([a,b])\to C_+([a,b])$, $\forall\ N\in\mathbb{N}$, be positive sublinear operators, such that $L_N(1)=1$, $\forall\ N\in\mathbb{N}$. Then*

$$|L_N(f)(x_0)-f(x_0)|\leq\frac{\omega_1\left(D^{\nu}_{x_0}f,\delta\right)}{\Gamma(\nu+1)}\cdot$$
$$\left[L_N\left(|\cdot-x_0|^{\nu}\right)(x_0)+\frac{L_N\left(|\cdot-x_0|^{\nu+1}\right)(x_0)}{(\nu+1)\delta}\right], \tag{36}$$

$\delta>0$, $\forall\ N\in\mathbb{N}$.

3 Applications

We give

Theorem 3. *Let $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x \in [0,1]$, $f : [0,1] \to \mathbb{R}_+$ and $f \in C_{x+}^{\nu}([0,1]) \cap C_{x-}^{\nu}([0,1])$. Assume that $f^{(k)}(x) = 0$, $k = 1,...,p-1$, and $\left(D_{x+}^{\nu} f\right)(x) = \left(D_{x-}^{\nu} f\right)(x) = 0$. Then*

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(D_x^{\nu} f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot \left[\frac{6}{\sqrt{N+1}} + \frac{1}{(\nu+1)} \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\nu}{\nu+1}} \right], \tag{37}$$

$\forall N \in \mathbb{N}$.

We get $\lim_{N\to+\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof. By [7] we get that

$$B_N^{(M)}\left(|\cdot - x|^{\nu}\right)(x) \leq \frac{6}{\sqrt{N+1}}, \ \forall x \in [0,1], \tag{38}$$

$\forall N \in \mathbb{N}$, $\forall \nu > 1$.

Also $B_N^{(M)}$ maps $C_+([0,1])$ into itself, $B_N^{(M)}(1) = 1$, and it is positive sublinear operator.

We apply Theorem 2 and (36), we get

$$\left| B_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(D_x^{\nu} f, \delta\right)}{\Gamma(\nu+1)} \left[\frac{6}{\sqrt{N+1}} + \frac{\frac{6}{\sqrt{N+1}}}{(\nu+1)\,\delta} \right]. \tag{39}$$

Choose $\delta = \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{\nu+1}}$, then $\delta^{\nu+1} = \frac{6}{\sqrt{N+1}}$, and apply it to (39). Clearly we derive (37).

We continue with

Remark. The truncated Favard-Szász-Mirakjan operators are given by

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \ \ x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{40}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, see also [1], p. 11.

By [1], p. 178-179, we get that

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \ \ \forall x \in [0,1], \ \forall N \in \mathbb{N}. \tag{41}$$

Clearly it holds

$$T_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{3}{\sqrt{N}}, \ \ \forall x \in [0,1], \ \forall N \in \mathbb{N}, \forall \beta > 0. \tag{42}$$

The operators $T_N^{(M)}$ are positive sublinear operators mapping $C_+([0,1])$ into itself, with $T_N^{(M)}(1) = 1$.

We continue with

Theorem 4. *Same assumptions as in Theorem 3. Then*

$$\left| T_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(D_x^{\nu} f, \left(\frac{3}{\sqrt{N}}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot \left[\frac{3}{\sqrt{N}} + \frac{1}{(\nu+1)} \left(\frac{3}{\sqrt{N}}\right)^{\frac{\nu}{\nu+1}} \right], \ \ \forall N \in \mathbb{N}. \tag{43}$$

We get $\lim_{N\to+\infty} T_N^{(M)}(f)(x) = f(x)$.

*Proof.*Use of Theorem 2, similar to the proof of Theorem 3.

We make

*Remark.*Next we study the truncated Max-product Baskakov operators (see [1], p. 11)

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \ f \in C_+([0,1]), \ N \in \mathbb{N}, \tag{44}$$

where

$$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}. \tag{45}$$

From [1], pp. 217-218, we get ($x \in [0,1]$)

$$\left(U_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \ N \geq 2, N \in \mathbb{N}. \tag{46}$$

Let $\lambda \geq 1$, clearly then it holds

$$\left(U_N^{(M)}\left(|\cdot - x|^{\lambda}\right)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \ \forall N \geq 2, N \in \mathbb{N}. \tag{47}$$

Also it holds $U_N^{(M)}(1) = 1$, and $U_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself.

We give

Theorem 5.*Same assumptions as in Theorem 3. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{\nu} f, \left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot \tag{48}$$

$$\left[\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}} + \frac{1}{(\nu+1)}\left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\nu}{\nu+1}}\right], \ \forall N \geq 2, N \in \mathbb{N}.$$

We get $\lim_{N \to +\infty} U_N^{(M)}(f)(x) = f(x)$.

*Proof.*Use of Theorem 2, similar to the proof of Theorem 3.

We continue with

*Remark.*Here we study the Max-product Meyer-Köning and Zeller operators (see [1], p. 11) defined by

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \ \forall N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{49}$$

$s_{N,k}(x) = \binom{N+k}{k} x^k$, $x \in [0,1]$.

By [1], p. 253, we get that

$$Z_N^{(M)}(|\cdot - x|)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \ \forall x \in [0,1], \ \forall N \geq 4, N \in \mathbb{N}. \tag{50}$$

We have that (for $\lambda \geq 1$)

$$Z_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}} := \rho(x), \tag{51}$$

$\forall x \in [0,1]$, $N \geq 4$, $N \in \mathbb{N}$.

Also it holds $Z_N^{(M)}(1) = 1$, and $Z_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself.

We give

Theorem 6. *Same assumptions as in Theorem 3. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^\nu f, (\rho(x))^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)}. \tag{52}$$

$$\left[\rho(x) + \frac{1}{(\nu+1)}(\rho(x))^{\frac{\nu}{\nu+1}}\right], \forall\, N \in \mathbb{N},\ N \geq 4.$$

We get $\lim_{N\to+\infty} Z_N^{(M)}(f)(x) = f(x)$, *where* $\rho(x)$ *is as in (51).*

Proof. Use of Theorem 2, similar to the proof of Theorem 3.

We continue with

Remark. Here we deal with the Max-product truncated sampling operators (see [1], p. 13) defined by

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \tag{53}$$

and

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{54}$$

$\forall\, x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ a continuous function.

Following [1], p. 343, and making the convention $\frac{\sin(0)}{0} = 1$ and denoting $s_{N,k}(x) = \frac{\sin(Nx-k\pi)}{Nx-k\pi}$, we get that $s_{N,k}\left(\frac{k\pi}{N}\right) = 1$, and $s_{N,k}\left(\frac{j\pi}{N}\right) = 0$, if $k \neq j$, furthermore $W_N^{(M)}(f)\left(\frac{j\pi}{N}\right) = f\left(\frac{j\pi}{N}\right)$, for all $j \in \{0,...,N\}$.

Clearly $W_N^{(M)}(f)$ is a well-defined function for all $x \in [0,\pi]$, and it is continuous on $[0,\pi]$, also $W_N^{(M)}(1) = 1$.

By [1], p. 344, $W_N^{(M)}$ are positive sublinear operators.

Call $I_N^+(x) = \left\{k \in \{0,1,...,N\}; s_{N,k}(x) > 0\right\}$, and set $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$.

We see that

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k\in I_N^+(x)} s_{N,k}(x)\, f(x_{N,k})}{\bigvee_{k\in I_N^+(x)} s_{N,k}(x)}. \tag{55}$$

By [1], p. 346, we have

$$W_N^{(M)}(|\cdot - x|)(x) \leq \frac{\pi}{2N},\quad \forall\, N \in \mathbb{N},\ \forall\, x \in [0,\pi]. \tag{56}$$

Notice also $\left|x_{N,k} - x\right| \leq \pi$, $\forall\, x \in [0,\pi]$.

Therefore ($\lambda \geq 1$) it holds

$$W_N^{(M)}\left(|\cdot - x|^\lambda\right)(x) \leq \frac{\pi^{\lambda-1}\pi}{2N} = \frac{\pi^\lambda}{2N},\quad \forall\, x \in [0,\pi], \forall\, N \in \mathbb{N}. \tag{57}$$

We continue with

Theorem 7. *Let* $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ *and* $f \in C_{x+}^\nu([0,\pi]) \cap C_{x-}^\nu([0,\pi])$. *Assume that* $f^{(k)}(x) = 0$, $k = 1,...,p-1$, *and* $\left(D_{x+}^\nu f\right)(x) = \left(D_{x-}^\nu f\right)(x) = 0$. *Then*

$$\left|W_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^\nu f, \left(\frac{\pi^{\nu+1}}{2N}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)}.$$

$$\left[\frac{\pi^\nu}{2N} + \frac{1}{(\nu+1)}\left(\frac{\pi^{\nu+1}}{2N}\right)^{\frac{\nu}{\nu+1}}\right],\quad \forall\, N \in \mathbb{N}. \tag{58}$$

We have that $\lim_{N\to+\infty} W_N^{(M)}(f)(x) = f(x)$.

Proof. Applying (36) for $W_N^{(M)}$ and using (57), we get

$$\left|W_N^{(M)}(f)(x)-f(x)\right| \le \frac{\omega_1\left(D_x^{\nu} f,\delta\right)}{\Gamma(\nu+1)}\left[\frac{\pi^{\nu}}{2N}+\frac{\frac{\pi^{\nu+1}}{2N}}{(\nu+1)\delta}\right]. \tag{59}$$

Choose $\delta=\left(\frac{\pi^{\nu+1}}{2N}\right)^{\frac{1}{\nu+1}}$, then $\delta^{\nu+1}=\frac{\pi^{\nu+1}}{2N}$, and $\delta^{\nu}=\left(\frac{\pi^{\nu+1}}{2N}\right)^{\frac{\nu}{\nu+1}}$. We use the last into (59) and we obtain (58).

We make

Remark. Here we continue with the Max-product truncated sampling operators (see [1], p. 13) defined by

$$K_N^{(M)}(f)(x)=\frac{\bigvee_{k=0}^{N}\frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N}\frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{60}$$

$\forall\, x\in[0,\pi]$, $f:[0,\pi]\to\mathbb{R}_+$ a continuous function.

Following [1], p. 350, and making the convention $\frac{\sin(0)}{0}=1$ and denoting $s_{N,k}(x)=\frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}$, we get that $s_{N,k}\left(\frac{k\pi}{N}\right)=1$, and $s_{N,k}\left(\frac{j\pi}{N}\right)=0$, if $k\neq j$, furthermore $K_N^{(M)}(f)\left(\frac{j\pi}{N}\right)=f\left(\frac{j\pi}{N}\right)$, for all $j\in\{0,...,N\}$.

Since $s_{N,j}\left(\frac{j\pi}{N}\right)=1$ it follows that $\bigvee_{k=0}^{N}s_{N,k}\left(\frac{j\pi}{N}\right)\ge 1>0$, for all $j\in\{0,1,...,N\}$. Hence $K_N^{(M)}(f)$ is well-defined function for all $x\in[0,\pi]$, and it is continuous on $[0,\pi]$, also $K_N^{(M)}(1)=1$. By [1], p. 350, $K_N^{(M)}$ are positive sublinear operators.

Denote $x_{N,k}:=\frac{k\pi}{N}$, $k\in\{0,1,...,N\}$.

By [1], p. 352, we have

$$K_N^{(M)}(|\cdot-x|)(x)\le\frac{\pi}{2N},\quad \forall\, N\in\mathbb{N},\ \forall\, x\in[0,\pi]. \tag{61}$$

Notice also $\left|x_{N,k}-x\right|\le\pi$, $\forall\, x\in[0,\pi]$.

Therefore ($\lambda\ge 1$) it holds

$$K_N^{(M)}\left(|\cdot-x|^{\lambda}\right)(x)\le\frac{\pi^{\lambda-1}\pi}{2N}=\frac{\pi^{\lambda}}{2N},\quad \forall\, x\in[0,\pi],\ \forall\, N\in\mathbb{N}. \tag{62}$$

We give

Theorem 8. *All as in Theorem 7. Then*

$$\left|K_N^{(M)}(f)(x)-f(x)\right|\le\frac{\omega_1\left(D_x^{\nu}f,\left(\frac{\pi^{\nu+1}}{2N}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)}\cdot$$
$$\left[\frac{\pi^{\nu}}{2N}+\frac{1}{(\nu+1)}\left(\frac{\pi^{\nu+1}}{2N}\right)^{\frac{\nu}{\nu+1}}\right],\quad \forall\, N\in\mathbb{N}. \tag{63}$$

We have that $\lim_{N\to+\infty}K_N^{(M)}(f)(x)=f(x)$.

Proof. As in Theorem 7.

We make

Remark. We mention the interpolation Hermite-Fejer polynomials on Chebyshev knots of the first kind (see [1], p. 4): Let $f:[-1,1]\to\mathbb{R}$ and based on the knots $x_{N,k}=\cos\left(\frac{(2(N-k)+1)}{2(N+1)}\pi\right)\in(-1,1)$, $k\in\{0,...,N\}$, $-1<x_{N,0}<x_{N,1}<...<x_{N,N}<1$, which are the roots of the first kind Chebyshev polynomial $T_{N+1}(x)=\cos((N+1)\arccos x)$, we define (see Fejér [8])

$$H_{2N+1}(f)(x)=\sum_{k=0}^{N}h_{N,k}(x)f\left(x_{N,k}\right), \tag{64}$$

where

$$h_{N,k}(x) = \left(1 - x \cdot x_{N,k}\right) \left(\frac{T_{N+1}(x)}{(N+1)\left(x - x_{N,k}\right)} \right)^2, \tag{65}$$

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see p. 12 of [1]) are defined by

$$H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f\left(x_{N,k}\right)}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \ \forall N \in \mathbb{N}, \tag{66}$$

where $f : [-1,1] \to \mathbb{R}_+$ is continuous.

Call

$$E_N(x) := H_{2N+1}^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) \left|x_{N,k} - x\right|}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \ x \in [-1,1]. \tag{67}$$

Then by [1], p. 287 we obtain that

$$E_N(x) \leq \frac{2\pi}{N+1}, \ \forall x \in [-1,1], \ N \in \mathbb{N}. \tag{68}$$

For $m > 1$, we get

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) \left|x_{N,k} - x\right|^m}{\bigvee_{k=0}^{N} h_{N,k}(x)} =$$

$$\frac{\bigvee_{k=0}^{N} h_{N,k}(x) \left|x_{N,k} - x\right| \left|x_{N,k} - x\right|^{m-1}}{\bigvee_{k=0}^{N} h_{N,k}(x)} \leq 2^{m-1} \frac{\bigvee_{k=0}^{N} h_{N,k}(x) \left|x_{N,k} - x\right|}{\bigvee_{k=0}^{N} h_{N,k}(x)} \tag{69}$$

$$\leq \frac{2^m \pi}{N+1}, \ \forall x \in [-1,1], \ N \in \mathbb{N}.$$

Hence it holds

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^m \pi}{N+1}, \ \forall x \in [-1,1], \ m > 1, \forall N \in \mathbb{N}. \tag{70}$$

Furthermore we have

$$H_{2N+1}^{(M)}(1)(x) = 1, \ \forall x \in [-1,1], \tag{71}$$

and $H_{2N+1}^{(M)}$ maps continuous functions to continuous functions over $[-1,1]$ and for any $x \in \mathbb{R}$ we have $\bigvee_{k=0}^{N} h_{N,k}(x) > 0$.

We also have $h_{N,k}\left(x_{N,k}\right) = 1$, and $h_{N,k}(x_{N,j}) = 0$, if $k \neq j$, furthermore it holds $H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, for all $j \in \{0,...,N\}$, see [1], p. 282.

$H_{2N+1}^{(M)}$ are positive sublinear operators, [1], p. 282.

We give

Theorem 9. *Let $\nu > 1$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $x \in [-1,1]$, $f : [-1,1] \to \mathbb{R}_+$ and $f \in C_{x+}^{\nu}([-1,1]) \cap C_{x-}^{\nu}([-1,1])$. Assume that $f^{(k)}(x) = 0$, $k = 1,...,p-1$, and $\left(D_{x+}^{\nu} f\right)(x) = \left(D_{x-}^{\nu} f\right)(x) = 0$. Then*

$$\left|H_{2N+1}^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{\nu} f, \left(\frac{2^{\nu+1}\pi}{N+1}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot \tag{72}$$

$$\left[\frac{2^{\nu}\pi}{N+1} + \frac{1}{(\nu+1)} \left(\frac{2^{\nu+1}\pi}{N+1}\right)^{\frac{\nu}{\nu+1}}\right], \ \forall N \in \mathbb{N}.$$

Furthermore it holds $\lim_{N \to +\infty} H_{2N+1}^{(M)}(f)(x) = f(x)$.

Proof. Use of Theorem 2, (36) and (70). Choose $\delta := \left(\frac{2^{\nu+1}\pi}{N+1}\right)^{\frac{1}{\nu+1}}$, etc.

We continue with

*Remark.*Here we deal with Lagrange interpolation polynomials on Chebyshev knots of second kind plus the endpoints ± 1 (see [1], p. 5). These polynomials are linear operators attached to $f : [-1,1] \to \mathbb{R}$ and to the knots $x_{N,k} = \cos\left(\left(\frac{N-k}{N-1}\right)\pi\right) \in [-1,1]$, $k = 1,...,N$, $N \in \mathbb{N}$, which are the roots of $\omega_N(x) = \sin(N-1)t \sin t$, $x = \cos t$. Notice that $x_{N,1} = -1$ and $x_{N,N} = 1$. Their formula is given by ([1], p. 377)

$$L_N(f)(x) = \sum_{k=1}^{N} l_{N,k}(x) f(x_{N,k}), \tag{73}$$

where

$$l_{N,k}(x) = \frac{(-1)^{k-1}\omega_N(x)}{(1+\delta_{k,1}+\delta_{k,N})(N-1)(x-x_{N,k})}, \tag{74}$$

$N \geq 2$, $k = 1,...,N$, and $\omega_N(x) = \prod_{k=1}^{N}(x - x_{N,k})$ and $\delta_{i,j}$ denotes the Kronecher's symbol, that is $\delta_{i,j} = 1$, if $i = j$, and $\delta_{i,j} = 0$, if $i \neq j$.

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ± 1, are defined by ([1], p. 12)

$$L_N^{(M)}(f)(x) = \frac{\bigvee_{k=1}^{N} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k=1}^{N} l_{N,k}(x)}, \quad x \in [-1,1], \tag{75}$$

where $f : [-1,1] \to \mathbb{R}_+$ continuous.

First we see that $L_N^{(M)}(f)(x)$ is well defined and continuous for any $x \in [-1,1]$. Following [1], p. 289, because $\sum_{k=1}^{N} l_{N,k}(x) = 1$, $\forall\, x \in \mathbb{R}$, for any x there exists $k \in \{1,...,N\} : l_{N,k}(x) > 0$, hence $\bigvee_{k=1}^{N} l_{N,k}(x) > 0$. We have that $l_{N,k}(x_{N,k}) = 1$, and $l_{N,k}(x_{N,j}) = 0$, if $k \neq j$. Furthermore it holds $L_N^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, all $j \in \{1,...,N\}$, and $L_N^{(M)}(1) = 1$.

Call $I_N^+(x) = \{k \in \{1,...,N\}; l_{N,k}(x) > 0\}$, then $I_N^+(x) \neq \emptyset$.

So for $f \in C_+([-1,1])$ we get

$$L_N^{(M)}(f)(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \geq 0. \tag{76}$$

Notice here that $|x_{N,k} - x| \leq 2$, $\forall\, x \in [-1,1]$.

By [1], p. 297, we get that

$$L_N^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k=1}^{N} l_{N,k}(x)|x_{N,k} - x|}{\bigvee_{k=1}^{N} l_{N,k}(x)} =$$

$$\frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)|x_{N,k} - x|}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{\pi^2}{6(N-1)}, \tag{77}$$

$N \geq 3$, $\forall\, x \in (-1,1)$, N is odd.

We get that $(m > 1)$

$$L_N^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)|x_{N,k} - x|^m}{\bigvee_{k \in I_N^+(x)} l_{N,k}(x)} \leq \frac{2^{m-1}\pi^2}{6(N-1)}, \tag{78}$$

$N \geq 3$ odd, $\forall\, x \in (-1,1)$.

$L_N^{(M)}$ are positive sublinear operators, [1], p. 290.

We give

Theorem 10.*Same assumptions as in Theorem 9. Then*

$$\left|L_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^\nu f, \left(\frac{2^\nu \pi^2}{6(N-1)}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)}. \tag{79}$$

$$\left[\frac{2^{\nu-1}\pi^2}{6(N-1)} + \frac{1}{(\nu+1)}\left(\frac{2^\nu \pi^2}{6(N-1)}\right)^{\frac{\nu}{\nu+1}}\right], \quad \forall\, N \in \mathbb{N} : N \geq 3, \textit{ odd}.$$

It holds $\lim_{N \to +\infty} L_N^{(M)}(f)(x) = f(x)$.

*Proof.*By Theorem 2, choose $\delta := \left(\frac{2^{\nu}\pi^2}{6(N-1)}\right)^{\frac{1}{\nu+1}}$, use of (36) and (78). At ± 1 the left hand side of (79) is zero, thus (79) is trivially true.

We make

*Remark.*Let $f \in C_+([-1,1])$, $N \geq 4$, $N \in \mathbb{N}$, N even.
By [1], p. 298, we get

$$L_N^{(M)}(|\cdot - x|)(x) \leq \frac{4\pi^2}{3(N-1)} = \frac{2^2\pi^2}{3(N-1)}, \quad \forall x \in (-1,1). \tag{80}$$

Hence $(m > 1)$

$$L_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^{m+1}\pi^2}{3(N-1)}, \quad \forall x \in (-1,1). \tag{81}$$

We present

Theorem 11.*Same assumptions as in Theorem 9. Then*

$$\left|L_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(D_x^{\nu} f, \left(\frac{2^{\nu+2}\pi^2}{3(N-1)}\right)^{\frac{1}{\nu+1}}\right)}{\Gamma(\nu+1)} \cdot \tag{82}$$

$$\left[\frac{2^{\nu+1}\pi^2}{3(N-1)} + \frac{1}{(\nu+1)}\left(\frac{2^{\nu+2}\pi^2}{3(N-1)}\right)^{\frac{\nu}{\nu+1}}\right], \quad \forall N \in \mathbb{N},\ N \geq 4,\ N \text{ is even.}$$

It holds $\lim_{N \to +\infty} L_N^{(M)}(f)(x) = f(x)$.

*Proof.*By Theorem 2, use of (36) and (81). Choose $\delta = \left(\frac{2^{\nu+2}\pi^2}{3(N-1)}\right)^{\frac{1}{\nu+1}}$, etc.

We need

Definition 3.*Let $x, x_0 \in \mathbb{R}$, $x \geq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $[\cdot]$ is the integral part, $\alpha = \nu - p$.*
Let $f \in C_b^p(\mathbb{R})$, i.e. $f \in C^p(\mathbb{R})$ with $\left\|f^{(p)}\right\|_\infty < +\infty$, where $\|\cdot\|_\infty$ is the supremum norm.
Clearly $\left(J_\nu^{x_0} f\right)(x)$ can be defined via (8) over $[x_0, +\infty)$.
We define the subspace $C_{x_0+}^{\nu}(\mathbb{R})$ of $C_b^p(\mathbb{R})$:

$$C_{x_0+}^{\nu}(\mathbb{R}) := \left\{f \in C_b^p(\mathbb{R}) : J_{1-\alpha}^{x_0} f^{(p)} \in C^1([x_0, +\infty))\right\}.$$

For $f \in C_{x_0+}^{\nu}(\mathbb{R})$, we define the left generalized ν-fractional derivative of f over $[x_0, +\infty)$ as

$$D_{x_0+}^{\nu} f = \left(J_{1-\alpha}^{x_0} f^{(p)}\right)'. \tag{83}$$

When $\nu > 1$, clearly then the left generalized fractional Taylor formula ($f \in C_{x_0+}^{\nu}(\mathbb{R})$) (12) is valid.

We need

Definition 4.*Let $x, x_0 \in \mathbb{R}$, $x \leq x_0$, $\nu > 0$, $\nu \notin \mathbb{N}$, $p = [\nu]$, $\alpha = \nu - p$. Let $f \in C_b^p(\mathbb{R})$. Clearly $\left(J_{x_0-}^{\nu} f\right)(x)$ can be defined via (13) over $(-\infty, x_0]$.*
We define the subspace of $C_{x_0-}^{\nu}(\mathbb{R})$ of $C_b^p(\mathbb{R})$:

$$C_{x_0-}^{\nu}(\mathbb{R}) := \left\{f \in C_b^p(\mathbb{R}) : \left(J_{x_0-}^{1-\alpha} f^{(p)}\right) \in C^1((-\infty, x_0])\right\}.$$

For $f \in C_{x_0-}^{\nu}(\mathbb{R})$, we define the right generalized ν-fractional derivative of f over $(-\infty, x_0]$ as

$$D_{x_0-}^{\nu} f = (-1)^{p-1}\left(J_{x_0-}^{1-\alpha} f^{(p)}\right)'. \tag{84}$$

When $\nu > 1$, clearly then the right generalized fractional Taylor formula ($f \in C_{x_0-}^{\nu}(\mathbb{R})$) (19) is valid.

We need

Definition 5.([9], p. 41) *Let $I \subset \mathbb{R}$ be an interval of finite or infinite length, and $f : I \to \mathbb{R}$ a bounded or uniformly continuous function. We define the first modulus of continuity*

$$\omega_1 (f, \delta)_I = \sup_{\substack{x,y \in I \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{85}$$

Clearly, it holds $\omega_1 (f, \delta)_I < +\infty$.

We also have

$$\omega_1 (f, r\delta)_I \leq (r+1)\, \omega_1 (f, \delta)_I, \quad any\ r \geq 0. \tag{86}$$

convention Let a real number $m > 1$, from now on we assume that $D^m_{x_0-} f$ is either bounded or uniformly continuous function on $(-\infty, x_0]$, similarly from now on we assume that $D^m_{x_0+} f$ is either bounded or uniformly continuous function on $[x_0, +\infty)$.

We need

Definition 6.*Let $D^m_{x_0} f$ (real number $m > 1$) denote any of $D^m_{x_0-} f$, $D^m_{x_0+} f$ and $\delta > 0$. We set*

$$\omega_1 \left(D^m_{x_0} f, \delta\right)_{\mathbb{R}} := \max \left\{ \omega_1 \left(D^m_{x_0-} f, \delta\right)_{(-\infty, x_0]}, \omega_1 \left(D^m_{x_0+} f, \delta\right)_{[x_0, +\infty)} \right\}, \tag{87}$$

where $x_0 \in \mathbb{R}$. Notice that $\omega_1 \left(D^m_{x_0} f, \delta\right)_{\mathbb{R}} < +\infty$.

We give

Theorem 12.*Let $m > 1$, $m \notin \mathbb{N}$, $p = [m]$, $x_0 \in \mathbb{R}$, and $f \in C^m_{x_0+} (\mathbb{R}) \cap C^m_{x_0-} (\mathbb{R})$. Assume that $f^{(k)} (x_0) = 0$, $k = 1, ..., p-1$, and $\left(D^m_{x_0+} f\right)(x_0) = \left(D^m_{x_0-} f\right)(x_0) = 0$. The convention 3 is imposed. Then*

$$|f(x) - f(x_0)| \leq \frac{\omega_1 \left(D^m_{x_0} f, \delta\right)_{\mathbb{R}}}{\Gamma(m+1)} \left[|x - x_0|^m + \frac{|x - x_0|^{m+1}}{(m+1)\,\delta} \right], \quad \delta > 0, \tag{88}$$

for all $x \in \mathbb{R}$.

*Proof.*Similar to Theorem 1.

*Remark.*Let $b : \mathbb{R} \to \mathbb{R}_+$ be a centered (it takes a global maximum at 0) bell-shaped function, with compact support $[-T, T]$, $T > 0$ (that is $b(x) > 0$ for all $x \in (-T, T)$) and $I = \int_{-T}^{T} b(x)\, dx > 0$.

The Cardaliaguet-Euvrard neural network operators are defined by (see [10])

$$C_{N,\alpha} (f)(x) = \sum_{k=-N^2}^{N^2} \frac{f\left(\frac{k}{n}\right)}{IN^{1-\alpha}} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right), \tag{89}$$

$0 < \alpha < 1$, $N \in \mathbb{N}$ and typically here $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded or uniformly continuous on $\mathbb{R}$.

$CB(\mathbb{R})$ denotes the continuous and bounded function on $\mathbb{R}$, and

$$CB_+ (\mathbb{R}) = \{ f : \mathbb{R} \to [0, \infty);\ f \in CB(\mathbb{R}) \}.$$

The corresponding max-product Cardaliaguet-Euvrard neural network operators will be given by

$$C^{(M)}_{N,\alpha} (f)(x) = \frac{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k=-N^2}^{N^2} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{90}$$

$x \in \mathbb{R}$, typically here $f \in CB_+ (\mathbb{R})$, see also [10].

Next we follow [10].

For any $x \in \mathbb{R}$, denoting

$$J_{T,N}(x) = \left\{ k \in \mathbb{Z};\ -N^2 \leq k \leq N^2,\ N^{1-\alpha}\left(x - \frac{k}{N}\right) \in (-T,T) \right\},$$

we can write

$$C_{N,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) f\left(\frac{k}{N}\right)}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{91}$$

$x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, where $J_{T,N}(x) \neq \emptyset$. Indeed, we have $\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right) > 0$, $\forall\, x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

We have that $C_{N,\alpha}^{(M)}(1)(x) = 1$, $\forall\, x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

See in [10] there: Lemma 2.1, Corollary 2.2 and Remarks.
We need

Theorem 13.([10]) *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T,T]$, $T > 0$, $0 < \alpha < 1$ and $C_{N,\alpha}^{(M)}$ be defined as in (90).*

(i) If $|f(x)| \leq c$ for all $x \in \mathbb{R}$ then $\left|C_{N,\alpha}^{(M)}(f)(x)\right| \leq c$, for all $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ and $C_{N,\alpha}^{(M)}(f)(x)$ is continuous at any point $x \in \mathbb{R}$, for all $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(ii) If $f, g \in CB_+(\mathbb{R})$ satisfy $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $C_{N,\alpha}^{(M)}(f)(x) \leq C_{N,\alpha}^{(M)}(g)(x)$ for all $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iii) $C_{N,\alpha}^{(M)}(f+g)(x) \leq C_{N,\alpha}^{(M)}(f)(x) + C_{N,\alpha}^{(M)}(g)(x)$ for all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$;

(iv) For all $f, g \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$, we have

$$\left|C_{N,\alpha}^{(M)}(f)(x) - C_{N,\alpha}^{(M)}(g)(x)\right| \leq C_{N,\alpha}^{(M)}(|f - g|)(x);$$

(v) $C_{N,\alpha}^{(M)}$ is positive homogeneous, that is $C_{N,\alpha}^{(M)}(\lambda f)(x) = \lambda C_{N,\alpha}^{(M)}(f)(x)$ for all $\lambda \geq 0$, $x \in \mathbb{R}$, $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$ and $f \in CB_+(\mathbb{R})$.

We make

*Remark.*We have that

$$E_{N,\alpha}(x) := C_{N,\alpha}^{(M)}(|\cdot - x|)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)\left|x - \frac{k}{N}\right|}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)}, \tag{92}$$

$\forall\, x \in \mathbb{R}$, and $N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

We mention from [10] the following:

Theorem 14.([10]) *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T,T]$, $T > 0$ and $0 < \alpha < 1$. In addition, suppose that the following requirements are fulfilled:*
(i) There exist $0 < m_1 \leq M_1 < \infty$ such that $m_1(T - x) \leq b(x) \leq M_1(T - x)$, $\forall\, x \in [0,T]$;
(ii) There exist $0 < m_2 \leq M_2 < \infty$ such that $m_2(x + T) \leq b(x) \leq M_2(x + T)$, $\forall\, x \in [-T,0]$.
Then for all $f \in CB_+(\mathbb{R})$, $x \in \mathbb{R}$ and for all $N \in \mathbb{N}$ satisfying $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, we have the estimate

$$\left|C_{N,\alpha}^{(M)}(f)(x) - f(x)\right| \leq c\,\omega_1\left(f, N^{\alpha-1}\right)_{\mathbb{R}}, \tag{93}$$

where

$$c := 2\left(\max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} + 1\right),$$

and

$$\omega_1 (f, \delta)_{\mathbb{R}} := \sup_{\substack{x,y \in \mathbb{R}: \\ |x-y| \leq \delta}} |f(x) - f(y)|. \tag{94}$$

We make

*Remark.*In [10], was proved that

$$E_{N,\alpha}(x) \leq \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \;\; \forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{95}$$

That is

$$C_{N,\alpha}^{(M)}(|\cdot - x|)(x) \leq \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \;\; \forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{96}$$

From (92) we have that $\left|x - \frac{k}{N}\right| \leq \frac{T}{N^{1-\alpha}}$.

Hence ($m > 1$) ($\forall x \in \mathbb{R}$ and $N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$)

$$C_{N,\alpha}^{(M)}(|\cdot - x|^m)(x) = \frac{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)\left|x - \frac{k}{N}\right|^m}{\bigvee_{k \in J_{T,N}(x)} b\left(N^{1-\alpha}\left(x - \frac{k}{N}\right)\right)} \leq \tag{97}$$

$$\left(\frac{T}{N^{1-\alpha}}\right)^{m-1} \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} N^{\alpha-1}, \;\; \forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}.$$

Then ($m > 1$) it holds

$$C_{N,\alpha}^{(M)}(|\cdot - x|^m)(x) \leq$$

$$T^{m-1} \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} \frac{1}{N^{m(1-\alpha)}}, \;\; \forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{98}$$

Call

$$\theta := \max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} > 0. \tag{99}$$

Consequently ($m > 1$) we derive

$$C_{N,\alpha}^{(M)}(|\cdot - x|^m)(x) \leq \frac{\theta T^{m-1}}{N^{m(1-\alpha)}}, \;\; \forall N > \max\left\{T + |x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}. \tag{100}$$

We need

Theorem 15.*All here as in Theorem 12, where $x = x_0 \in \mathbb{R}$ is fixed. Let b be a centered bell-shaped function, continuous and with compact support $[-T, T]$, $T > 0$, $0 < \alpha < 1$ and $C_{N,\alpha}^{(M)}$ be defined as in (90). Then*

$$\left|C_{N,\alpha}^{(M)}(f)(x) - f(x)\right| \leq$$

$$\frac{\omega_1(D_x^m f, \delta)_{\mathbb{R}}}{\Gamma(m+1)}\left[C_{N,\alpha}^{(M)}(|\cdot - x|^m)(x) + \frac{C_{N,\alpha}^{(M)}\left(|\cdot - x|^{m+1}\right)(x)}{(m+1)\delta}\right], \tag{101}$$

$\forall N \in \mathbb{N} : N > \max\left\{T + |x|, T^{-\frac{1}{\alpha}}\right\}$.

*Proof.*By Theorem 12 and (88) we get

$$|f(\cdot)-f(x)| \leq \frac{\omega_1\left(D_x^m f,\delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[|\cdot-x|^m+\frac{|\cdot-x|^{m+1}}{(m+1)\delta}\right], \quad \delta>0, \tag{102}$$

true over $\mathbb{R}$.

As in Theorem 13 and using similar reasoning and $C_{N,\alpha}^{(M)}(1)=1$, we get

$$\left|C_{N,\alpha}^{(M)}(f)(x)-f(x)\right| \leq C_{N,\alpha}^{(M)}(|f(\cdot)-f(x)|)(x) \overset{(102)}{\leq}$$

$$\frac{\omega_1\left(D_x^m f,\delta\right)_{\mathbb{R}}}{\Gamma(m+1)}\left[C_{N,\alpha}^{(M)}\left(|\cdot-x|^m\right)(x)+\frac{C_{N,\alpha}^{(M)}\left(|\cdot-x|^{m+1}\right)(x)}{(m+1)\delta}\right], \tag{103}$$

$\forall\, N\in\mathbb{N}: N>\max\left\{T+|x|, T^{-\frac{1}{\alpha}}\right\}$.

We continue with

Theorem 16.*Here all as in Theorem 12, where $x=x_0\in\mathbb{R}$ is fixed and $m>1$. Also the same assumptions as in Theorem 14. Then*

$$\left|C_{N,\alpha}^{(M)}(f)(x)-f(x)\right| \leq \frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f,\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}\cdot$$

$$\left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}}+\frac{1}{(m+1)}\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{m}{m+1}}\right], \tag{104}$$

$\forall\, N\in\mathbb{N}: N>\max\left\{T+|x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$.

We have that $\lim_{N\to+\infty} C_{N,\alpha}^{(M)}(f)(x)=f(x)$.

*Proof.*We apply Theorem 15. In (101) we choose

$$\delta:=\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}},$$

thus $\delta^{m+1}=\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}$, and

$$\delta^m=\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{m}{m+1}}. \tag{105}$$

Therefore we have

$$\left|C_{N,\alpha}^{(M)}(f)(x)-f(x)\right| \overset{(100)}{\leq} \frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f,\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}\cdot \tag{106}$$

$$\left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}}+\frac{1}{(m+1)\delta}\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right]=$$

$$\frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f,\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}\left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}}+\frac{1}{(m+1)\delta}\delta^{m+1}\right]\overset{(105)}{=}$$

$$\frac{1}{\Gamma(m+1)}\omega_1\left(D_x^m f,\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{1}{m+1}}\right)_{\mathbb{R}}\cdot$$

$$\left[\frac{\theta T^{m-1}}{N^{m(1-\alpha)}}+\frac{1}{(m+1)}\left(\frac{\theta T^m}{N^{(m+1)(1-\alpha)}}\right)^{\frac{m}{m+1}}\right], \tag{107}$$

$\forall\, N\in\mathbb{N}: N>\max\left\{T+|x|, \left(\frac{2}{T}\right)^{\frac{1}{\alpha}}\right\}$, proving the inequality (104).

We finish with (case of $\alpha = 1.5$)

Corollary 1. *Let* $x \in [0,1]$, $f : [0,1] \to \mathbb{R}_+$ *and* $f \in C_{x+}^{1.5}([0,1]) \cap C_{x-}^{1.5}([0,1])$. *Assume that* $f'(x) = 0$, *and* $\left(D_{x+}^{1.5} f\right)(x) = \left(D_{x-}^{1.5} f\right)(x) = 0$. *Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{4\omega_1\left(D_x^{1.5} f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{2}{5}}\right)}{3\sqrt{\pi}}$$

$$\left[\frac{6}{\sqrt{N+1}} + \frac{2}{5}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{3}{5}}\right], \quad \forall N \in \mathbb{N}. \tag{108}$$

Proof. By Theorem 3, apply (37).

Due to lack of space we do not give other example applications.

4 Conclusion

In this article we determined the rate of convergence of general sublinear positive operators to the unit in the presence of Canavati fractional smoothness. We gave applications to a great variety of max-product operators. The results are quantitative and the produced Jackson type inequalities involve the modulus of continuity of Canavati fractional order derivative under initial conditions. Our approach in the applications results in a higher order of convergence.

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